Chapter 2

Polynomial and Rational Functions

2.7 Polynomial and Rational Inequalities

Chapter 2

Polynomial and Rational Functions

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Chapter 2.7

Objectives

- Solve polynomial inequalities.
- Solve rational inequalities.
- Solve problems modeled by polynomial or rational inequalities.

Definition of a Polynomial Inequality

- A polynomial inequality is any inequality that can be written as one of these forms:

$$f(x) < 0 \qquad f(x) \leq 0 \qquad f(x) \geq 0 \qquad f(x) > 0$$

where f is a polynomial function.

Procedure for Solving Polynomial Inequalities

1. Express the inequality in the form $f(x) < 0$ or $f(x) > 0$, where f is a polynomial function.
2. Solve the equation $f(x) = 0$. The real solutions are **boundary** (critical) points.
3. Locate these boundary (critical) points on a number line, thereby dividing the number line into **intervals**.
4. Choose one representative number, called a **test value**, within each interval and evaluate f at that number.
   a. If the value is positive, $f(x) > 0$ for all numbers, x, in the interval.
   b. If the value is negative, $f(x) < 0$ for all numbers, x, in the interval.
5. The solution set is the interval or intervals that satisfy the inequality.

Procedure for Solving Polynomial Inequalities

This procedure is valid if $<$ is replaced by $\leq$ or $>$ is replaced by $\geq$.

However, if the inequality involves $\leq$ or $\geq$, **include** the boundary (critical) points [the solutions of $f(x) = 0$] in the solution set.

Example: Solving a Polynomial Inequality

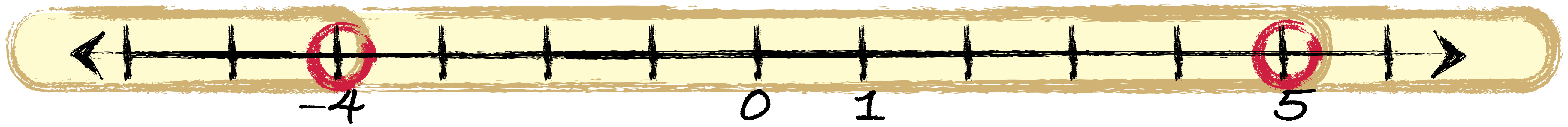
Solve and graph the solution set for: $x^2 - x > 20$

1. Express the inequality in the form $f(x) > 0$, $x^2 - x - 20 > 0.$

2. Solve the equation $f(x) = 0$. $x^2 - x - 20 = 0$

   $x^2 - x - 20 = (x + 4)(x - 5) = 0$. Boundary (critical) values are -4 and 5.

3. Locate these boundary points on a number line.



$(-\infty, -4)$ $(-4, 5)$ $(5, \infty)$

Example: Solving a Polynomial Inequality

Solve and graph the solution set: $x^2 - x > 20$, $x^2 - x - 20 > 0$

4. Choose one test value within each interval and evaluate.

+ + + + + +	- - - - - - - - - - - - - -	+ + + + + +
$(-\infty, -4)$	$(-4, 5)$	$(5, \infty)$
$(-5)^2 - -5 - 20 = 10.$	$(0)^2 - 0 - 20 = -20.$	$(6)^2 - 6 - 20 = 10.$
> 0	< 0	> 0

5. The solution set is the interval or intervals that satisfy the inequality $x^2 - x - 20 > 0$.

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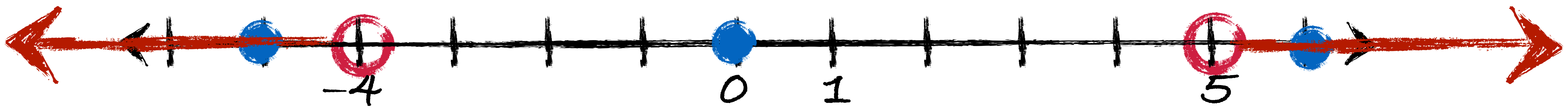
$\{x \mid x < -4 \text{ or } x > 5\}$

Example: Solving a Polynomial Inequality

Solve and graph the solution set: $x^2 - x > 20$, $x^2 - x - 20 > 0$

Once you have determined the critical values of the function, you can use the **test values** in the original inequality as well.

4. Choose one **test value** within each interval and evaluate.



$(-\infty, -4)$	$(-4, 5)$	$(5, \infty)$
$(-5)^2 - -5 > 20$	$(0)^2 - 0 > 20$	$(6)^2 - 6 > 20$
$30 > 20$	$-20 < 0$	$30 > 0$

The result better be the same.

