Chapter 3

Exponential and Logarithmic Functions

3.3 Properties of Logarithms

Chapter 3

Homework

3.3 p421 7, 11, 17, 19, 25, 27, 35, 39, 47, 51, 59, 67, 79

Chapter 3-3

Objectives

- Use the product rule.
- Use the quotient rule.
- Use the power rule.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base property.

Why Logarithms

- Large numbers are difficult for people to assimilate. Donald Trump claims to be worth in excess of $10 billion, the U.S. budget debt is in the trillions of dollars, but what do those numbers represent?

- Well a million seconds is about 11.5 days, a billion seconds is about 31 years, and one trillion seconds is about 31,710 years. That is the difference between a year's vacation time, the time it takes to get the kids out of the house, and the entirety of human existence.

- To change these numbers into manageable size we use logarithms. The log of 1,000,000 is 6, log 1,000,000,000 = 9, and log 1,000,000,000,000 = 12. 6, 9, 12 are much easier to handle. We accomplish this through the use of **exponents**.

The Product Rule

- Let b, M, and N be positive real numbers with b ≠ 1. Then:

$$\log_b(MN) = \log_b M + \log_b N$$

  - The logarithm of a product is the sum of the logarithms.
  - This can be easily verified by converting to exponential form.
    - Let $\log_b(M) = x$ $\log_b(N) = y$
    - Then $b^x = M$ $b^y = N$

$$\log_b(MN) = \log_b\left(b^x \cdot b^y\right) = \log_b\left(b^{x+y}\right) = x + y = \log_b M + \log_b N$$

Example: Using the Product Rule

- Use the product rule to expand each logarithmic expression:

$$\log_b(MN) = \log_b M + \log_b N$$

$$\log_6(7 \bullet 11) = \log_6 7 + \log_6 11$$

$$\log 100x = \log 100 + \log x = 2 + \log x$$

$$\log_6 648 = \log_6 (216 \cdot 3) = \log_6 216 + \log_6 3 = 3 + \log_6 3$$

- Solve for y: $\ln y = \ln x + \ln c = \ln(x \cdot c) \quad y = x \cdot c$

The Quotient Rule

- Let b, M, and N be positive real numbers with b ≠ 1. Then:

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

- The logarithm of a quotient is the **difference** of the logarithms.

- This is simply a rewrite of the previous rule, remembering subtraction is addition of the opposite.

Example: Using the Quotient Rule

- Use the quotient rule to expand each logarithmic expression:

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

$$\log_8\left(\frac{23}{x}\right) = \log_8 23 - \log_8 x$$

$$\ln\left(\frac{e^5}{11}\right) = \ln e^5 - \ln 11$$

$$= 5 - \ln 11$$

$$\log\left(\frac{\sqrt{x}\sqrt[3]{y^2}}{z^4}\right) = \log\sqrt{x} + \log\sqrt[3]{y^2} - \log z^4$$

- We will finish this in a couple of slides.

The Power Rule

- Let b, M, and N be positive real numbers with b ≠ 1. Then:

$$\log_b M^p = p \log_b M$$

- The logarithm of a number with an exponent is the product of the exponent and the logarithm of that number.
- This can again be shown by converting to exponential form.
  - Let $\log_b M = a$
  - Then $b^a = M$

$$M^p = \left(b^a\right)^p = b^{pa} \qquad \log_b M^p = pa = p \log_b M$$

Example: Using the Power Rule

- Use the power rule to expand each logarithmic expression:

$$\log_b M^p = p\log_b M$$

$$\log_6 3^9 = 9\log_6 3$$

$$\ln\sqrt[3]{x} = \ln x^{\frac{1}{3}} = \frac{1}{3}\ln x$$

$$\log(x+4)^2 = 2\log(x+4)$$

$$\log_{25} 5^3 = 3\log_{25} 5 = 3\log_{25} 25^{\frac{1}{2}} = \frac{3}{2}\log_{25} 25 = \frac{3}{2}$$

