Chapter 2

Polynomial and Rational Functions

2.2 Quadratic Functions

Chapter 2

Homework

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Chapter 2

Objectives

- Recognize characteristics of parabolas.
- Graph parabolas.
- Determine a quadratic function’s minimum or maximum value.
- Solve problems involving a quadratic function’s minimum or maximum value.

- A polynomial function is of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots a_1 x^1 + a_0$$

- n is a non-negative integer (it is 0 in the last term)
- The degree of the function is n
- There are n+1 terms

The Standard Form of a Quadratic Function

- The **general form** of the quadratic function is

$$f(x) = ax^2 + bx + c$$

- The **standard form** of the quadratic function is

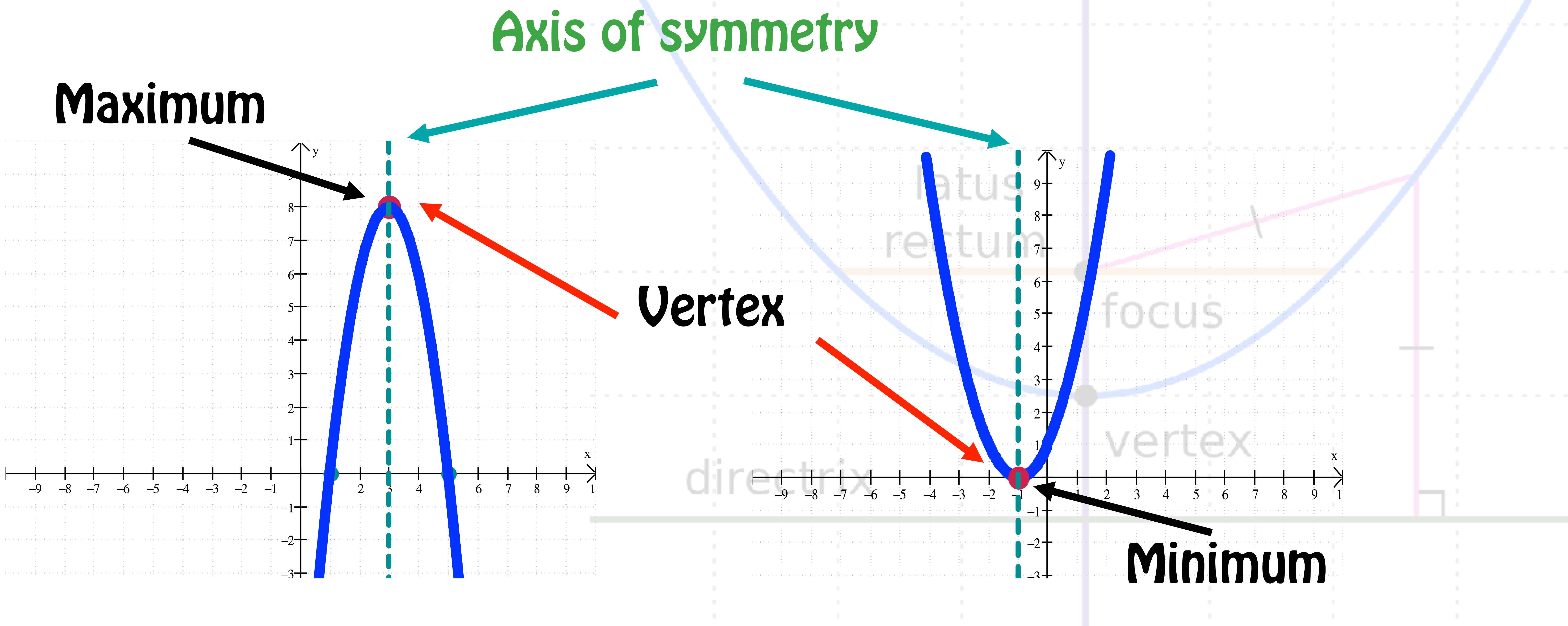
$$f(x) = a(x-h)^2 + k$$

- The graph of the quadratic function, f, is a **parabola**.

- The parabola is symmetric with respect to the line $x = h$ (axis of symmetry). If $a > 0$ the parabola **opens up** (concave up), if $a < 0$, the parabola **opens down** (concave down).

Graphs of Quadratic Functions   Parabolas

$$f(x) = ax^2 + bx + c$$



Graphing Quadratic Functions with Equations in Standard Form

To graph $f(x) = a(x - h)^2 + k$.