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Solving a Polynomial Inequality

Solve and graph the solution set on a number line: $x^3 - 4x^2 - x \le -4$

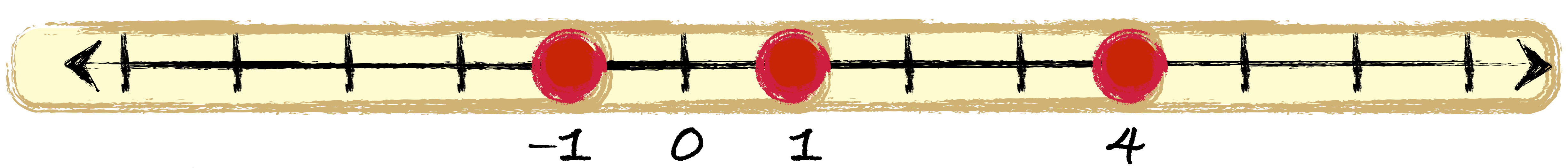
1. Express the inequality in the form $f(x) \le 0$, $x^3 - 4x^2 - x + 4 \le 0.$

2. Solve the equation $f(x) = 0$. $x^3 - 4x^2 - x + 4 = 0$

$$x^3 - 4x^2 - x + 4 = x^2(x-4) + -1(x-4) = (x^2-1)(x-4) = (x-1)(x+1)(x-4) = 0.$$

Critical values are -1, 1, and 4.

3. Locate these boundary points on a number line.

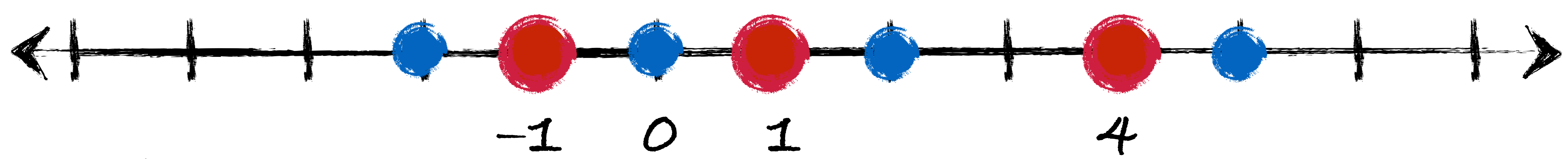


$(-\infty, -1]$ $[-1, 1]$ $[1, 4]$ $[4, \infty)$

Since the inequality is ≤, the boundaries are included.

Solving a Polynomial Inequality

Solve and graph the solution set on a number line: $x^3 - 4x^2 - x \le -4$



$(-\infty, -1]$ $[-1, 1]$ $[1, 4]$ $[4, \infty)$

4. Choose one **test value** within each interval and evaluate.

$(-\infty, -1]$

$(-2)^3 - 4(-2)^2 - -2 \le -4$

$-8 - 16 + 2 \le -4$

$-22 \le -4$

$[-1, 1]$

$(0)^3 - 4(0)^2 - 0 \le -4$

$0 \le -4$

$[1, 4]$

$(2)^3 - 4(2)^2 - 2 \le -4$

$8 - 16 - 2 \le -4$

$-10 \le -4$

$[4, \infty)$

$(5)^3 - 4(5)^2 - 5 \le -4$

$125 - 100 - 5 \le -4$

$20 \le -4$

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TI-84

You can find the solutions by using the TI-84 as well.

Solve and graph the solution set on a number line: $x^3 - 4x^2 - x \le -4$

graph $y = x^3 - 4x^2 - x$ graph $y = -4$

Find the values for which the graph of $y = x^3 - 4x^2 - x$ are at or below the graph of $y = -4$.

To find the points of intersection

2nd | TRACE CALC | ∨ 5:Intersect | Set your first curve by moving the cursor. | ENTER | Set your second curve by moving the cursor. | ENTER | Guess? | ENTER

We find one point of intersection at $x = -1$, everything to the left is less than -4. Repeat for the other values of x.

TI-84

Solve and graph the solution set on a number line: $x^3 - 4x^2 - x \leq -4$

Now let us graph $y = x^3 - 4x^2 - x + 4$.

Find values for which the graph of $y = x^3 - 4x^2 - x + 4$
are at or below the x-axis ($y \leq 0$).

To find the zeros

2nd | TRACE CALC | ∨ | 2:zero | **Set your left bound by moving the cursor.** | ENTER | **Set your right bound by moving the cursor.** | ENTER | **Guess?** | ENTER

We find a zero at $x = -1$, everything to the left is less than 0. Repeat for the other values of x.

What have we duplicated with this graphing and finding zeros?

Definition of a Rational Inequality

A rational inequality is any inequality that can be put into one of the forms

$$f(x) < 0 \qquad f(x) \leq 0 \qquad f(x) \geq 0 \qquad f(x) > 0$$

where f is a rational function.