$$\log\left(\frac{\sqrt{x}\sqrt[3]{y^2}}{z^4}\right) = \log\sqrt{x} + \log\sqrt[3]{y^2} - \log z^4$$

$$= \log x^{\frac{1}{2}} + \log\left(y^2\right)^{\frac{1}{3}} - \log z^4$$

$$= \frac{1}{2}\log x + \frac{2}{3}\log y - 4\log z$$

Properties for Expanding Logarithmic Expressions

- For M > 0 and N > 0:
  - Product Rule $$\log_b(MN) = \log_b M + \log_b N$$
  - Quotient Rule $$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$
  - Power Rule $$\log_b M^p = p\log_b M$$
- Note: In every case the base remains consistent (the same).

Common Errors to Avoid

$$\log_b(M+N) \neq \log_b M + \log_b N$$

$$\log_b(M-N) \neq \log_b M - \log_b N$$

$$\log_b\left(\frac{M}{N}\right) \neq \frac{\log_b M}{\log_b N}$$

$$\log_b M^p \neq \log_b pM$$

$$\log_b MN^p \neq p\log_b MN$$

NONONONONONONONONONONONONO

Example: Expanding Logarithmic Expressions

- Expand each logarithmic expression as much as possible:

$$\log_b\left(x^4\sqrt[3]{y}\right) = \log_b\left(x^4y^{\frac{1}{3}}\right)$$

$$= \log_b\left(x^4\right) + \log_b\left(y^{\frac{1}{3}}\right)$$

$$= 4\log_b x + \frac{1}{3}\log_b y$$

Example

- Write the following expression in terms of logs of x and z.

$$\log\left(x\sqrt{\frac{\sqrt{x}}{z}}\right)$$

$$= \log x + \log\sqrt{\frac{\sqrt{x}}{z}}$$

$$= \log x + \log\left(\frac{\sqrt{x}}{z}\right)^{\frac{1}{2}}$$

$$= \log x + \frac{1}{2}\log\left(\frac{\sqrt{x}}{z}\right)$$

$$= \log x + \frac{1}{2}\left(\log\sqrt{x} - \log z\right)$$

$$= \log x + \frac{1}{2}\left(\log x^{\frac{1}{2}} - \log z\right)$$

$$= \log x + \frac{1}{2}\left(\frac{1}{2}\log x - \log z\right)$$

$$= \log x + \frac{1}{4}\log x - \frac{1}{2}\log z$$

Example: Expanding Logarithmic Expressions

- Use the product rule to expand each logarithmic expression as much as possible:

$$\log_5\left(\frac{\sqrt{x}}{25y^3}\right) = \log_5\left(\frac{x^{\frac{1}{2}}}{25y^3}\right) = \log_5 x^{\frac{1}{2}} - \log_5 25y^3$$

$$= \log_5 x^{\frac{1}{2}} - \left(\log_5 25 + \log_5 y^3\right)$$

$$= \frac{1}{2}\log_5 x - \left(\log_5 25 + 3\log_5 y\right)$$

$$= \frac{1}{2}\log_5 x - 2 - 3\log_5 y$$

- Write the following expression in terms of logs of x, y, and z.

$$\log\left(\sqrt{\frac{xy^2}{z^8}}\right)$$

$$= \log\left(\frac{xy^2}{z^8}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\log\left(\frac{xy^2}{z^8}\right)$$

$$= \frac{1}{2}\left(\log\left(xy^2\right) - \log z^8\right)$$

$$= \frac{1}{2}\left(\log x + \log y^2 - \log z^8\right)$$

$$= \frac{1}{2}\left(\log x + 2\log y - 8\log z\right)$$

$$= \frac{1}{2}\log x + \log y - 4\log z$$

Condensing Logarithmic Expressions

- For M > 0 and N > 0:
  - Product Rule

$$\log_b M + \log_b N = \log_b (MN)$$

  - Quotient Rule

$$\log_b M - \log_b N = \log_b \left(\frac{M}{N}\right)$$

  - Power Rule

$$p \log_b M = \log_b M^p$$

- Note: In every case the base remains consistent (the same).

