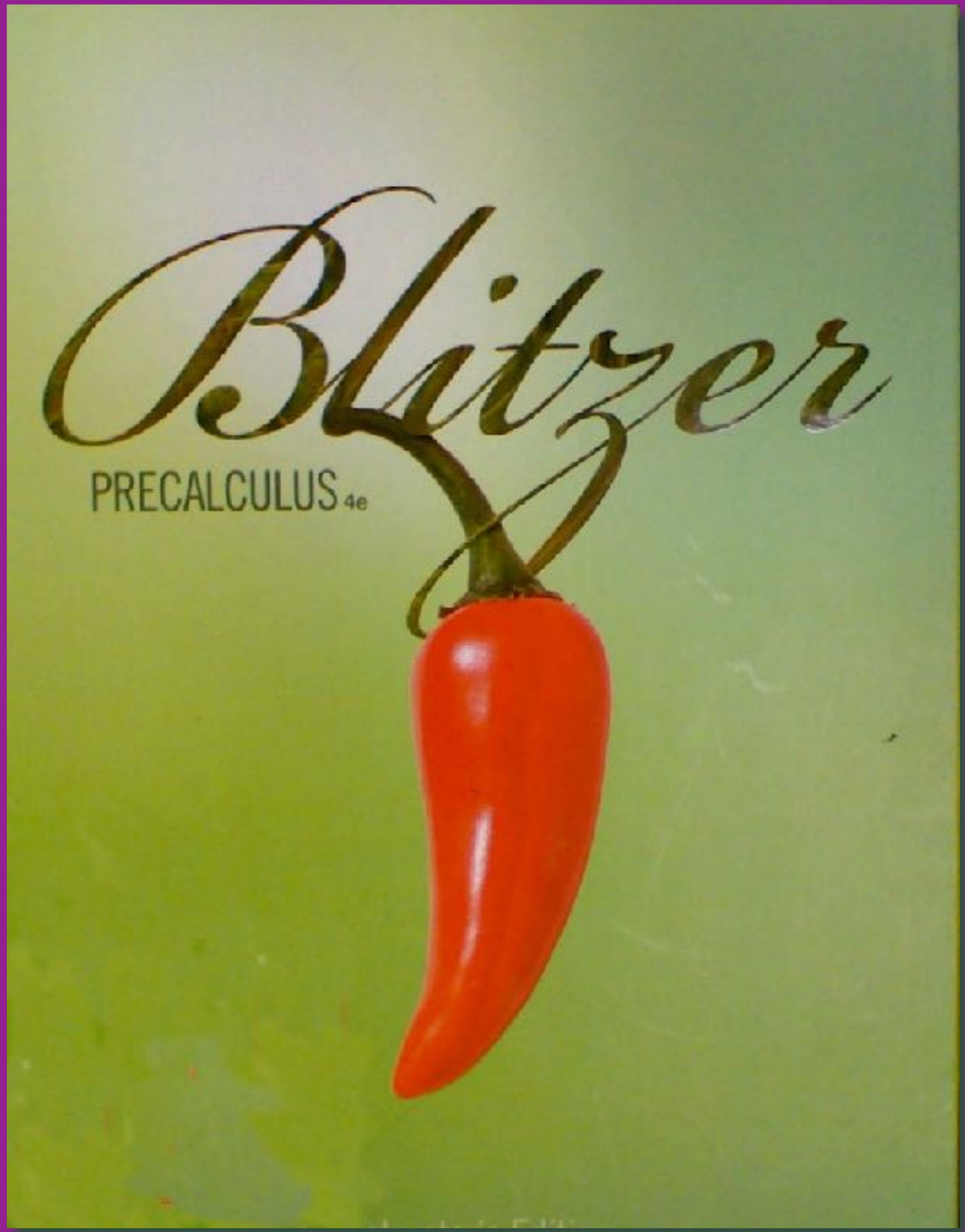
Chapter 2

Polynomial and Rational Functions

2.6 Rational Functions and Their Graphs



Chapter 2

Homework

<u>2.6</u> p 354 5, 7, 9-14, 23, 27, 31, 33, 35, 39, 43, 46, 47, 51, 53, 63, 69, 73, 75, 77

Objectives:

- Find the domains of rational functions.
- Use arrow notation.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Use transformations to graph rational functions.
- Graph rational functions.
- Identify slant asymptotes.
- Solve applied problems involving rational functions.

Rational Functions

- **Rational functions** are quotients of polynomial functions. This means that rational functions can be expressed as

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions and $q(x) \neq 0$.

- The **domain** of a rational function is the set of all real numbers except the x-values that make the denominator zero.

Finding the Domain of a Rational Function

- Find the domain of the rational function:

$$f(x) = \frac{x^2 - 25}{x - 5}$$

- Because division by 0 is undefined, we must exclude from the domain of each function values of *x* that cause the polynomial function in the denominator to be 0.

D: $(-\infty, 5) \cup (5, \infty)$ $\{x \mid x \neq 5\}$

Finding the Domain of a Rational Function

- Find the domain of the rational function:

$$g(x) = \frac{x}{x^2 - 25}$$

- Because division by 0 is undefined, we must exclude from the domain of each function values of *x* that cause the polynomial function in the denominator to be 0.

$$x^2 - 25 \neq 0$$
$$x \neq \pm 5$$

The domain of *g* consists of all real numbers except –5 or 5.

D: $(-\infty, -5) \cup (-5, 5) \cup (5, \infty)$ $\{x \mid x \neq -5 x \neq 5\}$

Finding the Domain of a Rational Function

- Find the domain of the rational function: $h(x) = \dfrac{x+5}{x^2+25}$

  - Because division by 0 is undefined, we must exclude from the domain of each function values of *x* that cause the polynomial function in the denominator to be 0.

$$x^2 + 25 \neq 0$$

$$x \neq \pm 5i$$

No real numbers cause the denominator of $h(x)$ to equal 0. The domain of *h* consists of all real numbers.

D:$(-\infty, \infty)$ $\{x \mid x \in \mathbb{R}\}$

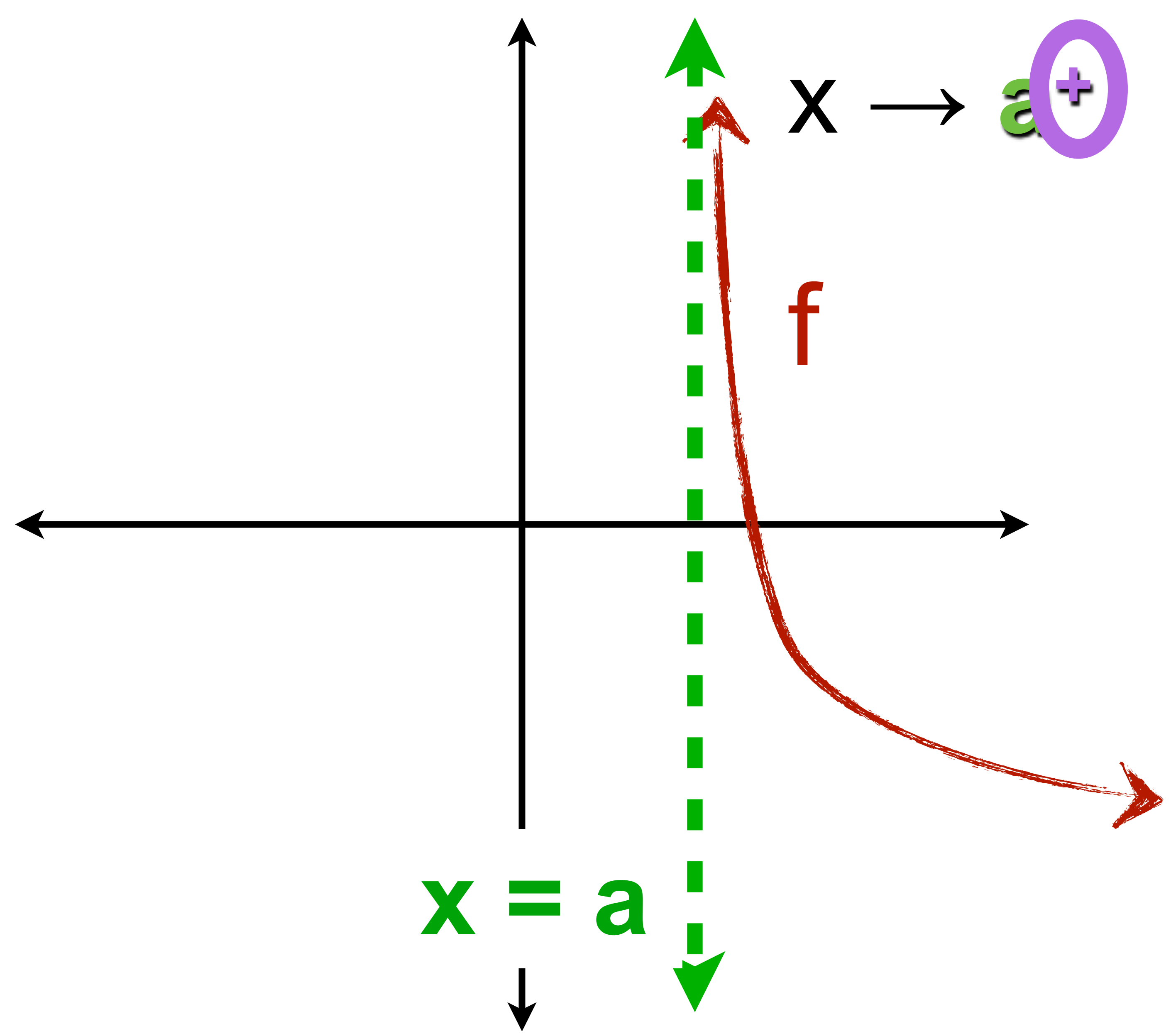
Arrow Notation

- We use arrow notation to describe the behavior of some functions.

$x \to a^{+}$	x approaches **a from the right**.
$x \to a^{-}$	x approaches **a from the left**.
$x \to \infty$	x approaches **infinity** (increases forever).
$x \to -\infty$	x approaches **negative infinity** (decreases forever).