Rational function $f(x) = \dfrac{p(x)}{q(x)}$.

Example: Solving a Rational Inequality

Solve and graph the solution set: $\dfrac{2x}{x+1} \geq 1$

1. Express the inequality in the simplified form: $\dfrac{p(x)}{q(x)} \geq 0$

$$\frac{2x}{x+1} - 1 \geq 0 \qquad \frac{2x}{x+1} - \frac{x+1}{x+1} \geq 0 \qquad \frac{x-1}{x+1} \geq 0$$

2. Set numerator ($p(x)=0$, x-intercepts) and denominator ($q(x)=0$, $f(x)$ is undefined) for critical values (boundaries).

$$\frac{x-1}{x+1} = 0 \qquad x - 1 = 0 \qquad x + 1 = 0$$

$$x = 1 \qquad x = -1$$

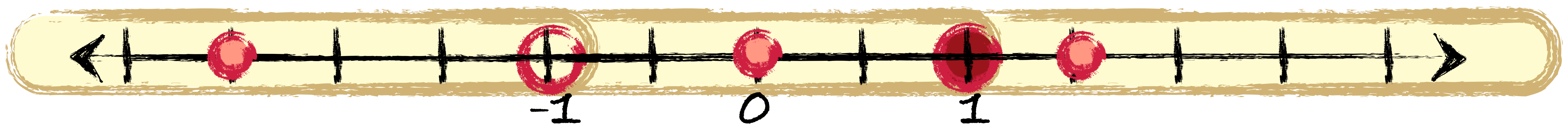
boundaries at -1, and 1

Example: Solving a Rational Inequality

Solve and graph the solution set: $\frac{2x}{x+1} \geq 1$ or $\frac{x-1}{x+1} \geq 0$

3. Locate boundary points on number line to create 3 intervals.

boundaries at -1, and 1



Note: x cannot be -1
but can be 1

+ + + + +	- - - - - -	+ + + + +
$(-\infty, -1)$	$(-1, 1]$	$[1, \infty)$

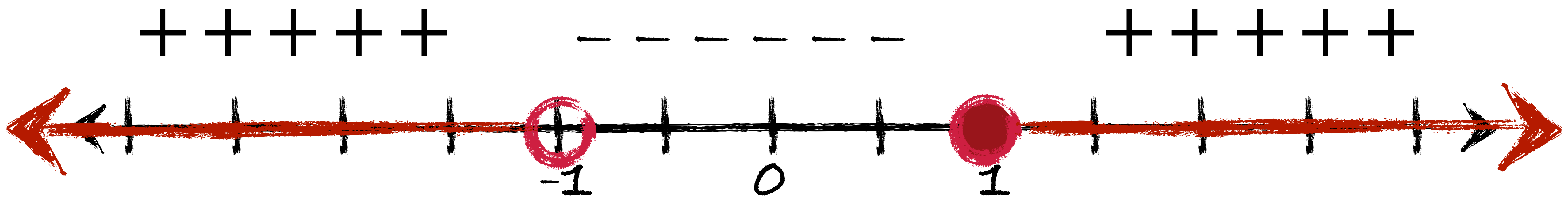
4. Choose one test value within each interval and evaluate.

$$\frac{-4-1}{-4+1} = \frac{-5}{-3} = \frac{5}{3} \qquad \frac{0-1}{0+1} = \frac{-1}{1} = -1 \qquad \frac{2-1}{2+1} = \frac{1}{3}$$

Example: Solving a Rational Inequality

Solve and graph the solution set: $\dfrac{2x}{x+1} \geq 1$ or $\dfrac{x-1}{x+1} \geq 0$

5. Write and graph the solution set that satisfies the given inequality.



$(-\infty, -1)$ $[1, \infty)$

$\cup$

$\{x \mid x < -1 \text{ or } x \geq 1\}$

Example: Solving a Rational Inequality

Solve and graph the solution set: $\frac{(2-x)(x+3)}{x+1} \le 0$

1. Express the inequality in the simplified form. $\frac{p(x)}{q(x)} \ge 0$

$$\frac{(2-x)(x+3)}{x+1} \le 0$$

2. Set numerator (p(x)=0, x-intercepts) and denominator (q(x)=0, f(x) is undefined) for boundaries.

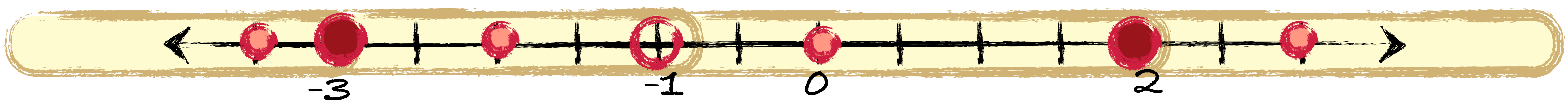
$x+1=0$	$2-x=0$	$x+3=0$
$x=-1$	$x=2$	$x=-3$