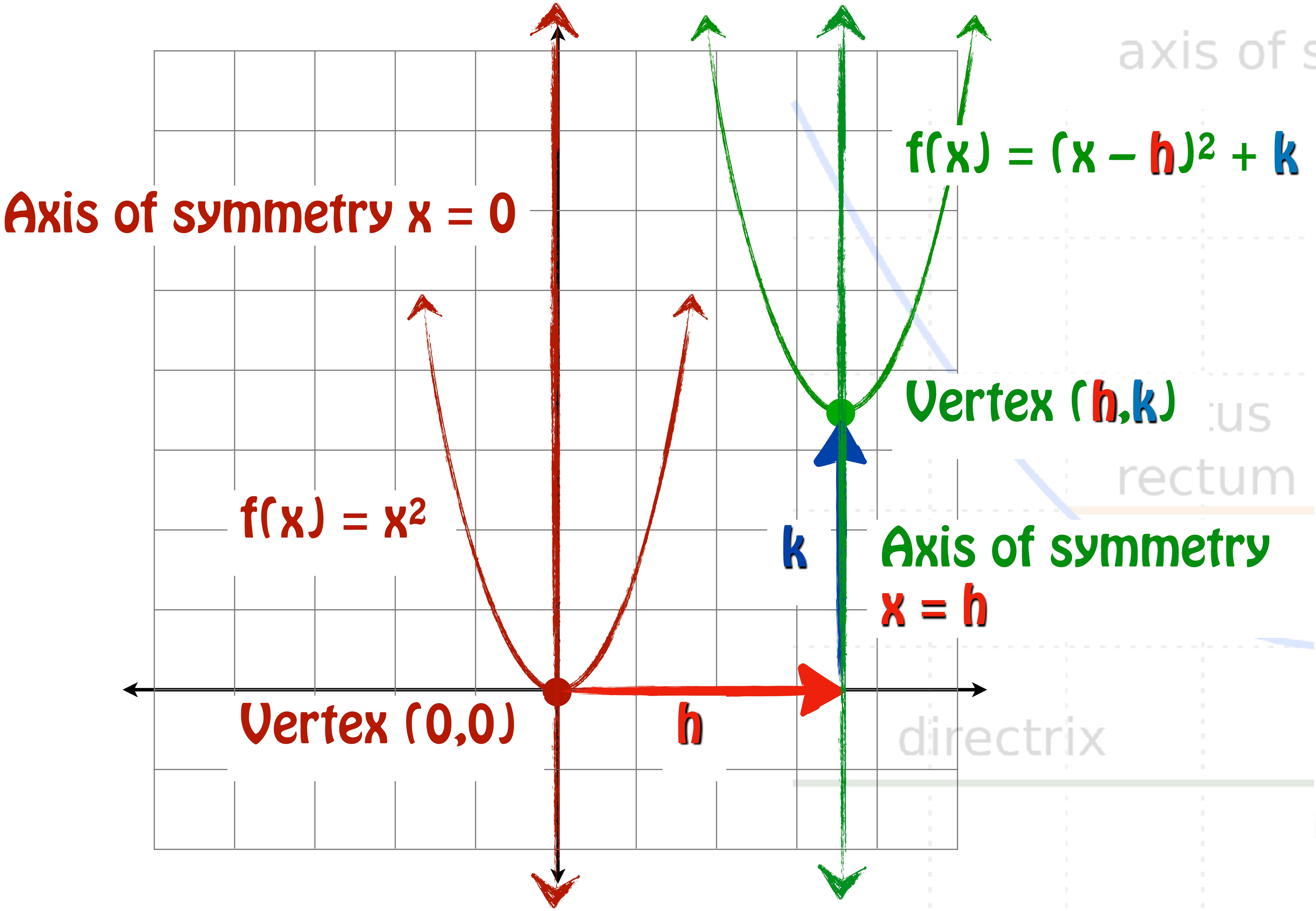
1. Determine the direction. $a > 0$, up $a < 0$ down

2. Determine the vertex. (h, k)

3. Find the intercepts. The zeros, $f(x) = 0$, are the x-intercepts. $f(0)$ determines the y-intercept

4. Find the additional points as necessary (remember the symmetry), plot the points and graph. Keep in mind the graph is a parabola (u-shaped).

Seeing the Transformation



STUDY TIP

The standard form of a quadratic function identifies four basic transformations of the graph of $y = x^2$.

a. The factor $|a|$ produces a vertical stretch or shrink.