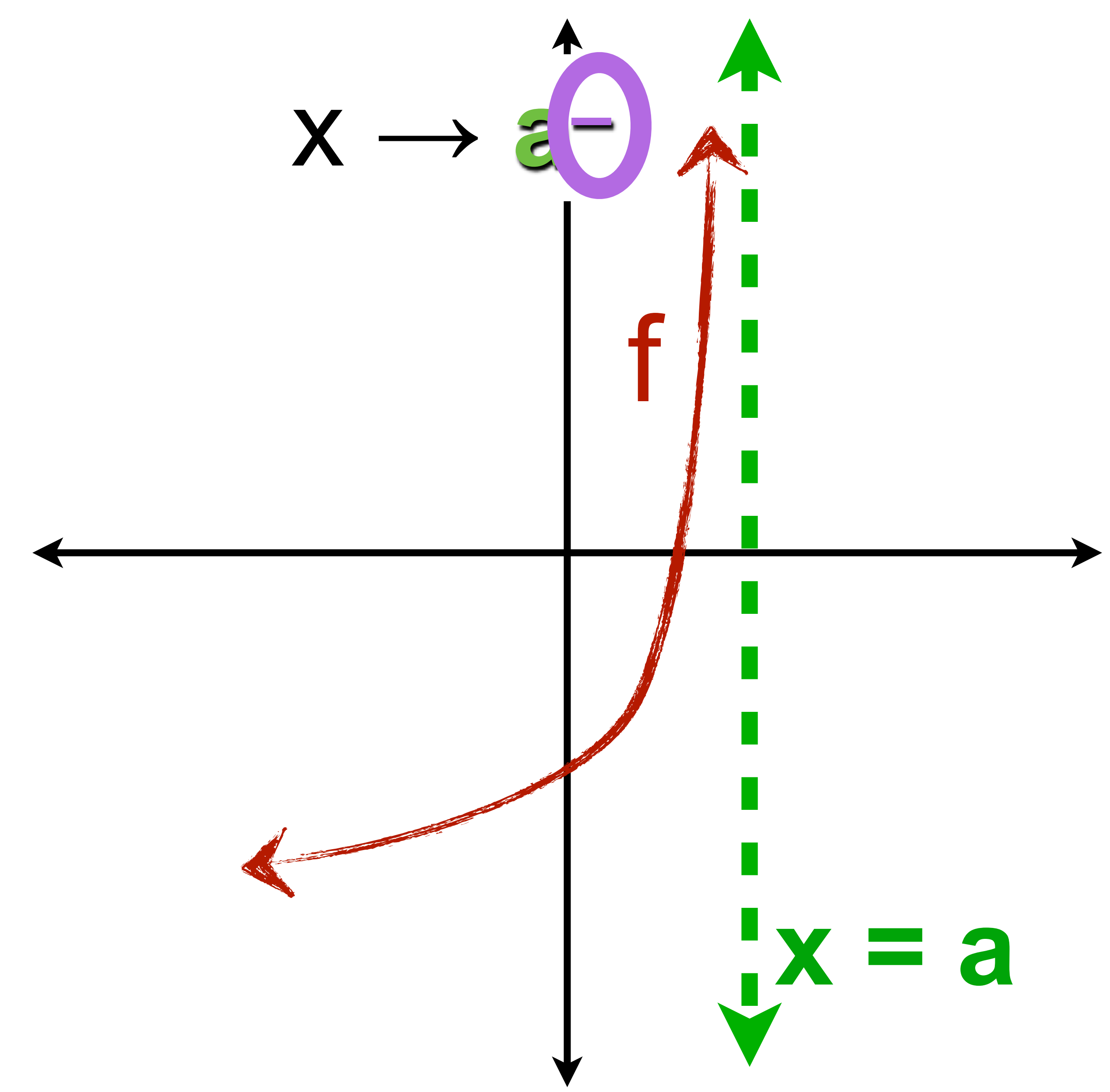
Definition of a Vertical Asymptote

- The line $x = a$ is a vertical asymptote of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches **a** (**$x \to a$**).



x approaches **a from the right**.

As $x \to a^{+}$, $f(x) \to \infty$

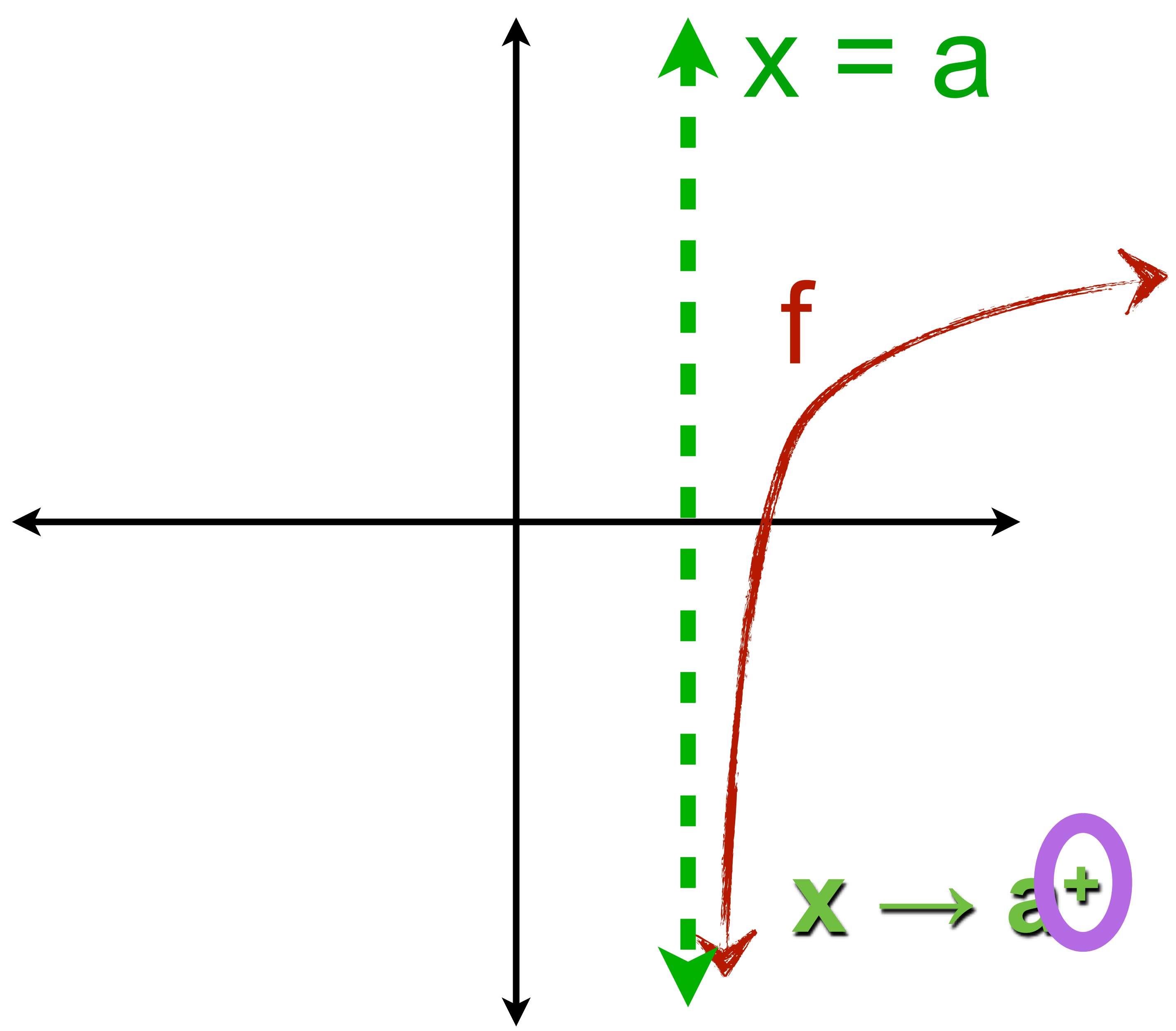


x approaches **a from the left**.

As $x \to a^{-}$, $f(x) \to \infty$

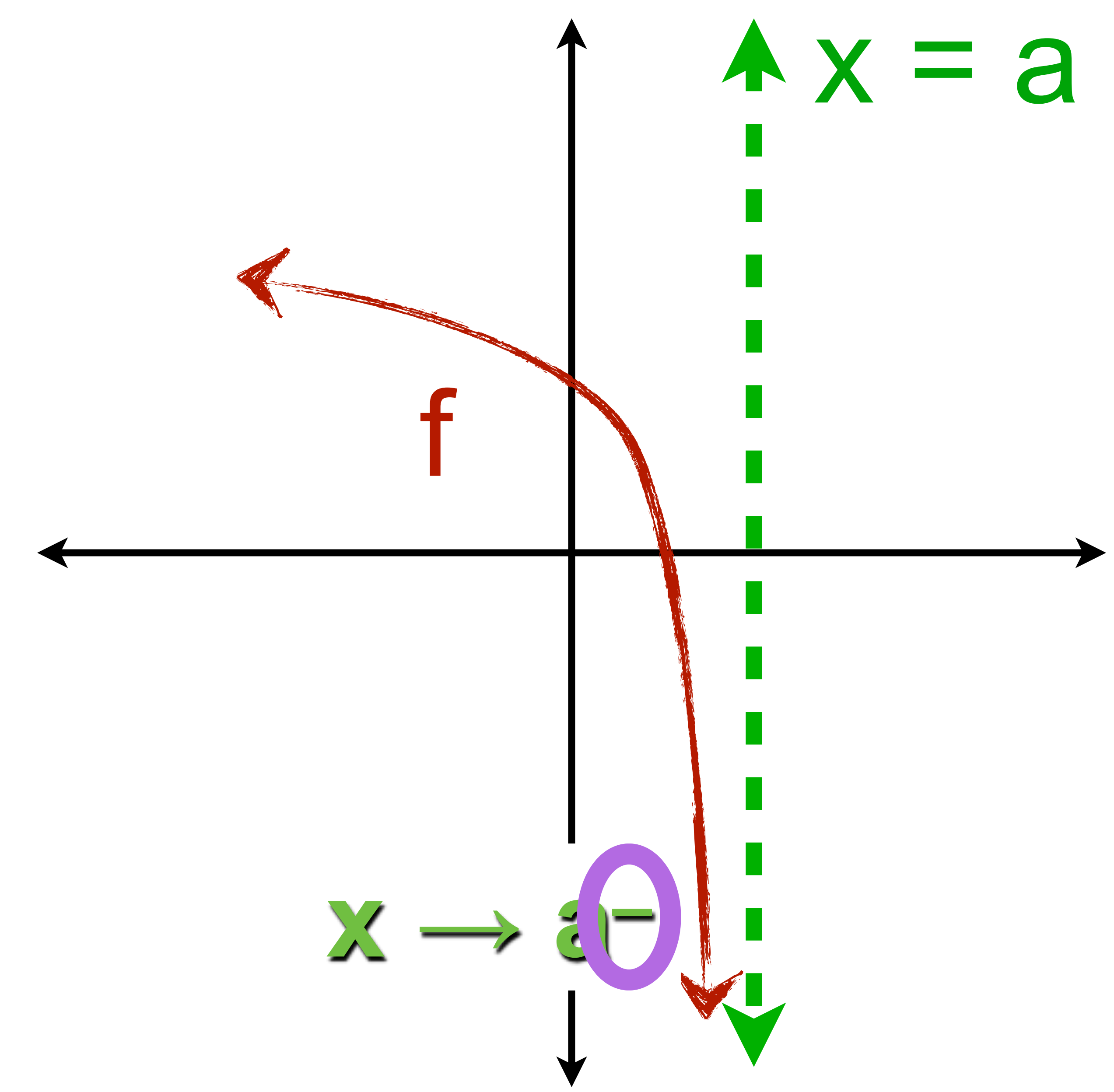
Definition of a Vertical Asymptote

- The line $x = a$ is a vertical asymptote of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a.



x approaches a from the right.

As $x \to a^+$, $f(x) \to -\infty$



x approaches a from the left.

As $x \to a^-$, $f(x) \to -\infty$

Locating Vertical Asymptotes

- If $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function in which $p(x)$ and $q(x)$ have **no common factors** and a is a zero of $q(x)$, **the denominator**, then $x = a$ is a vertical asymptote of the graph of f.

- In other words, values of **x** that make the **denominator** of a **simplified** rational function equal zero define the **vertical asymptote**, $x = a$.