Example: Condensing Logarithmic Expressions

- Write as a single logarithm

$$\log 25 + \log 4 = \log(25 \bullet 4) = \log 100 = 2$$

$$\log(7x+6) - \log x = \log\left(\frac{7x+6}{x}\right)$$

$$2\ln x + \frac{1}{3}\ln(x+5) = \ln x^2 + \ln(x+5)^{\frac{1}{3}}$$

$$= \ln x^2 + \ln\sqrt[3]{x+5}$$

$$= \ln x^2\sqrt[3]{x+5}$$

- Find x if $$2\log_b 5+\frac{1}{2}\log_b 9-\log_b 3=\log_b x$$

$$\log_b 5^2+\log_b 9^{\frac{1}{2}}-\log_b 3=\log_b x$$

$$\log_b 25+\log_b 3-\log_b 3=\log_b x \quad \Rightarrow \quad x=25$$

- or, if you prefer $$\log_b\left(\frac{25\cdot 3}{3}\right)=\log_b x$$

The Change-of-Base Property

- For any logarithmic bases a and b, and any positive number M,

$$\log_b M = \frac{\log_a M}{\log_a b}$$

- The logarithm of M with base b is equal to the logarithm of M with any new base divided by the logarithm of b with that new base.

- The change of base formula allows you to estimate any log on your calculator without using logbase(), or to use a simpler calculator (i.e. TI 30XIIS).

The Change-of-Base Property

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows.

Base b	*Base 10*	*Base e*
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log x}{\log a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

The Change-of-Base Property: Introducing Common and Natural Logarithms

- Introducing Common Logarithms

$$\log_b M = \frac{\log M}{\log b}$$

- Introducing Natural Logarithms

$$\log_b M = \frac{\ln M}{\ln b}$$

Example: Changing Base to Common Logarithms

- Use your calculator and common logs to evaluate

$$\log_7 2506 = \frac{\log 2506}{\log 7} \approx \frac{3.39898}{.8451} \approx 4.022$$

- Use your calculator and natural logs to evaluate

$$\log_7 2506 = \frac{\ln 2506}{\ln 7} \approx \frac{7.8264}{1.9489} \approx 4.022$$

Finding the inverse of a function

- Remember, when finding the inverse of a function:

    1. Replace $f(x)$ with y in the equation.
    2. Exchange x and y in the equation.
    3. Solve the equation for y in terms of x.
    4. If the inverse is a function, replace y with $f^{-1}(x)$.

- These same rules apply when finding the inverse of an exponential equation (a log) or when finding the inverse of a log function (an exponential function).

Finding the Inverse of an Exponential Equation.

- Find the inverse of $f(x) = 2 \cdot 5^x - 3$

1. Replace f(x) with y in the equation. $y = 2 \cdot 5^x - 3$
2. Exchange x and y in the equation. $x = 2 \cdot 5^y - 3$
3. Solve the equation for y in terms of x. $x + 3 = 2 \cdot 5^y$ $\quad \frac{x+3}{2} = 5^y$ $\quad \log_5\left(\frac{x+3}{2}\right) = y$
4. If the inverse is a function, replace y with $f^{-1}(x)$. $f^{-1}(x) = \log_5\left(\frac{x+3}{2}\right)$

Finding the Inverse of an Exponential Equation.