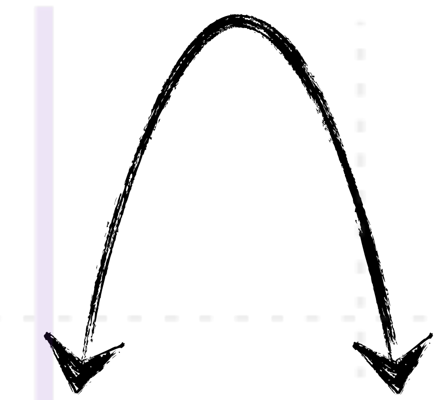
b. If $a < 0$, the graph is reflected in the x-axis.

c. The factor $(x - h)^2$ represents a horizontal shift of h units.

d. The term k represents a vertical shift of k units.

Example: Graphing a Quadratic Function in Standard Form

- Graph the quadratic function $f(x) = -(x - 1)^2 + 4$.



- Step 1 $a = -1$, $a < 0$. The parabola opens down.
- Step 2 The vertex (h, k) is $(1, 4)$.
- Step 3 The x-intercepts $0=-(x-1)^2+4$ $-4=-(x-1)^2$ $(x-1)^2=4$

  $x-1=\pm 2$ $x-1=\pm 2$ $x=1\pm 2$

  The x-intercepts are -1 and 3
- Step 4 The y-intercept $y=-(0-1)^2+4$ $y=-(-1)^2+4$ $y=-1+4$