- If the function is **not simplified**, then zeros of the denominator that are **NOT** zeros of the numerator determine the vertical asymptotes.

Finding the Vertical Asymptotes of a Rational Function

- Find the vertical asymptotes, if any, of the graph of the rational function:

$$f(x) = \frac{x}{x^2 - 1}$$

There are no common factors in the numerator and the denominator.

$$x^2 - 1 = 0 \qquad x = \pm 1$$

The zeros of the denominator are –1 and 1.

- Thus, the **lines $x = -1$ and $x = 1$** are the **vertical asymptotes** for the graph of *f*.

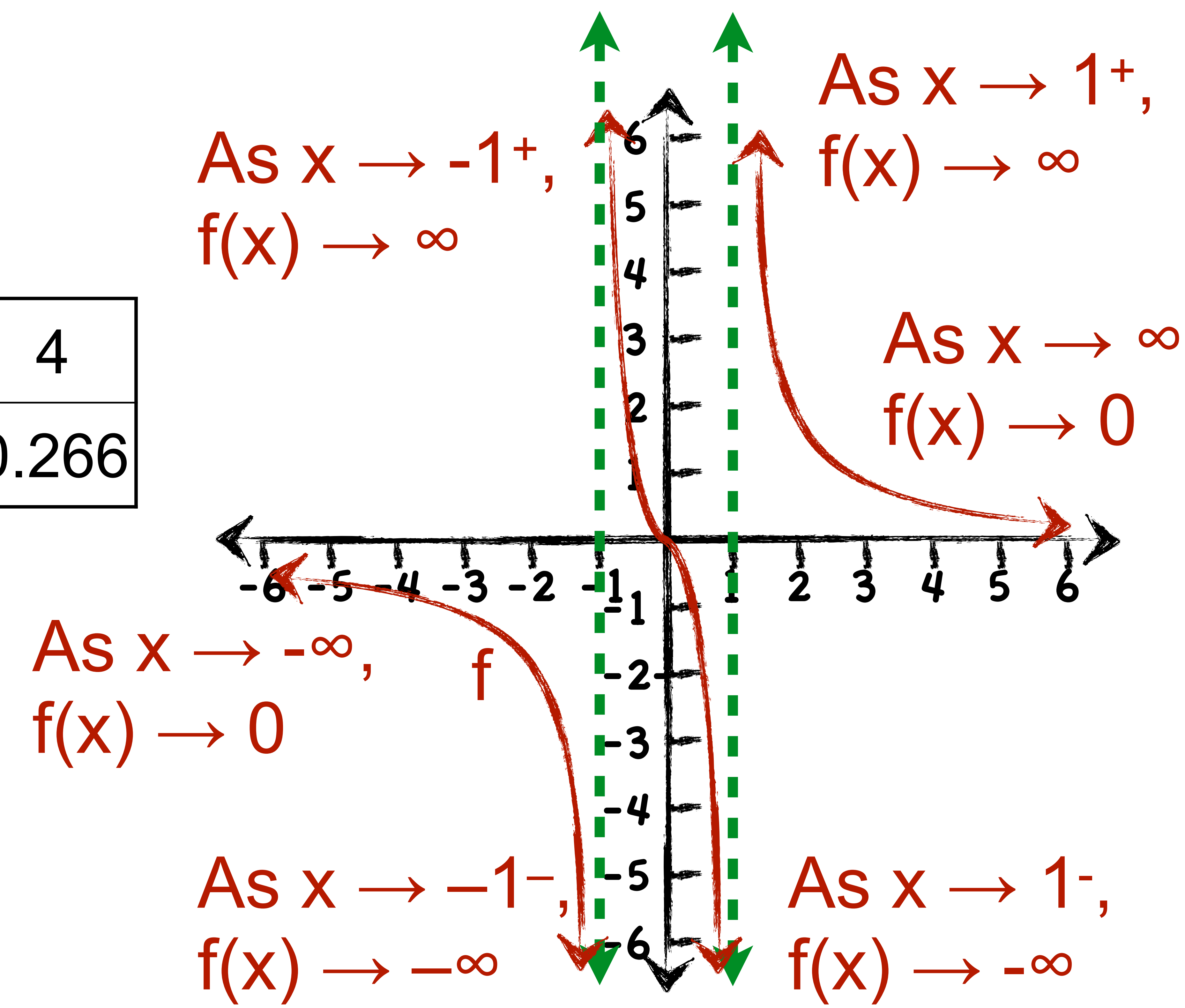
Finding the Vertical Asymptotes of a Rational Function

- Find the vertical asymptotes, if any, of the graph of the rational function:

$$f(x) = \frac{x}{x^2 - 1}$$

***x* = –1 and *x* = 1** are the **vertical asymptotes**

x	-4	-3	-2	-1	0	1	2	3	4
f(x)	-0.27	-0.375	-0.666	und	0	und	0.666	0.375	0.266



Finding the Vertical Asymptotes of a Rational Function

- Find the vertical asymptotes, if any, of the graph of the rational function:

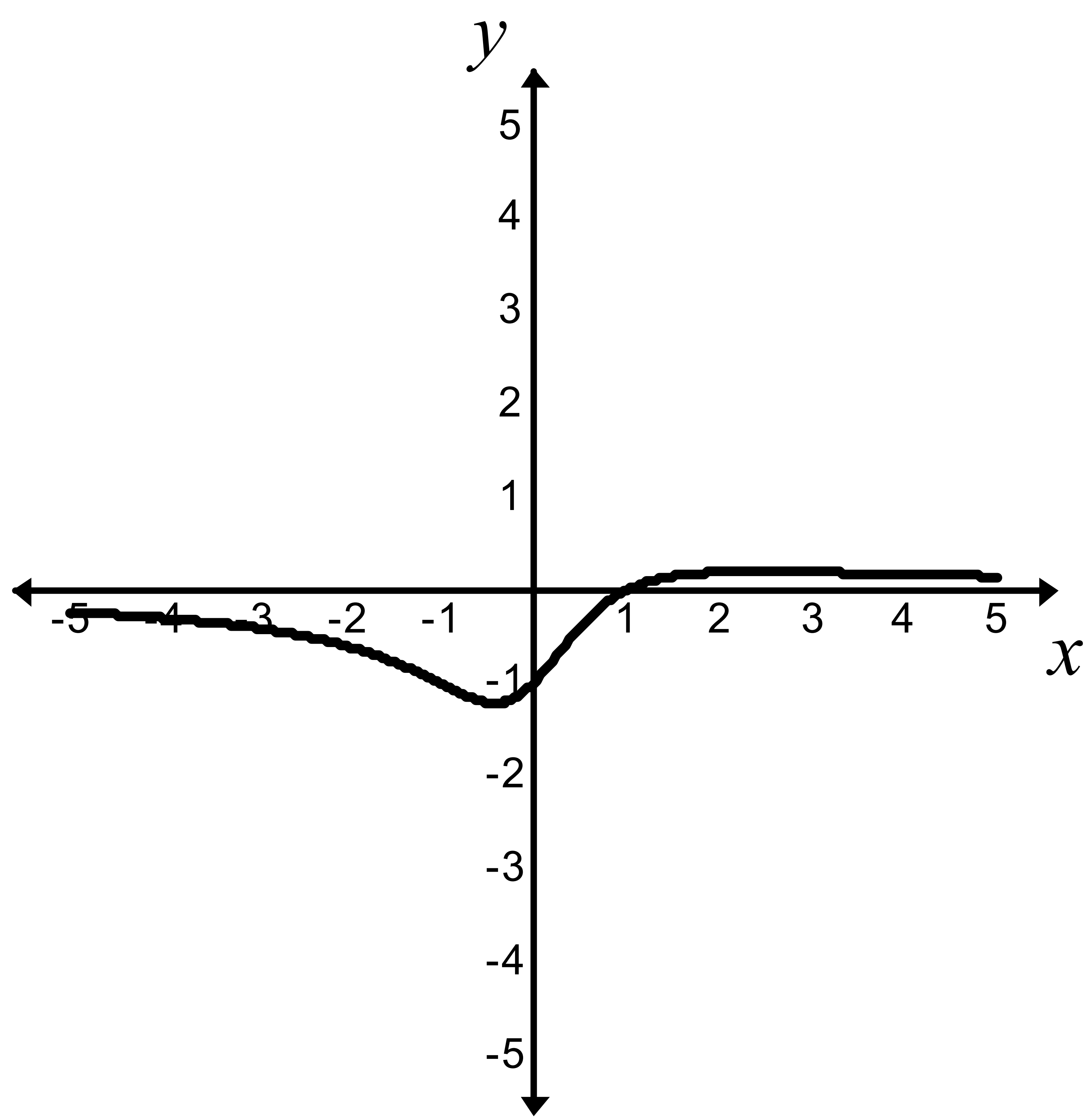
$$h(x) = \frac{x-1}{x^2+1}$$

- We cannot factor the denominator of $h(x)$ over the real numbers. The denominator has no real zeros. Thus, the graph of h has no vertical asymptotes.

Use your calculator to graph h(x).

$$y = (x-1) \div (x^2+1)$$

Do not forget the parentheses.



Holes

- There is a situation that requires care.

Values of **x** that make the denominator of a **simplified** rational function equal zero define the vertical asymptote, **x = a**.

A function **as originally written** need not have vertical asymptotes at values that make the denominator 0.

If the numerator and denominator have common factors, the value of **a** that makes the denominator 0 may **not** be a vertical asymptote **x = a**.