Example: Solving a Rational Inequality

Solve and graph the solution set: $\dfrac{(2-x)(x+3)}{x+1} \le 0$

$x=-1 \quad x=2 \quad x=-3$

3. Locate boundary points on number line to create 4 intervals.



(-∞, -3]   [-3, -1)   (-1, 2]   [2, ∞)

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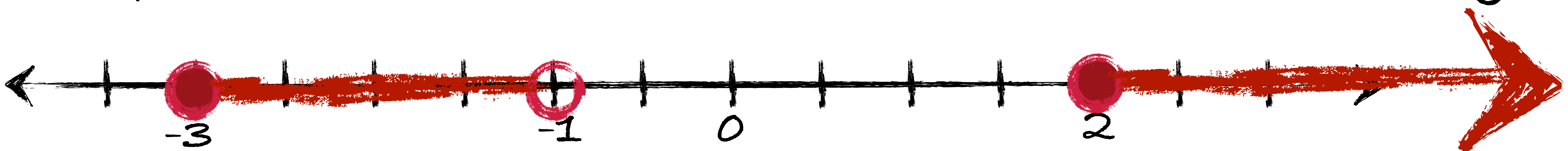
4. Choose one test value within each interval and evaluate.

$\dfrac{(2--4)(-4+3)}{-4+1}=2 \qquad \dfrac{(2--2)(-2+3)}{-2+1}=-4 \qquad \dfrac{(2-0)(0+3)}{0+1}=6 \qquad \dfrac{(2-3)(3+3)}{3+1}=-\dfrac{3}{2}$

Example: Solving a Rational Inequality

Solve and graph the solution set: $\dfrac{(2-x)(x+3)}{x+1} \le 0$

5. Write and graph the solution set that satisfies the given inequality.



(-3, -1) [2, ∞)

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∪

$\{x \mid -3 < x < -1 \text{ or } x \ge 2\}$

The Position Function for a Free-Falling Object Near Earth's Surface

An object that is falling or vertically projected into the air has its height above the ground, s(t), in feet given by

$$s(t) = -16t^2 + v_0 t + s_0$$

where v_0 is the original velocity (initial velocity) of the object, in feet per second, t is the time that the object is in motion, in seconds, and s_0 is the original height (initial height) of the object, in feet.

Example: Application

An object is propelled straight up from ground level with an initial velocity of 80 feet per second. It's height at time t is modeled by

$$s(t) = -16t^2 + 80t + 0$$

where the height, s(t), is measured in feet and the time, t, is measured in seconds.

In which time interval will the object be more than 64 feet above ground?

We must solve the inequality $s(t) > 64$ or $-16t^2 + 80t > 64$

Example: Application

We must solve the inequality s(t) > 64 or $-16t^2 + 80t > 64$

1. Express the inequality in the form f(x) > 0, $-16t^2 + 80t - 64 > 0$

2. Solve the equation $f(x) = 0$. $-16t^2 + 80t - 64 = 0$

$-16t^2 + 80t - 64 = 0 \qquad -16(t^2 + 5t - 4) = 0 \qquad (t - 1)(t - 4) = 0$

t = 1 sec and t = 4 sec are the boundaries

3. Locate these boundary points on a number line.

$(-\infty, 1]$ $[1, 4]$ $[4, \infty)$

Example: Application

We must solve the inequality $s(t) = -16t^2 + 80t - 64 > 0$

4. Choose one **test value** within each interval and evaluate.

$(-\infty, 1]$ $[1, 4]$ $[4, \infty)$

$-16(0)^2 + 80(0) - 64 = -64.$ < 0

$-16(2)^2 + 80(2) - 64 = 32.$ > 0

$-16(5)^2 + 80(5) - 64 = -64.$ < 0

5. The solution set is the interval or intervals that satisfy the inequality $-16t^2 + 80t - 64 > 0$.

$\{x \mid 1 < x < 4\}$

Example: Application

An object that is falling or vertically projected into the air has its height above the ground, $s(t)$, in feet given by

$$s(t) = -16t^2 + v_0t + s_0$$

where v_0 is the original velocity (initial velocity) of the object, in feet per second, t is the time that the object is in motion, in seconds, and s_0 is the original height (initial height) of the object, in feet.

In which time interval will the object be more than 64 feet above ground?

$$\{x \mid 1 < x < 4\}$$

The object will be more than 64 feet above the ground between 1 and 4 seconds, not inclusive, after release.