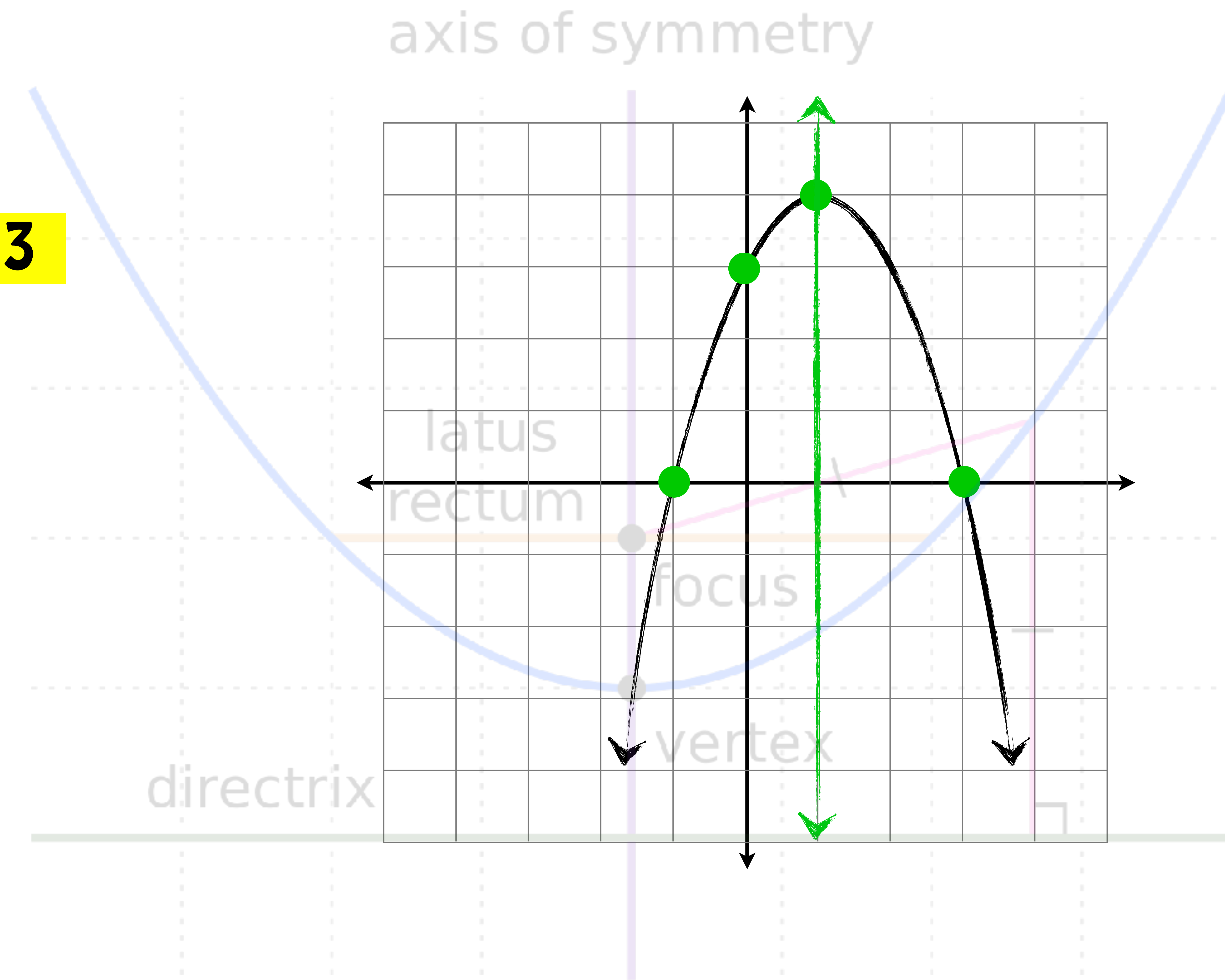
  The y-intercept is 3

Example: Graphing a Quadratic Function in Standard Form

The vertex (h, k) is (1, 4).

The x-intercepts are -1 and 3

The y-intercept is 3



General Form to Standard Form

To convert the general form $f(x) = ax^2 + bx + c$ to the standard form $f(x) = a(x - h)^2 + k$ we … … complete the square.

$$f(x) = ax^2 + bx + c$$

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$f(x) = a\left(x^2 + \frac{b}{a}x + \_\right) + c - \_$$

$$f(x) = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2$$

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - a\frac{b^2}{4a^2}$$

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac}{4a} - \frac{b^2}{4a}$$

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Note that the axis of symmetry is

$$x = -\frac{b}{2a}$$

The Vertex of a Parabola Whose Equation is $f(x) = ax^2 + bx + c$

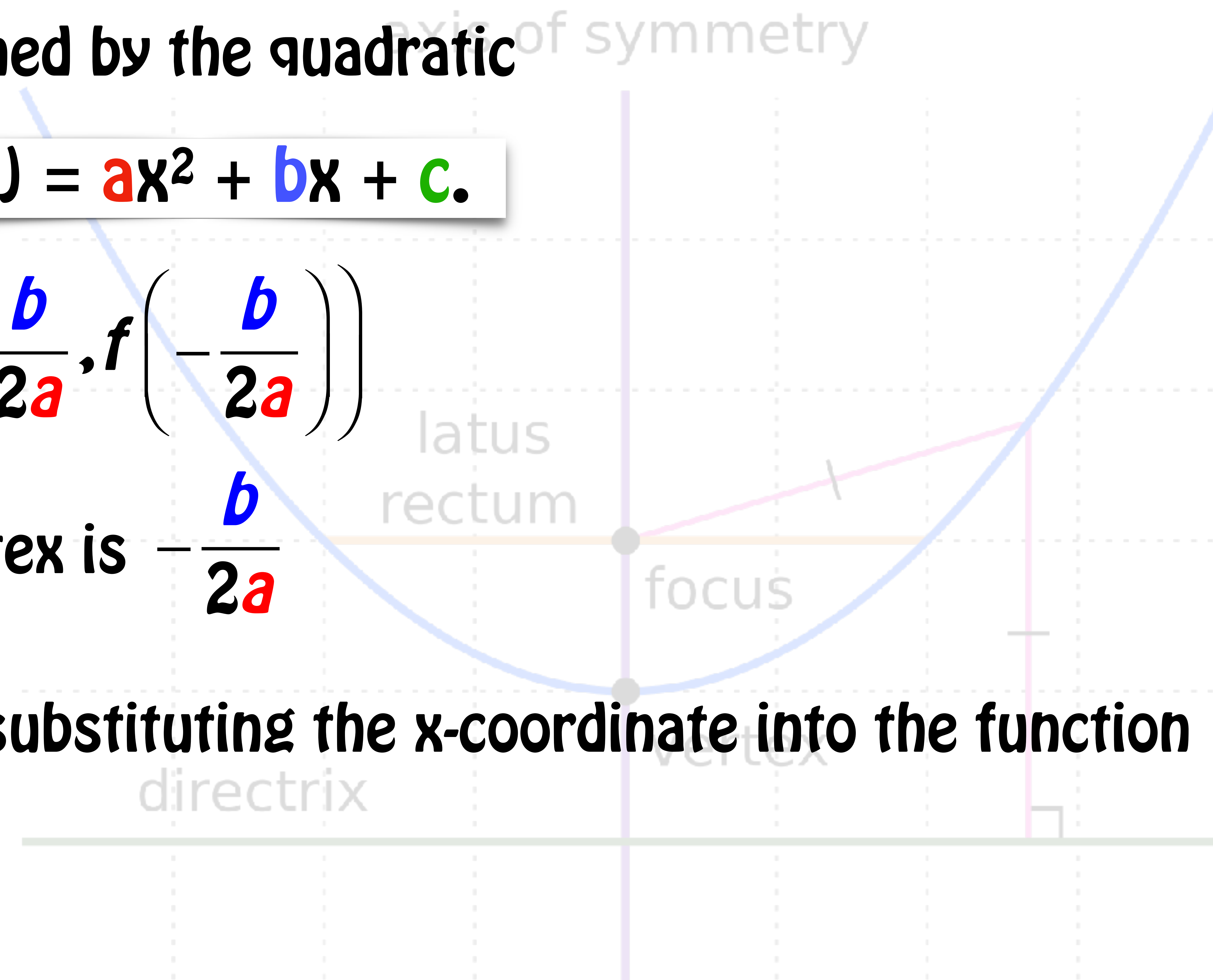
Consider the parabola defined by the quadratic