Consider $h(x) = \dfrac{x^2 + 5x + 4}{x + 1}$

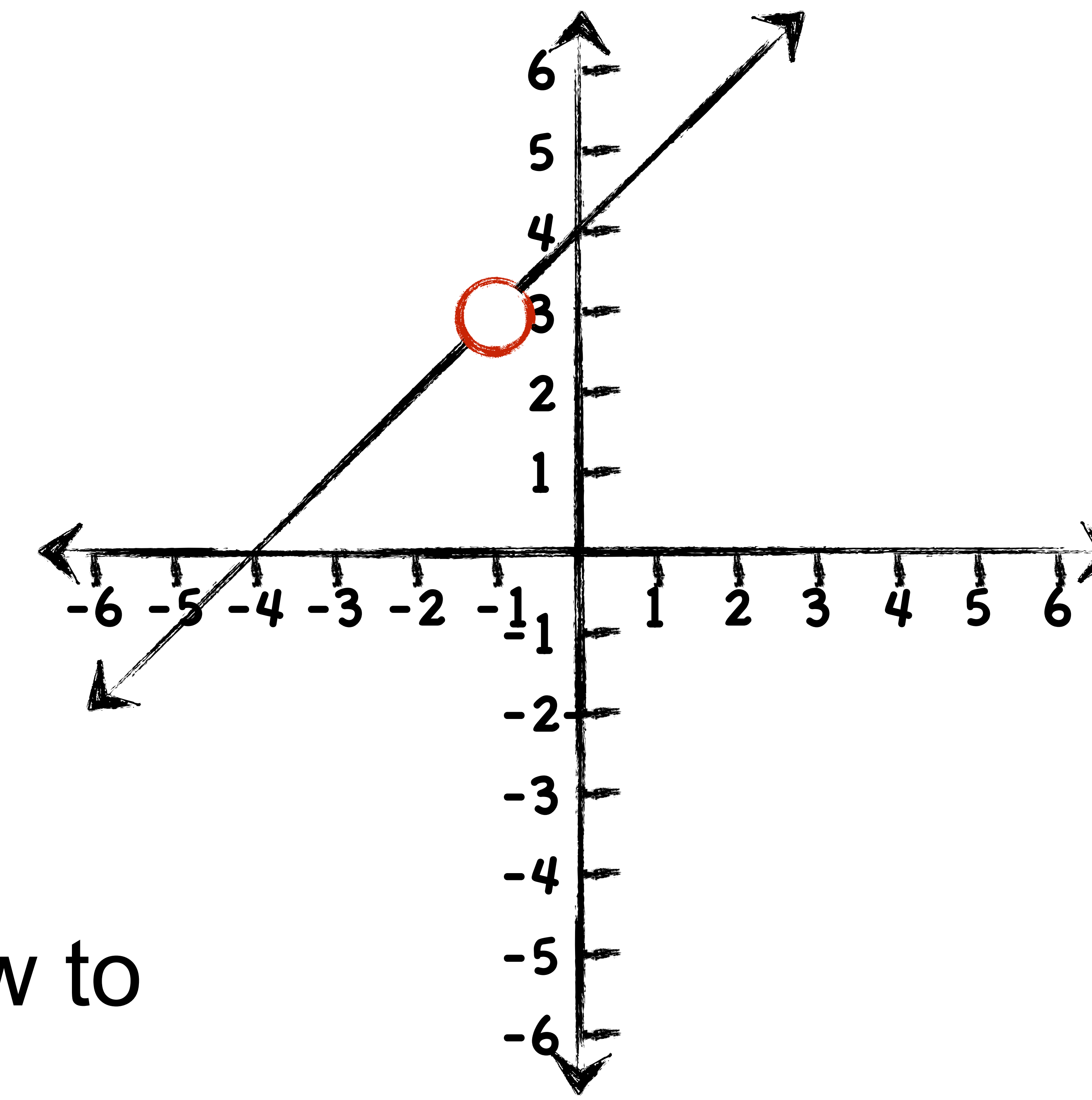
Because the denominator is 0 when x = –1, the domain is $\{x \mid x \neq -1\}$

Howsomever, upon further examination, the function is not **simplified**.

Holes

- Consider $h(x)=\dfrac{x^2+5x+4}{x+1}$

$$h(x)=\frac{x^2+5x+4}{x+1}=\frac{(x+1)(x+4)}{x+1}=x+4 \qquad x \neq -1$$



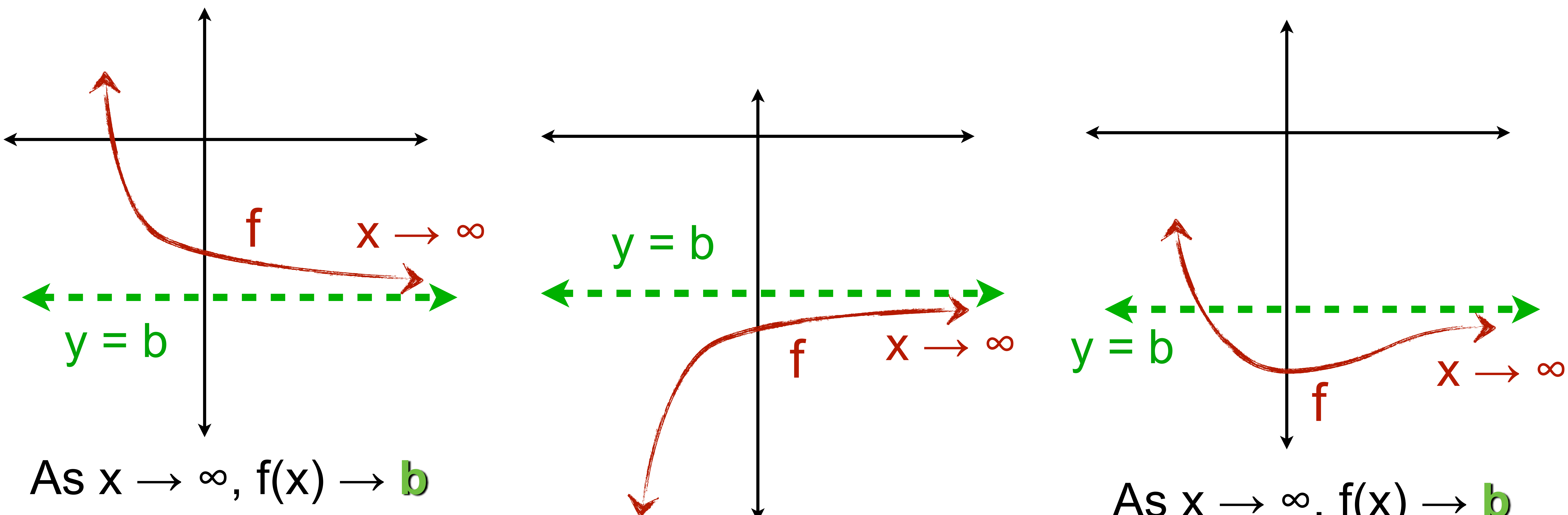
- When we graph this function we get a line with a **hole** instead of a vertical asymptote.

Try graphing in the TI-84 and adjust your window to try to see the hole.

When graphing rational functions it is a good idea to factor the numerator and denominator whenever possible. This will reveal what we will later call critical values.

Definition of a Horizontal Asymptote

- The line ***y* = *b*** is a horizontal asymptote of the graph of a function f if *f*(*x*) approaches ***b*** *as x* increases or decreases without bound.



As x → ∞, f(x) → **b**

As x → ∞, f(x) → **b**

As x → ∞, f(x) → **b**

Locating Horizontal Asymptotes

- Let f be the rational function given by

$$f(x)=\frac{a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x^1+a_0}{b_mx^m+b_{m-1}x^{m-1}+\ldots+b_1x^1+b_0}$$

Obviously a_n and b_m are not 0. The degree of the numerator is **n**, the degree of the denominator is **m**.

1. If **n** < **m**, the x-axis (y = 0) is the horizontal asymptote of the graph of f.
2. If **n** = **m**, the line $y=\frac{a_n}{b_m}$ is the horizontal asymptote of the graph of f.
3. If **n** > **m**, the graph of f has no horizontal asymptotes.

Locating Horizontal Asymptotes

Consider the case when |x| is very, very large (|x| >> 0).

$$f(x)=\frac{p(x)}{q(x)}=\frac{a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1 x^1 + a_0}{b_m x^m + b_{m-1}x^{m-1}+\ldots+b_1 x^1 + b_0}$$

Now divide numerator and denominator by x^m.

$$f(x)=\frac{p(x)}{q(x)}=\frac{a_n \frac{x^n}{x^m} + a_{n-1}\frac{x^{n-1}}{x^m}+\ldots+a_1 \frac{x^1}{x^m} + \frac{a_0}{x^m}}{b_m \frac{x^m}{x^m} + b_{m-1}\frac{x^{m-1}}{x^m}+\ldots+b_1 \frac{x^1}{x^m} + \frac{b_0}{x^m}}$$

Locating Horizontal Asymptotes

$$f(x)=\frac{p(x)}{q(x)}=\frac{a_n\frac{x^n}{x^m}+a_{n-1}\frac{x^{n-1}}{x^m}+\ldots+a_1\frac{x^1}{x^m}+\frac{a_0}{x^m}}{b_m\frac{x^m}{x^m}+b_{m-1}\frac{x^{m-1}}{x^m}+\ldots+b_1\frac{x^1}{x^m}+\frac{b_0}{x^m}}$$

Remember, we are considering as x → ∞ or x → -∞. So we can consider most terms in the numerator and denominator to be **very near 0**, except for the first terms in the numerator and denominator. So our function begins to look like:

$$f(x)=\frac{p(x)}{q(x)}=\frac{a_n\frac{x^n}{x^m}}{b_m\frac{x^m}{x^m}}$$

Locating Horizontal Asymptotes

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n \dfrac{x^n}{x^m}}{b_m \dfrac{x^m}{x^m}}$$

1. If $n < m$, $f(x) = \dfrac{a_n \dfrac{x^n}{x^m}}{b_m} \to \dfrac{0}{b_m}$

   the x-axis ($y = 0$) is the horizontal asymptote of the graph of f.

2. If $n = m$, $f(x) = \dfrac{a_n \cdot 1}{b_m \cdot 1} = \dfrac{a_n}{b_m}$

   the line $y = \dfrac{a_n}{b_m}$ is the horizontal asymptote of the graph of f.

3. If $n > m$, $f(x) = \dfrac{a_n}{b_m} x^{n-m}$

   the graph of f has no horizontal asymptotes.

Finding the Horizontal Asymptote of a Rational Function

- Find the horizontal asymptotes, if any, of the graph of the rational function:

$$f(x) = \frac{9x^2}{3x^2 + 1}$$

- The degree of the numerator, 2, is equal to the degree of the denominator, also 2. The leading coefficients of the numerator and the denominator, 9 and 3, are used to obtain the equation of the horizontal asymptote.

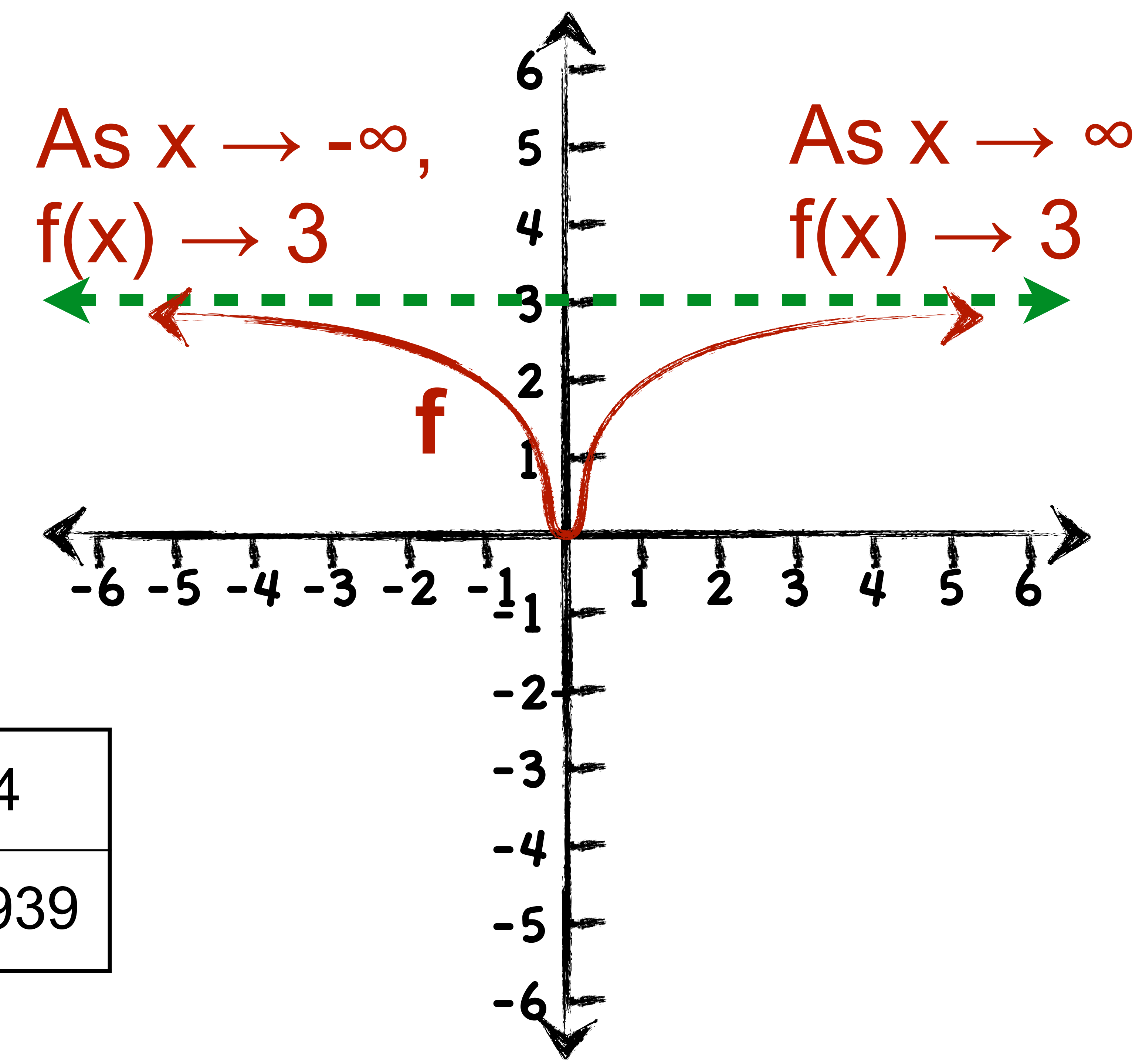
- The horizontal asymptote is $y = \frac{9}{3} = 3$