$$f(x) = ax^2 + bx + c.$$

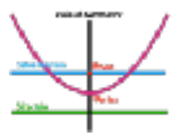
The parabola's vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

The x-coordinate is of the vertex is $-\frac{b}{2a}$

The y-coordinate is found by substituting the x-coordinate into the function and evaluating $f\left(-\frac{b}{2a}\right)$



Graphing Quadratic Functions with Equations in the Form $f(x)=ax^2+bx+c$



To graph $f(x) = ax^2 + bx + c$.

1. Determine the direction. $a > 0$, up $a < 0$ down
2. Determine the vertex. $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$
3. Find the intercepts. The zeros, $f(x) = 0$, are the x-intercepts. $f(0)$ determines the y-intercept
4. Find the additional points as necessary (remember the symmetry), plot the points and graph. Keep in mind the graph is a parabola (u-shaped).

Example

Write the $f(x) = 2x^2 + 8x + 2$ in standard form

$$f(x) = 2x^2 + 8x + 2$$

$$f(x) = 2x^2 + 8x + \_\_\_ + 2 - \_\_\_$$

$$f(x) = 2(x^2 + 4x + \_\_\_) + 2 - \_\_\_$$

$$f(x) = 2(x^2 + 4x + 4) + 2 - 8$$

$$f(x) = 2(x + 2)^2 - 6$$

The vertex is (-2,-6).

The parabola opens up.

$$0 = 2(x + 2)^2 - 6$$

$$6 = 2(x + 2)^2 \quad 3 = (x + 2)^2$$

$$\pm\sqrt{3} = x + 2 \quad x = -2 \pm \sqrt{3}$$

x-intercepts

$$f(0) = 2(2)^2 - 6 \quad f(0) = 2$$

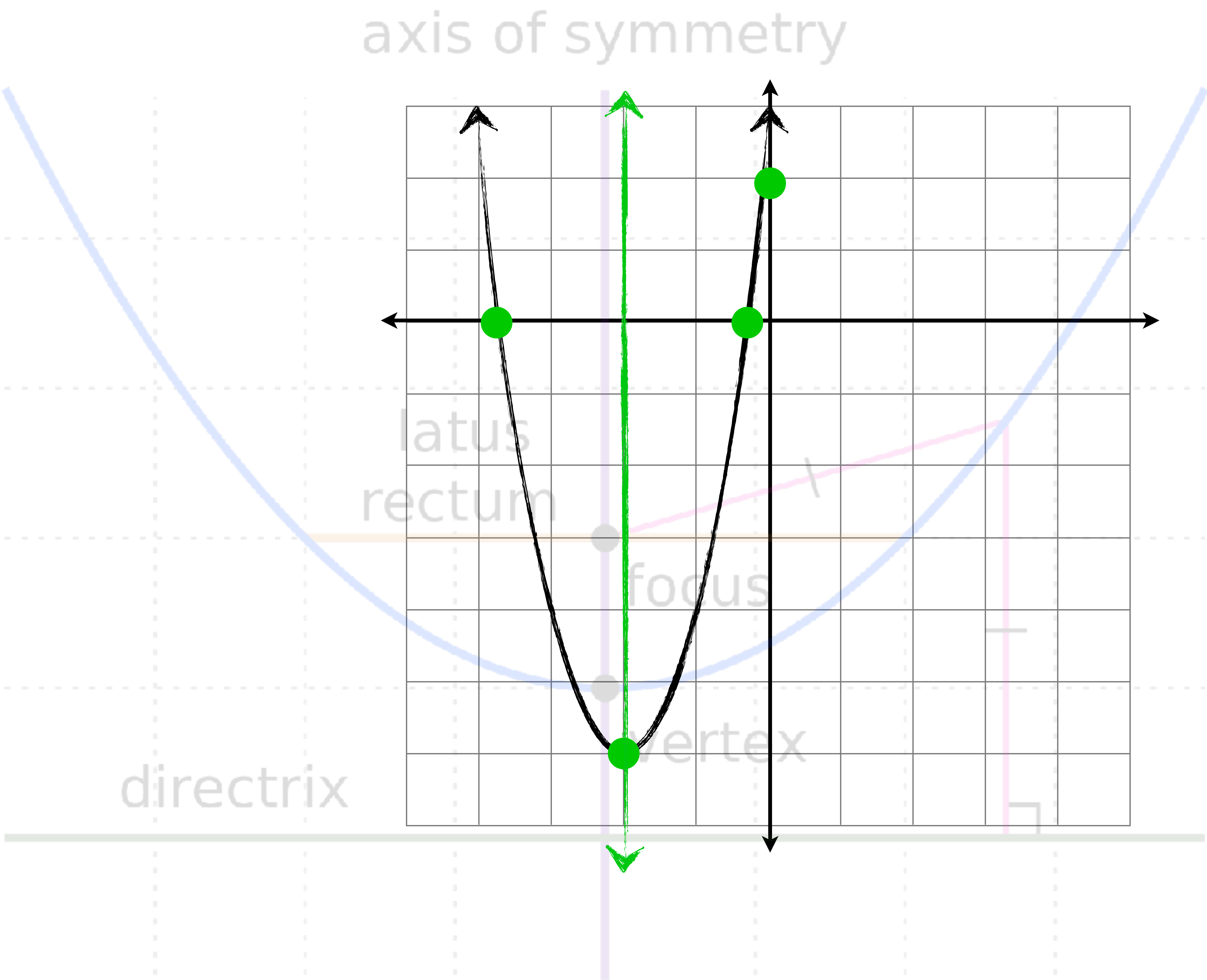
y-intercept

Graph

The vertex is (-2,-6).

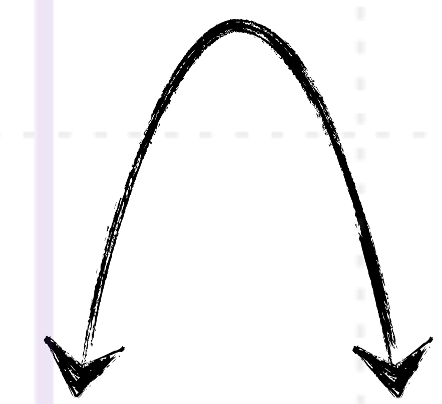
x-intercepts $x = -2 \pm \sqrt{3}$

y-intercept $f(0) = 2$



Graphing Quadratic Functions with Equations in the Form $f(x)=ax^2+bx+c$

- Graph the quadratic function $f(x) = -x^2 + 4x + 1$.
  - $a = -1$, $b = 4$, $c = 1$



- Step 1 $a = -1$, $a < 0$. The parabola opens down.
- Step 2 The vertex.

The x-coordinate is $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$

The y-coordinate is $f(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$

The vertex is (2,5).

Graphing Quadratic Functions with Equations in the Form $f(x)=ax^2+bx+c$

The vertex is (2,5).

Step 3 The x-intercepts $0 = -x^2 + 4x + 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{b^2 - 4(-1)(1)}}{2(-1)} = \frac{-4 \pm \sqrt{20}}{-2} = \frac{-4 \pm 2\sqrt{5}}{-2} = 2 \pm \sqrt{5}$$

The x-intercepts are $2-\sqrt{5}, 2+\sqrt{5}$

The y-intercepts $f(0) = -0^2 + 4(0) + 1$

The y-intercept is 1

Graphing Quadratic Functions with Equations in the Form $f(x)=ax^2+bx+c$

Finally; Graph $f(x) = -x^2 + 4x + 1$.