Finding the Horizontal Asymptote of a Rational Function

- Find the horizontal asymptotes, if any, of the graph of the rational function:

$$f(x) = \frac{9x^2}{3x^2+1}$$

- The horizontal asymptote is $y = \frac{9}{3} = 3$



x	-4	-3	-2	-1	0	1	2	3	4
f(x)	2.939	2.893	2.769	2.25	0	2.25	2.789	2.893	2.939

Finding the Horizontal Asymptote of a Rational Function

- Find the horizontal asymptotes, if any, of the graph of the rational function:

$$f(x) = \frac{9x}{3x^2 + 1}$$

- The degree of the numerator, 1, is less than the degree of the denominator, 2. Thus, the graph of *g* has the *x*-axis as a horizontal asymptote.
  - The horizontal asymptote is $y = 0$.
- When graphing rational functions it is a good idea to factor the numerator and denominator whenever possible. This will reveal what we will later call critical values.

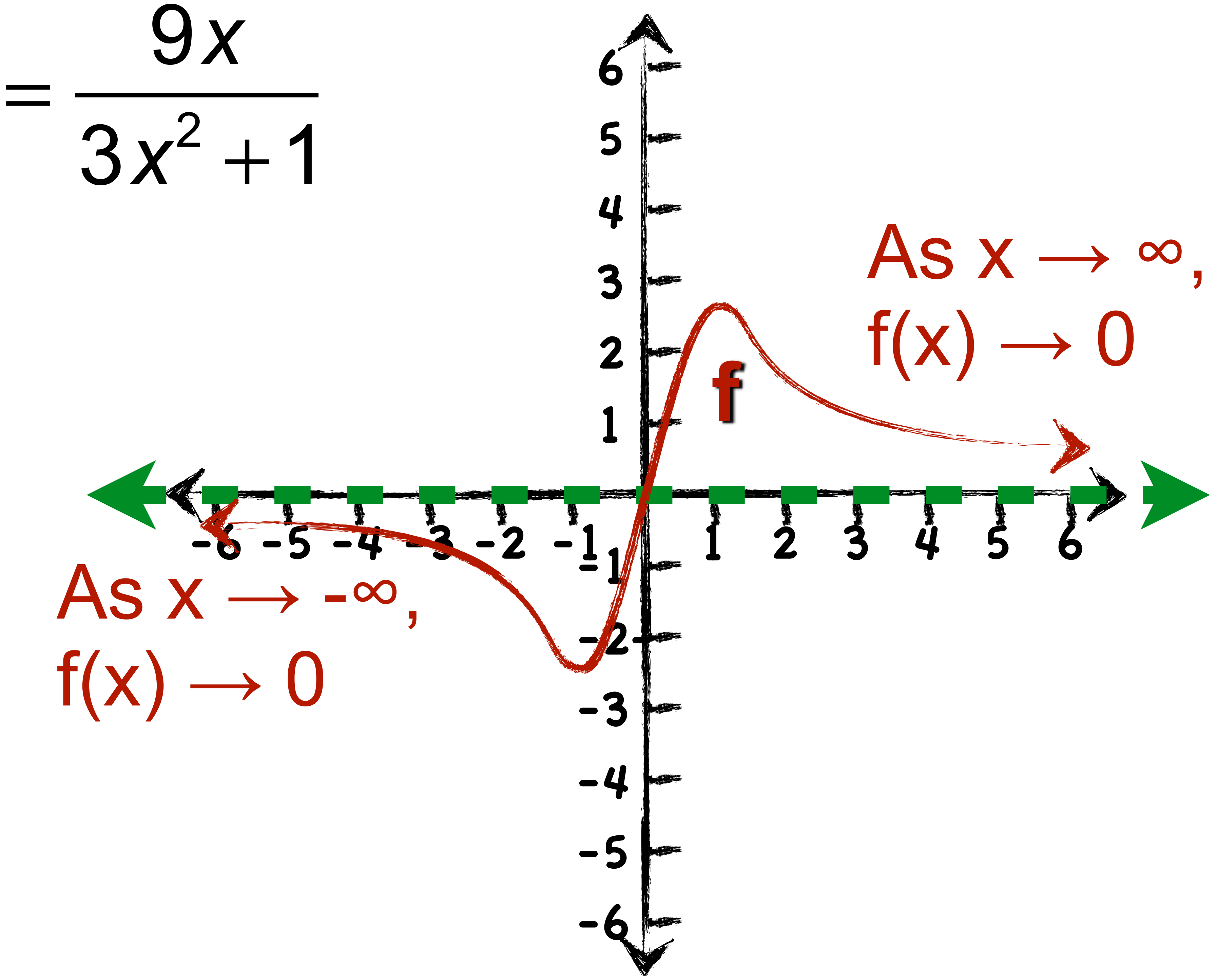
Finding the Horizontal Asymptote of a Rational Function

- Find the horizontal asymptotes, if any, of the graph of the rational function:

- The horizontal asymptote is $y = 0$.

$$f(x) = \frac{9x}{3x^2 + 1}$$

x	f(x)
-4	-0.7346
-3	-0.964
-2	-1.385
-1	-2.25
0	0
1	2.25
2	1.385
3	0.964
4	0.735



Finding the Horizontal Asymptote of a Rational Function

- Find the horizontal asymptotes, if any, of the graph of the rational function:

$$h(x) = \frac{9x^3}{3x^2+1}$$

3. If **n** > **m**, the graph of f has no horizontal asymptotes.

- The degree of the numerator, 3, is greater than the degree of the denominator, 2. Thus the graph of h has no horizontal asymptote.

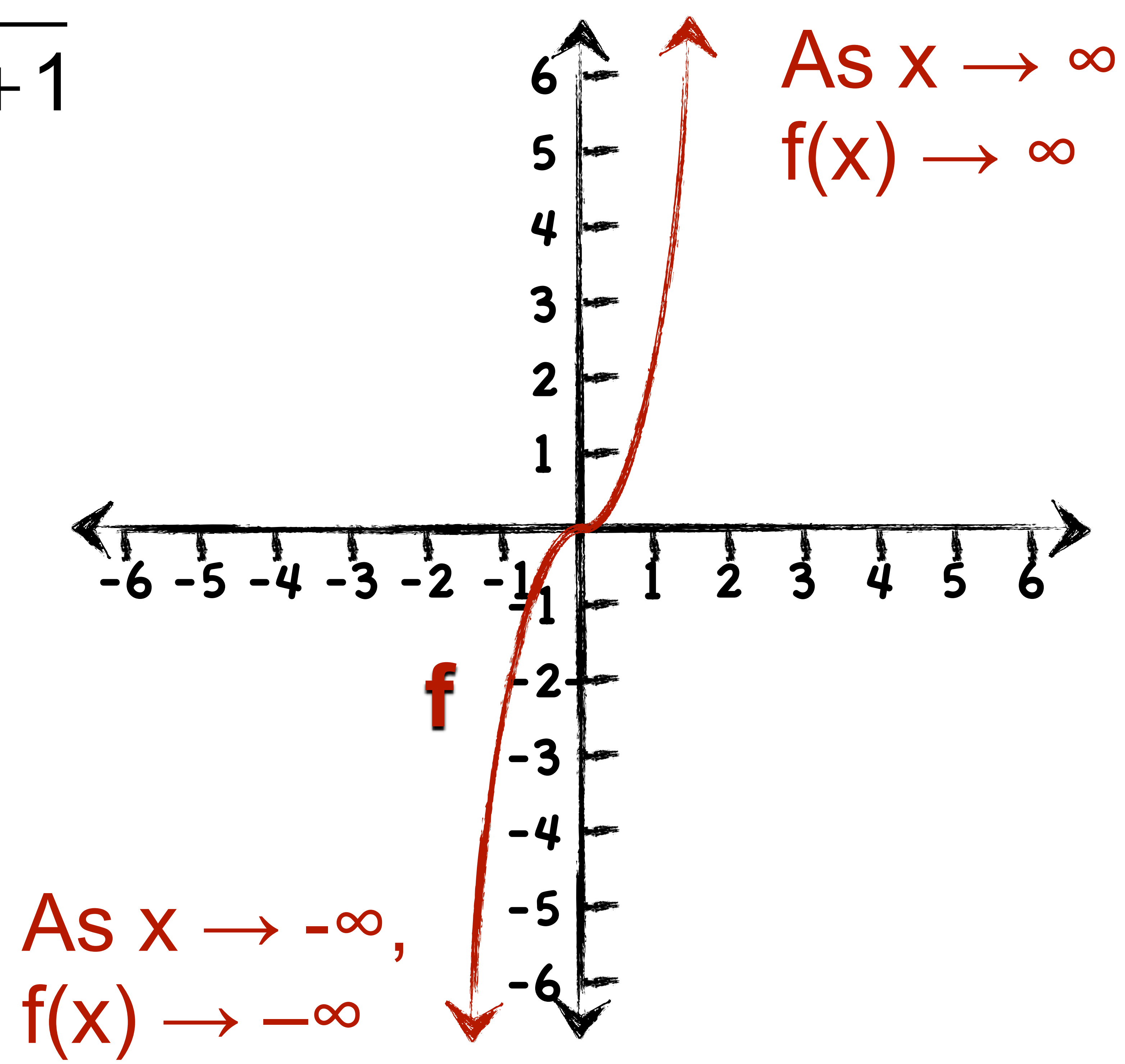
Finding the Horizontal Asymptote of a Rational Function

- Find the horizontal asymptotes, if any, of the graph of the rational function:

h has no horizontal asymptote

$$h(x) = \frac{9x^3}{3x^2+1}$$

x	f(x)
-4	-11.755
-3	-8.679
-2	-5.538
-1	-2.25
0	0
1	2.25
2	5.538
3	8.679
4	11.76



Asymptotes of Rational Functions

Asymptotes of a Rational Function

Let f be the rational function given by

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.
2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.
   - **a.** If $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
   - **b.** If $n = m$, the graph of f has the line $y = a_n/b_m$ (ratio of the leading coefficients) as a horizontal asymptote.
   - **c.** If $n > m$, the graph of f has no horizontal asymptote.

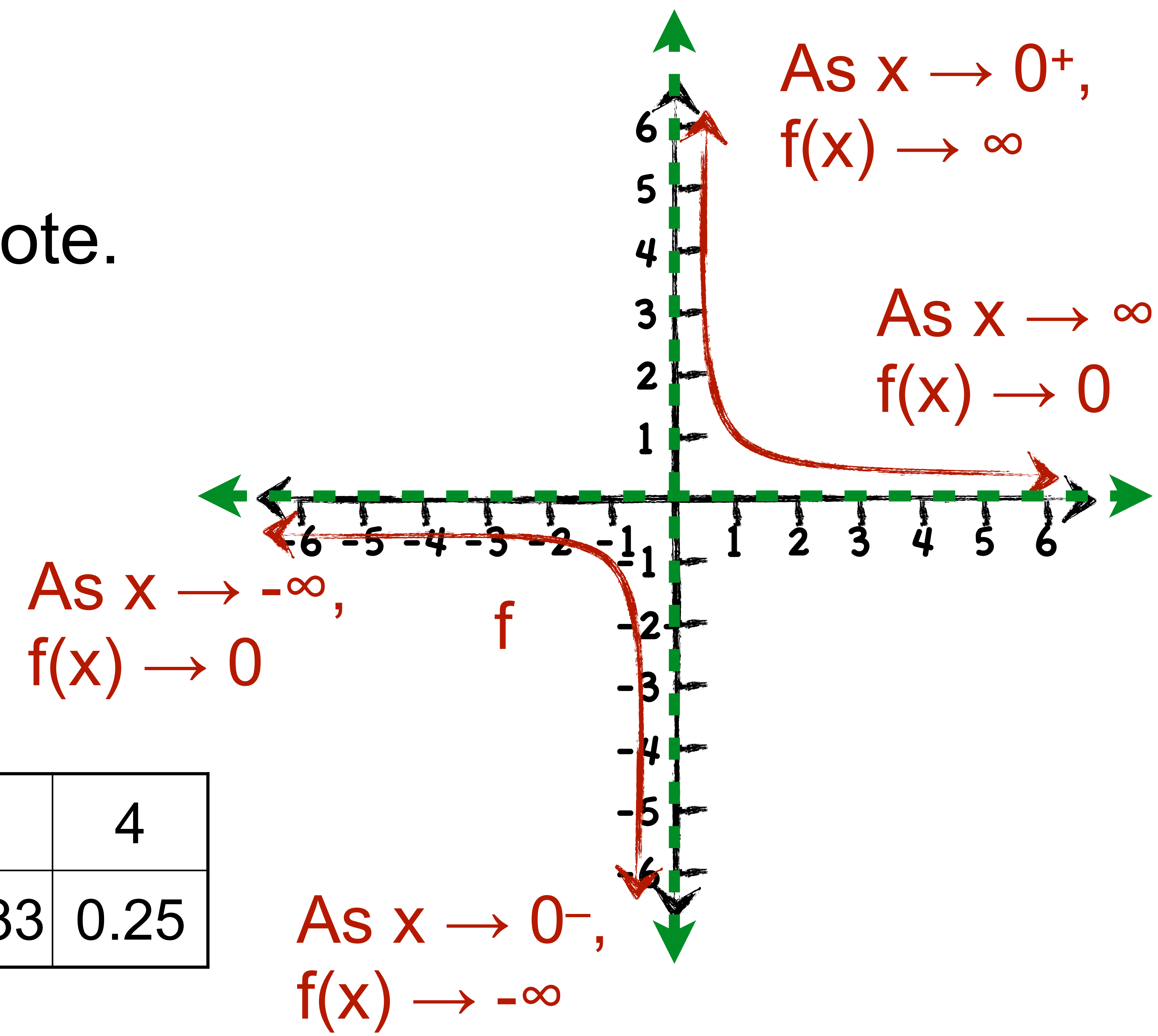
Basic Reciprocal Function $f(x) = \dfrac{1}{x}$

- $x = 0$ makes denominator 0.
  The vertical asymptote is $x = 0$.

  n < m, y = 0 is the horizontal asymptote.

  Odd function f(-x) = -f(x)

  So, origin symmetry



x	-4	-3	-2	-1	0	1	2	3	4
f(x)	-0.25	-0.333	-0.5	-1	und	1	0.5	0.333	0.25

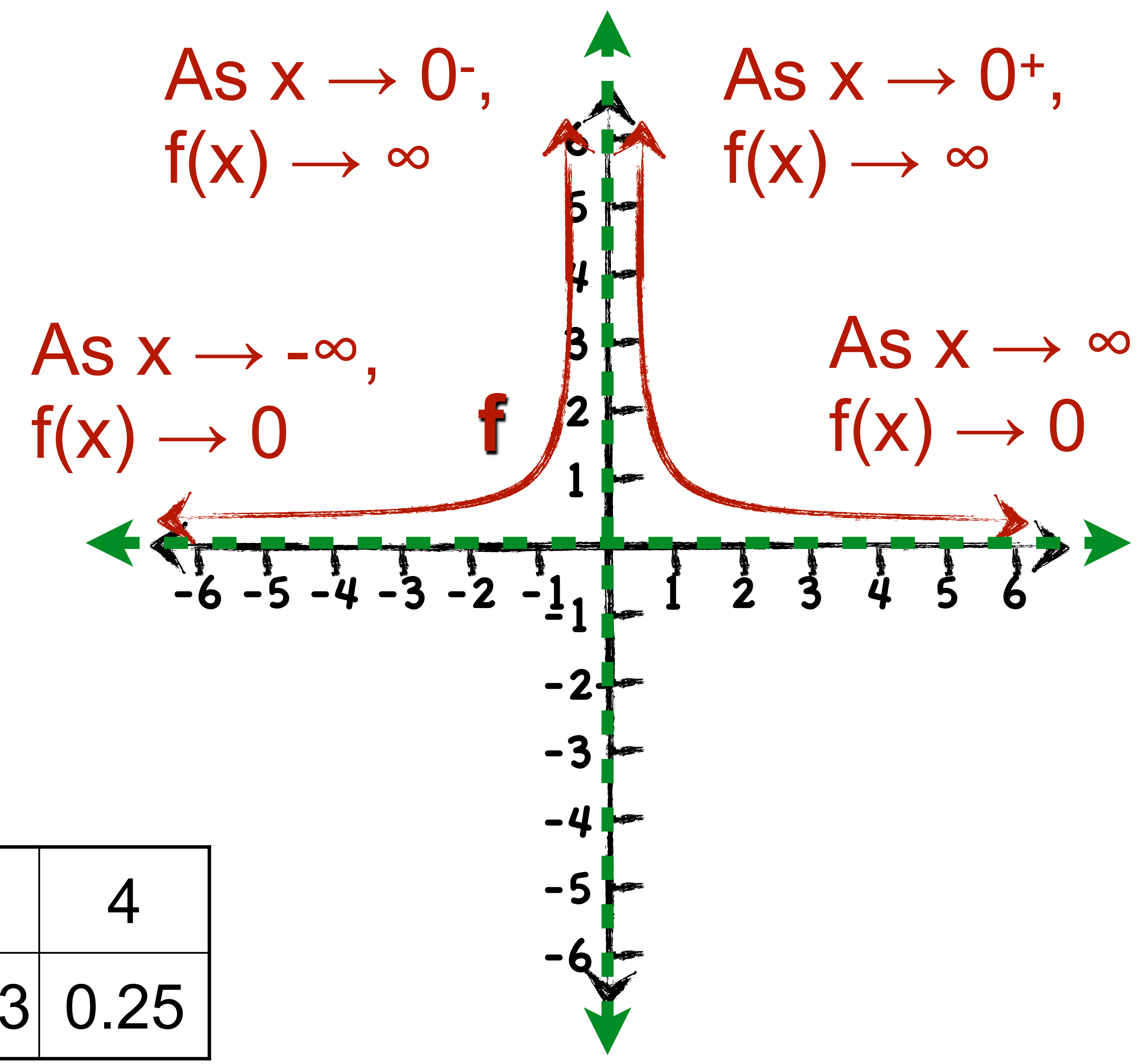
Basic Reciprocal Function $f(x) = \dfrac{1}{x^2}$

$x = 0$ makes denominator 0.
The vertical asymptote is $x = 0$.

n < m, y = 0 is the horizontal asymptote.

Even function f(-x) = -f(x)