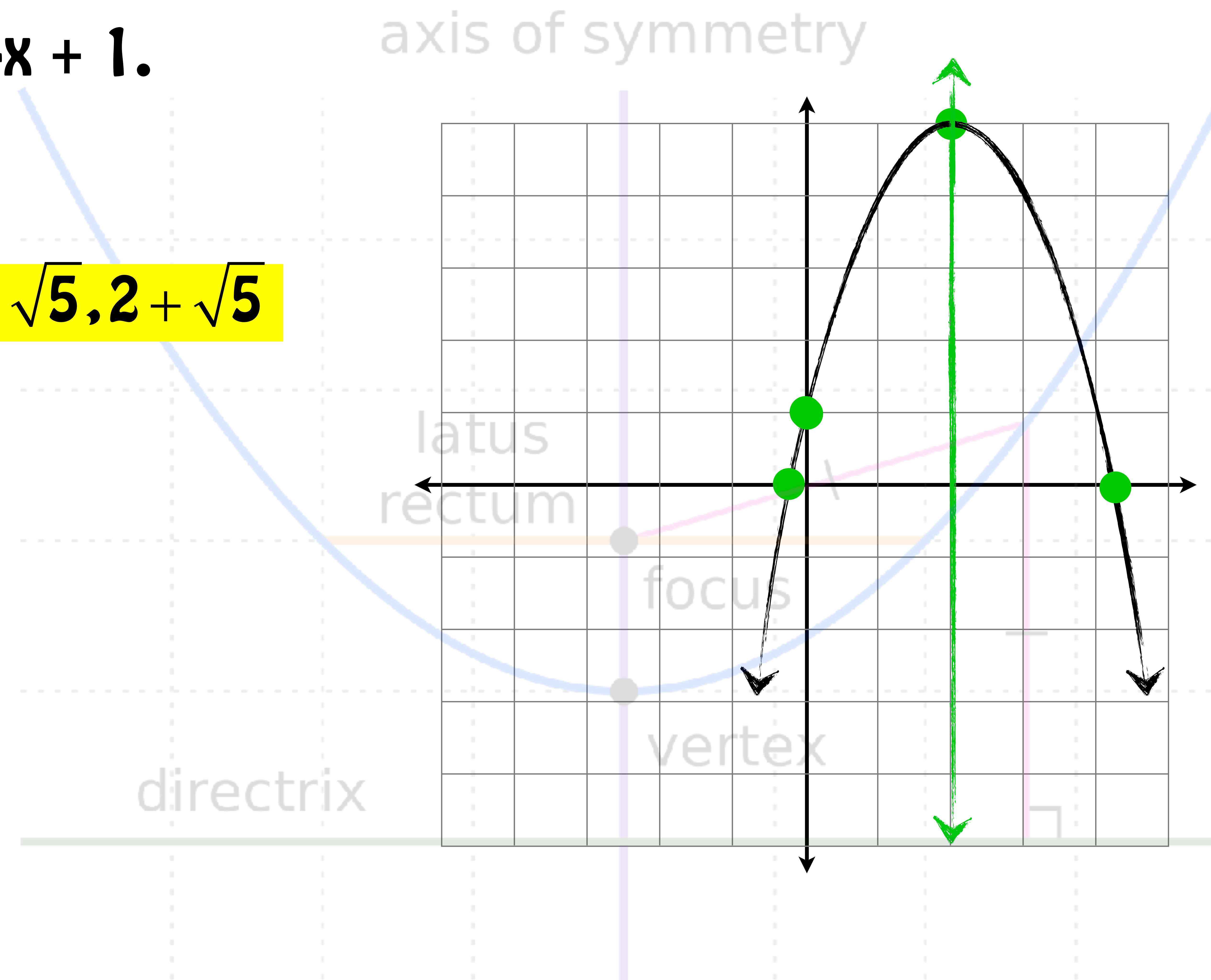
The vertex is (2,5).

The x-intercepts are $2-\sqrt{5}, 2+\sqrt{5}$

The y-intercept is 1

$$2-\sqrt{5} \approx 2-2.2 = -.2$$

$$2+\sqrt{5} \approx 2+2.2 = 4.2$$



Minimum and Maximum: Quadratic Functions

- Consider the quadratic function $f(x) = ax^2 + bx + c$.

1. If $a > 0$, then f has a **minimum** that occurs at $x = -\dfrac{b}{2a}$

   The **minimum** value is $f\left(-\dfrac{b}{2a}\right)$

2. If $a < 0$, then f has a **maximum** that occurs at $x = -\dfrac{b}{2a}$

   The **maximum** value is $f\left(-\dfrac{b}{2a}\right)$

- In either case, the value of x determines the **location** of the minimum or maximum value of the function f.
- y (or f(x)) is the value of the maximum or minimum of f.
  y determines the range of f.

The domain of any quadratic function includes all real numbers. If the vertex is the graph's maximum, the range includes all real numbers at or below the y-coordinate of the vertex. If the vertex is the graph's lowest point, the range includes all real numbers at or above the y-coordinate of the vertex.

Example: Obtaining Information about a Quadratic Function from Its Equation

- Consider the quadratic function $f(x) = 4x^2 - 16x + 1000$
  - Determine, without graphing, whether the function has a minimum value or a maximum value.

    $a = 4;\ a > 0.$ The function has a minimum value.

  - Find the minimum value. $x = -\frac{b}{2a} = -\frac{-16}{2(4)} = 2$

    $f(2) = 4(2)^2 - 16(2) + 1000 = 984$

    The minimum value is 984.

Example: Obtaining Information about a Quadratic Function from Its Equation

- Consider the quadratic function $f(x) = 4x^2 - 16x + 1000$
  - Identify the function's domain and range (without graphing).

    Like all quadratic functions, the domain is $(-\infty, \infty)$

  - We found that the vertex is at (2, 984).

    $a > 0$, the function has a minimum value.

    The range of the function is $[984, \infty)$

Solving Problems Involving Maximizing or Minimizing Quadratic Functions

1. Read the problem carefully and decide which quantity is to be maximized or minimized.

2. Write the quantity to minimized or maximized as a function in one variable.

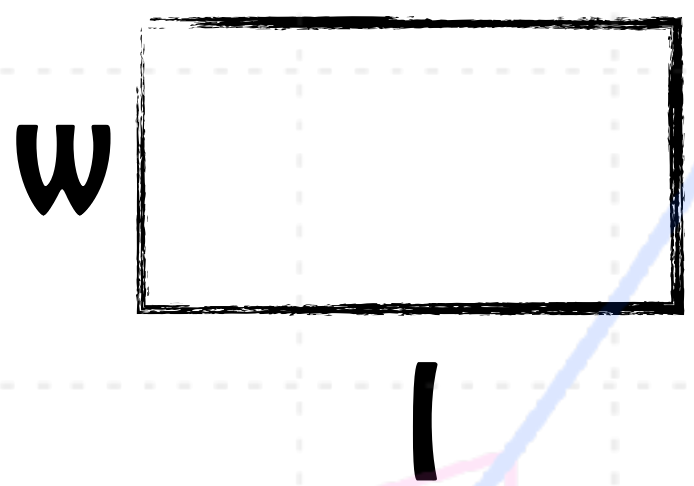
3. Write the function in the form $f(x) = ax^2 + bx + c$.

4. Find the vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ of the function.

5. If $a < 0$, the function has a maximum at the vertex, if $a > 0$ the function has a minimum at the vertex.

Example: Maximizing Area

You have 120 feet of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. What is the maximum area?



1. Given: 120 feet of fencing to enclose a rectangle.
2. Question: What is the maximum rectangular area?
3. Variable: Let w = width of area.
4. Perimeter = 2width + 2length

   Area = length x width

Example: Maximizing Area

- You have 120 feet of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. What is the maximum area?

Perimeter = 2width + 2length

$120 = 2w + 2\text{length}$ $\quad 60 - w = \text{length}$

Area = length x width

$\text{Area} = (60 - w) \bullet w \quad = (60 - w)w \quad A(w) = -w^2 + 60w$

Example: Maximizing Area

$A(w) = -w^2 + 60w$

- $a < 0$, so the function has a maximum at this value.

Find the vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ of the function.

$$w = -\frac{b}{2a} = -\frac{60}{2(-1)} = 30$$

This means that the area, A(w), of a rectangle with perimeter 120 feet is a maximum when one of the rectangle's dimensions, w, is 30 feet.

$$f\left(-\frac{b}{2a}\right) = -30^2 + 60(30) = 900$$

The minimum value of the function (Area) is 900 ft^2.

The dimensions of the rectangle are 30ft x 30ft.