So, y-axis symmetry



x	-4	-3	-2	-1	0	1	2	3	4
f(x)	0.25	0.333	0.5	1	und	1	0.5	0.333	0.25

Using Transformations to Graph a Rational Function

- Use the graph of $f(x)=\dfrac{1}{x}$ to graph $g(x)=\dfrac{1}{x+2}-1$

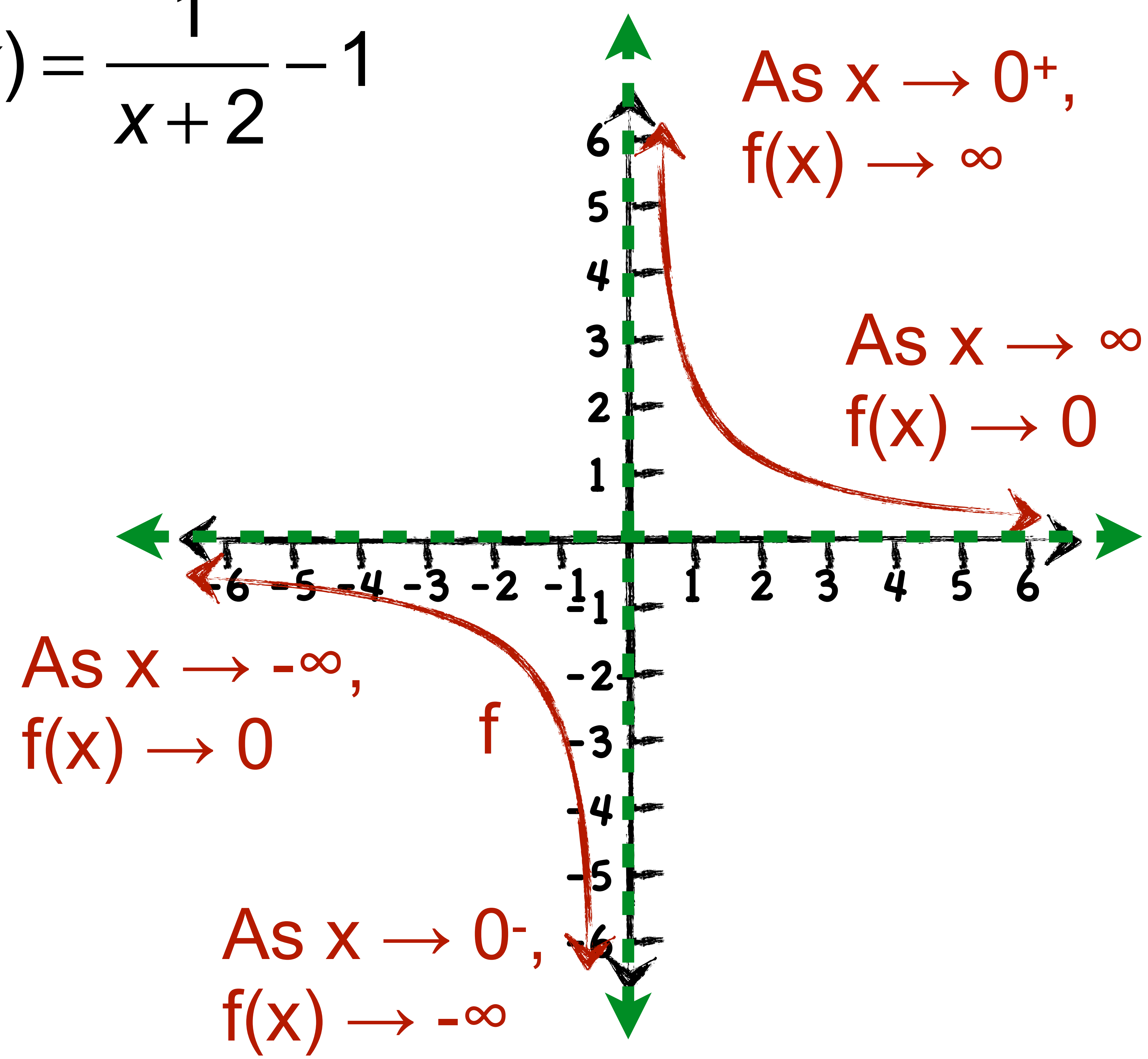
Begin with $f(x)=\dfrac{1}{x}$

x = -2 makes denominator 0.
The vertical asymptote is x = -2.

$$g(x)=\frac{1}{x+2}-1=\frac{1}{x+2}-\frac{x+2}{x+2}=\frac{-x-1}{x+2}$$

n = m, y = –1 is the horizontal asymptote.

f shifts 2 units left and 1 unit down.



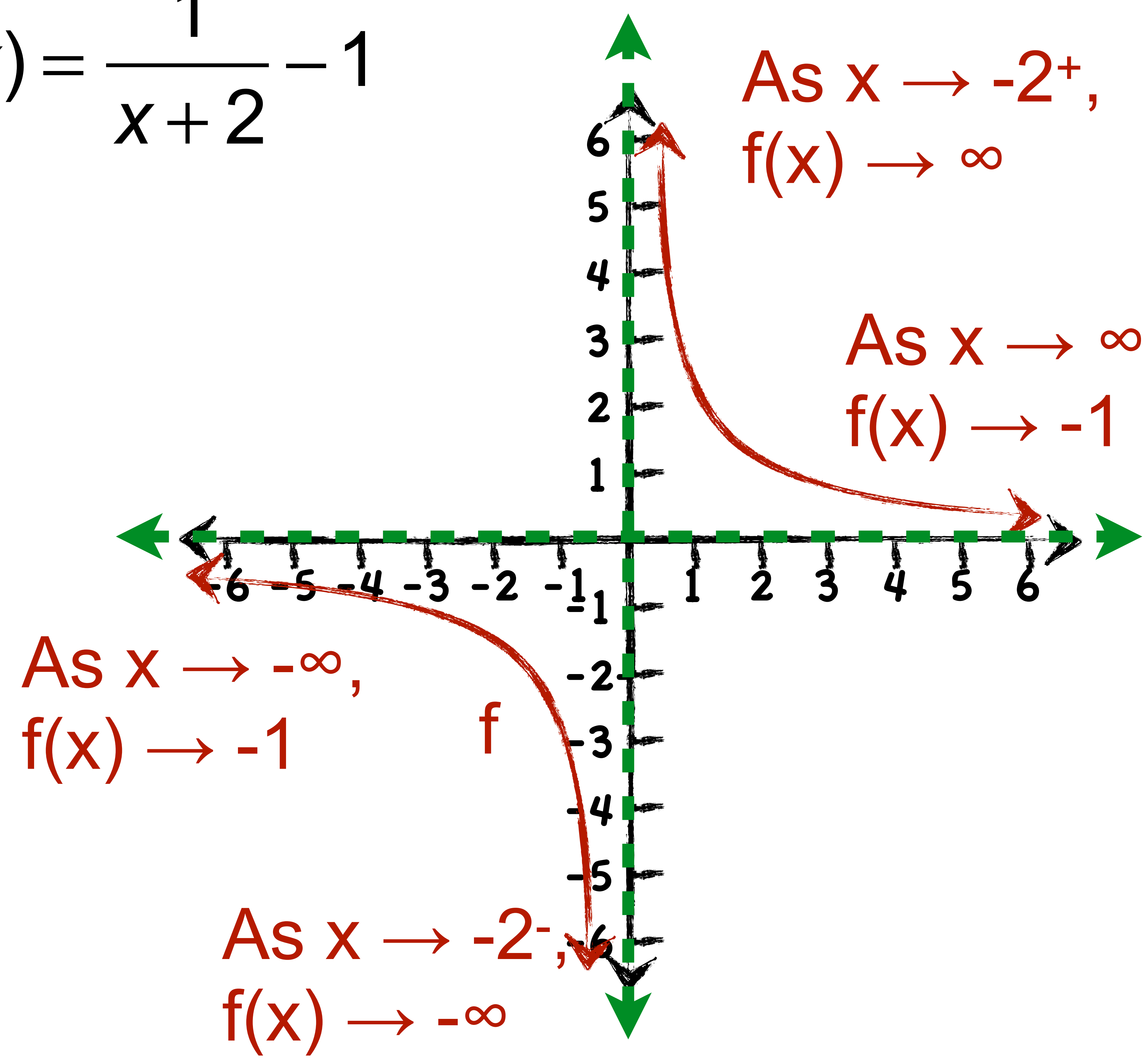
Using Transformations to Graph a Rational Function

- Use the graph of $f(x)=\dfrac{1}{x}$ to graph $g(x)=\dfrac{1}{x+2}-1$

f shifts 2 units left and 1 unit down.

and, of course, you can always...

x	f(x)
-4	-1.5
-3	-2
-2	und
-1	0
0	-0.5
1	-0.666
2	-0.75
3	-0.8
4	-0.833



Strategy for Graphing a Rational Function

$$f(x) = \frac{p(x)}{q(x)}$$ **p(x)** and **q(x)** have no common factors.

1. Determine whether the graph of *f* has symmetry.

   f(-x) = f(x) - y-axis symmetry f(-x) = -f(x) - origin symmetry
2. Find the *y*-intercept (if there is one) by evaluating *f*(0).
3. Find the *x*-intercepts (if there are any) by solving **p(x) = 0**.
4. Find any vertical asymptote(s) by solving the equation **q(x) = 0**.
5. Find the horizontal asymptote (if there is one) using the rules for determining the horizontal asymptote of a rational function.
6. Plot at least one point between and beyond each *x*-intercept and vertical asymptote.
7. Use the information obtained previously to graph the function between and beyond the vertical asymptotes.

Graphing a Rational Function

Graph: $f(x) = \dfrac{3x-3}{x-2}$

1. Determine whether the graph of *f* has symmetry.

f(-x) = f(x) - y-axis symmetry

f(-x) = -f(x) - origin symmetry

$$f(-x) = \frac{3(-x)-3}{(-x)-2} = \frac{-3x-3}{-x-2} = \frac{3x+3}{x+2}$$

Because $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, the graph has neither symmetry.

2. Find the *y*-intercept (if there is one) by evaluating *f*(0).

$$f(0) = \frac{3(0)-3}{(0)-2} = \frac{3}{2}$$

y-intercept = $\dfrac{3}{2}$

Graphing a Rational Function

Graph: $f(x) = \dfrac{3x - 3}{x - 2}$ $\quad$ y-intercept = $\dfrac{3}{2}$

3. Find the x-intercepts (if there are any) by solving $p(x) = 0$.

   $3x – 3 = 0 \quad x = 1 \qquad$ x-intercept = 1

4. Find any vertical asymptote(s) by solving the equation $q(x) = 0$.

   $x – 2 = 0 \quad x = 2 \qquad$ vertical asymptote $x = 2$

5. Find the horizontal asymptote.

   degree of numerator = degree of denominator $\quad$ horizontal asymptote $y = \dfrac{3}{1} = 3$

Graphing a Rational Function

Graph: $f(x) = \dfrac{3x - 3}{x - 2}$

y-intercept = $\dfrac{3}{2}$

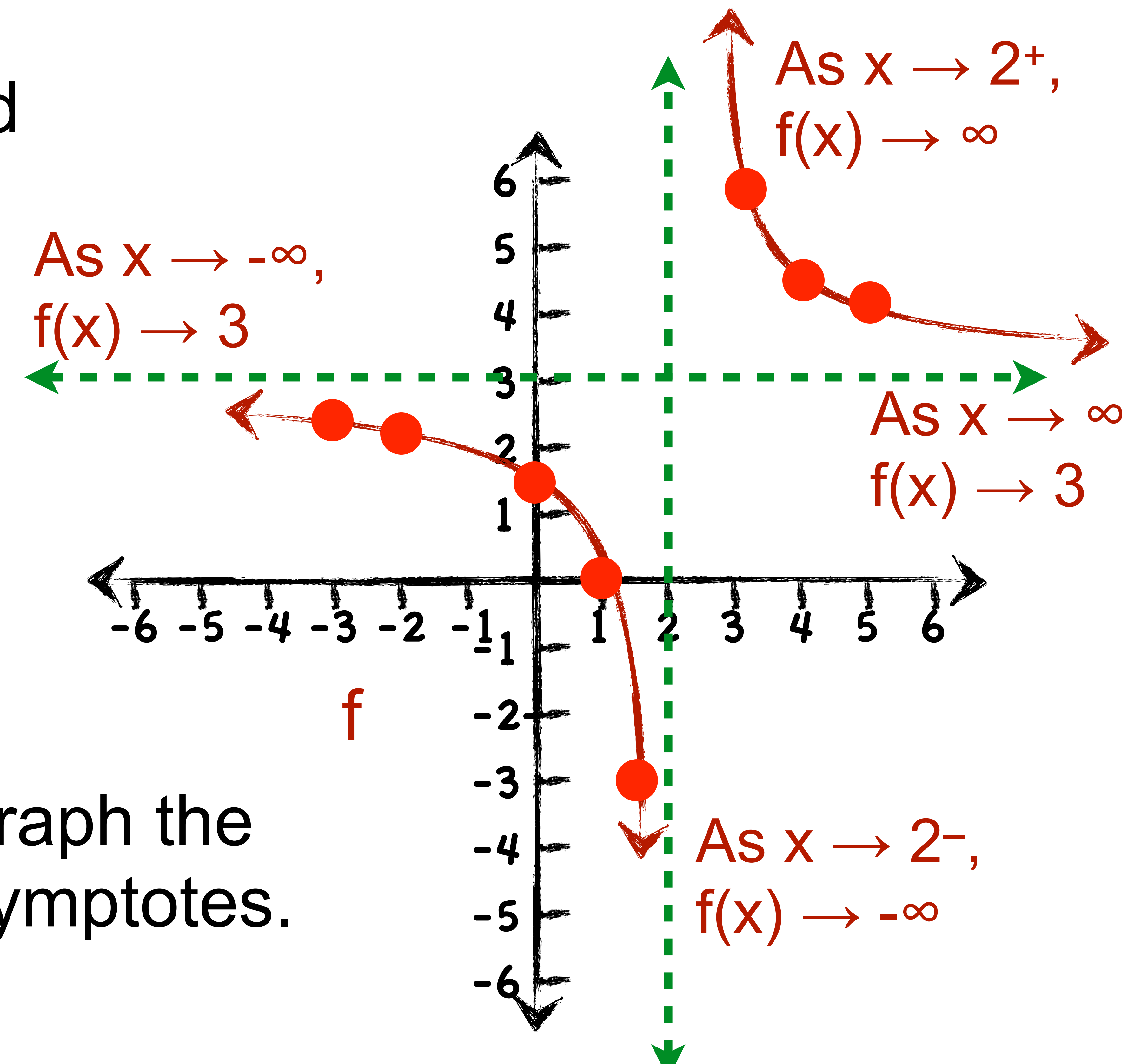
x-intercept = 1

vertical asymptote x = 2

horizontal asymptote y = $\dfrac{3}{1}$ = 3

6. Plot at least one point between and beyond each x-intercept and vertical asymptote.

x		
f(x)		



7. Use the information obtained previously to graph the function between and beyond the vertical asymptotes.

Slant (Oblique) Asymptotes

- The graph of a rational function $f(x) = \frac{p(x)}{q(x)}$ has a **slant asymptote** if the degree of the numerator (**n**) is **one greater** than the degree of the denominator (**m**).

In general, if $\boldsymbol{f(x) = \frac{p(x)}{q(x)}}$ ***p*** and ***q*** have no common factors, and the degree of ***p*** is **one greater** than the degree of ***q***, find the slant asymptotes by **dividing** ***q***(x) into ***p***(x).

When graphing rational functions it is a good idea to factor the numerator and denominator whenever possible. This will reveal what we will later call critical values.

Slant Asymptotes

Recall that we simplified f(x) when looking for horizontal asymptotes

3. If **n** > **m**, $f(x) = \frac{a_n}{b_m} x^{n-m}$ the graph of f has no horizontal asymptotes.

If **n** = **m+1** , then the function simplifies to the linear equation;

$$f(x) = \frac{a_n}{b_m} x$$

Of course, the function never actually gets to that linear state because the terms we simplified out never actually become 0. Thus a **slant asymptote**.

Finding Slant Asymptotes

- Find the slant asymptote of $f(x) = \dfrac{2x^2 - 5x + 7}{x - 2}$

- The degree of the numerator, n = 2, is exactly one more than the degree of the denominator, m = 1.

  To find the equation of the slant asymptote, simply divide.

$$\begin{array}{c|ccc} 2 & 2 & -5 & 7 \\ & & 4 & -2 \\ \hline & 2 & -1 & \boxed{5} \end{array}$$

The quotient is $2x - 1 + \dfrac{5}{x-2}$

- The equation of the slant asymptote is y = 2x – 1
  - We ignore the remainder because we are looking for a **linear approximation** of the function.

Finding Slant Asymptotes

- Find the slant asymptote of $f(x) = \dfrac{2x^2 - 5x + 7}{x - 2}$
- The equation of the slant asymptote is 2x – 1

1. Determine whether the graph of *f* has symmetry.

   f(-x) = f(x) - y-axis symmetry   f(-x) = -f(x) - origin symmetry

$$f(-x) = \frac{2(-x)^2 - 5(-x) + 7}{(-x) - 2} = \frac{2x^2 + 5x + 7}{-x - 2}$$

Because *f*(–x) ≠ *f*(*x*) and f(–x) ≠ –*f*(*x*), the graph has neither symmetry.

Finding Slant Asymptotes

- Find the slant asymptote of $f(x) = \dfrac{2x^2 - 5x + 7}{x - 2}$
- The equation of the slant asymptote is 2x – 1

2. Find the *y*-intercept (if there is one) by evaluating *f*(0).

$$f(0) = \frac{2(0^2) - 5(0) + 7}{(0) - 2} = \frac{7}{2} \qquad y\text{-intercept} = \frac{7}{-2}$$

3. Find the *x*-intercepts (if there are any) by solving *p*(*x*) = 0.

$$2x^2 - 5x + 7 = 0 \qquad \sqrt{(-5)^2 - 4(2)(7)} = \sqrt{-31} \qquad \text{no x-intercept}$$

Finding Slant Asymptotes

- Find the slant asymptote of $f(x) = \dfrac{2x^2 - 5x + 7}{x - 2}$
- The equation of the slant asymptote is 2x – 1

y-intercept = $\dfrac{7}{-2}$ no x-intercept

4. Find any vertical asymptote(s) by solving the equation $q(x) = 0$.

   x – 2 = 0 x = 2 vertical asymptote x = 2

5. Find the horizontal asymptote.

   degree of numerator > degree of denominator no horizontal asymptote

Example: Finding the Slant Asymptotes

- Find the slant asymptote of $f(x) = \dfrac{2x^2 - 5x + 7}{x - 2}$
- The equation of the slant asymptote is 2x – 1

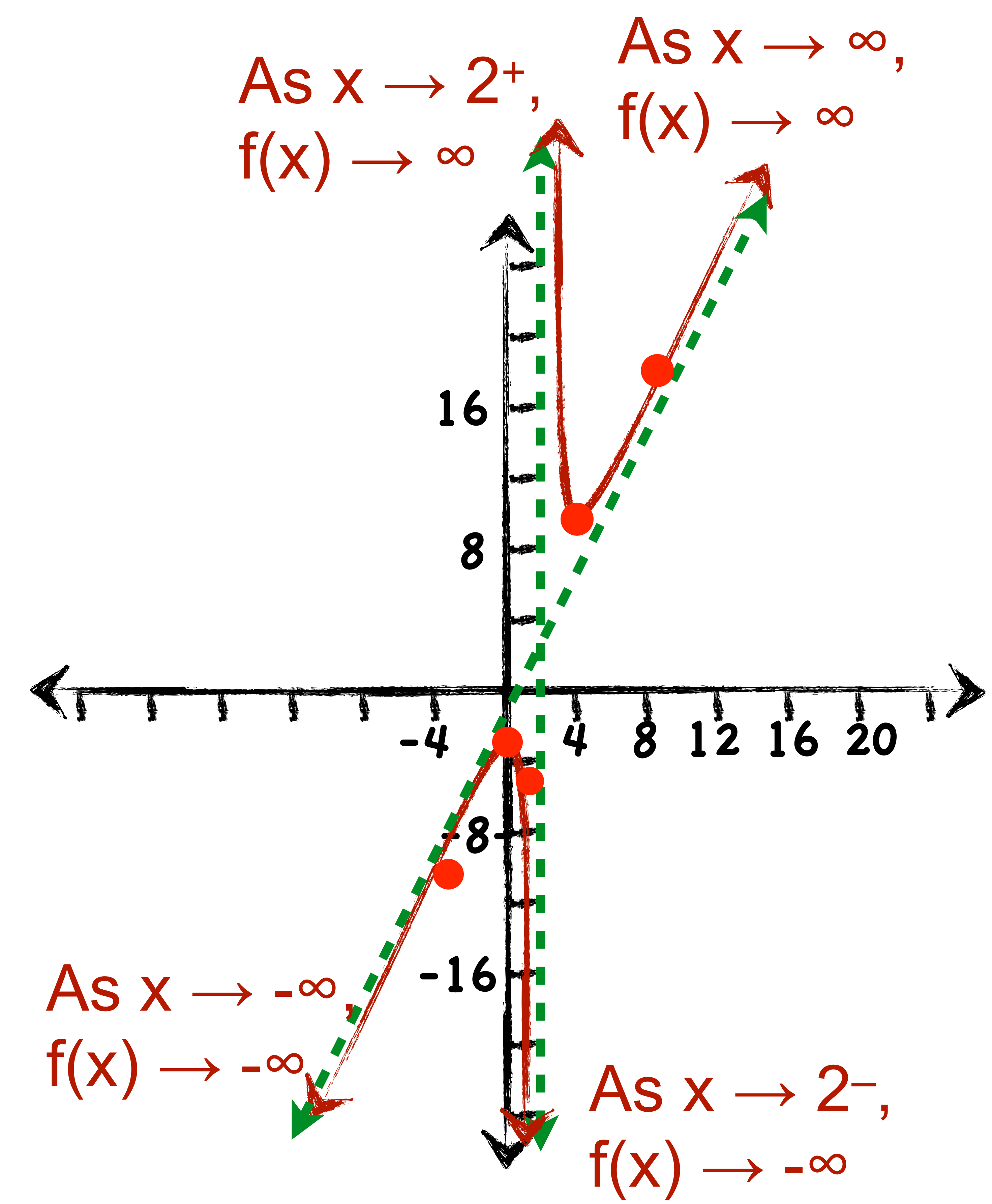
y-intercept = $\dfrac{7}{-2}$ no x-intercept

vertical asymptote x = 2 no horizontal asymptote

6. Plot at least one point between and beyond each *x*-intercept and vertical asymptote.

x		
f(x)		

7. Use the information obtained to graph the function.



Example: Application

- A company is planning to manufacture wheelchairs that are light, fast, and beautiful. The fixed monthly cost will be \$500,000 and it will cost \$400 to produce each radically innovative chair.

1. Write the cost function, C, of producing x wheelchairs.

$$C(x) = 400(x) + 500{,}000$$

2. Write the average cost function, $\overline{C}(x)$ of producing x wheelchairs.

$$\overline{C}(x) = \frac{400x + 500{,}000}{x}$$

3. Find and interpret $\overline{C}(1000)$

$$\overline{C}(1000) = \frac{400(1000) + 500{,}000}{1000} = 900$$

It costs an average of \$900/chair to produce 1000 chairs.

Example: Application

- A company is planning to manufacture wheelchairs that are light, fast, and beautiful. The fixed monthly cost will be \$500,000 and it will cost \$400 to produce each radically innovative chair.

Find and interpret $\overline{C}(10000)$

$$\overline{C}(10000) = \frac{400(10000) + 500{,}000}{10000} = 450$$

It costs an average of \$450/chair to produce 10000 chairs.

Find and interpret $\overline{C}(100{,}000)$

$$\overline{C}(100{,}000) = \frac{400(10000) + 500{,}000}{100000} = 405$$

It costs an average of \$405/chair to produce 100,000 chairs.

Example: Application

- A company is planning to manufacture wheelchairs that are light, fast, and beautiful. The fixed monthly cost will be \$500,000 and it will cost \$400 to produce each radically innovative chair.

Find the horizontal asymptote for the average cost function, and describe what the horizontal asymptote represents.

$$\overline{C}(x) = \frac{400x + 500{,}000}{x}$$

degree of numerator = degree of denominator

horizontal asymptote $y = \dfrac{400}{1} = 400$

As the number of chairs produced gets large, the average cost to produce a chair approaches \$400.