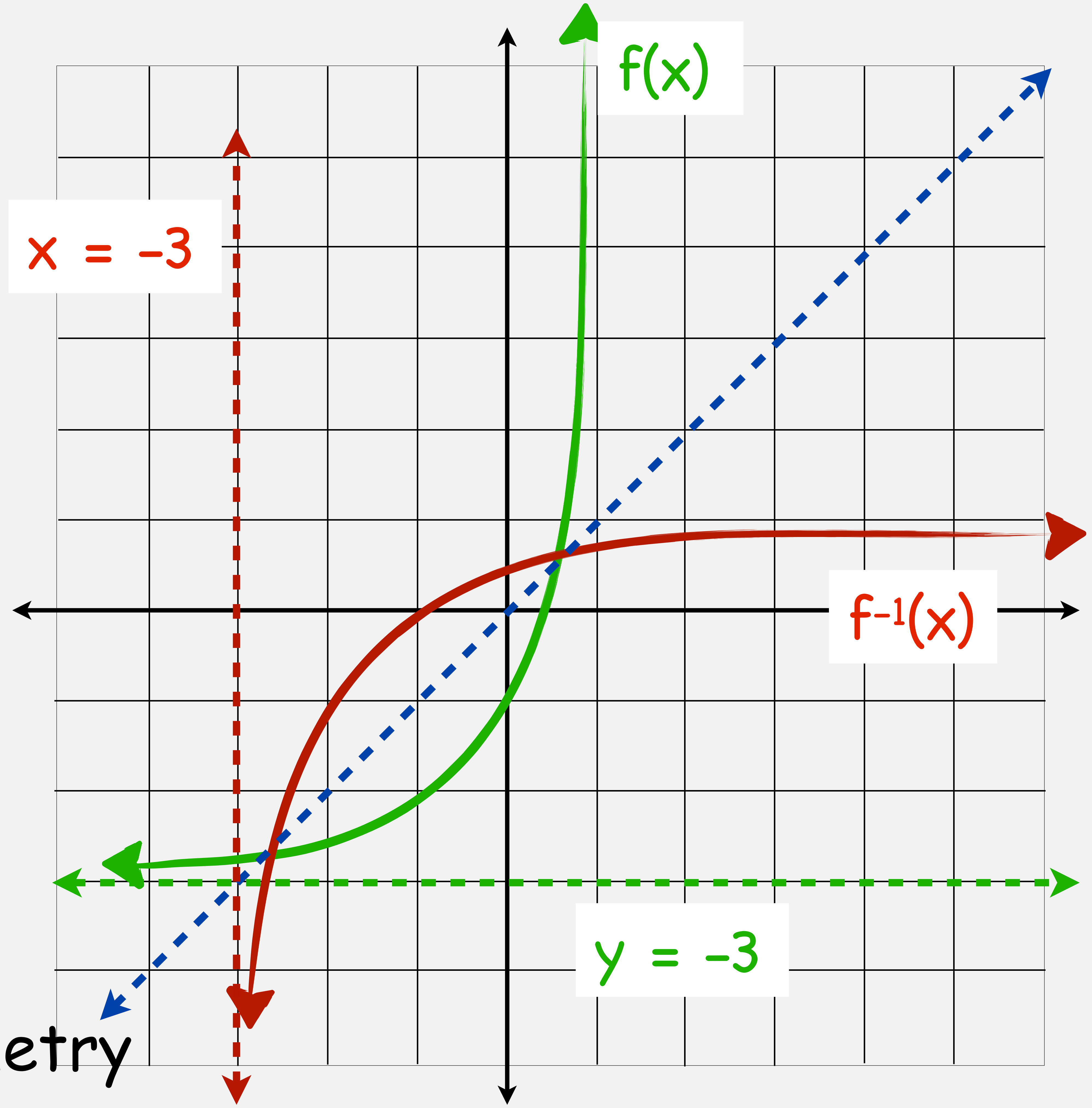
- We can graph both functions

$f(x) = 2 \cdot 5^x - 3$

x	f(x)
-2	-2.92
-1	-2.6
0	-1
1	7
2	47

$f^{-1}(x) = \log_5\left(\frac{x+3}{2}\right)$

x	$f^{-1}(x)$
-3	oops
-2	-0.4307
-1	0
0	0.2519
2	0.5693
7	1



- Note asymptotes, intercepts, and axis of symmetry

Finding Asymptotes and Intercepts

$f(x) = 2 \cdot 5^x - 3$

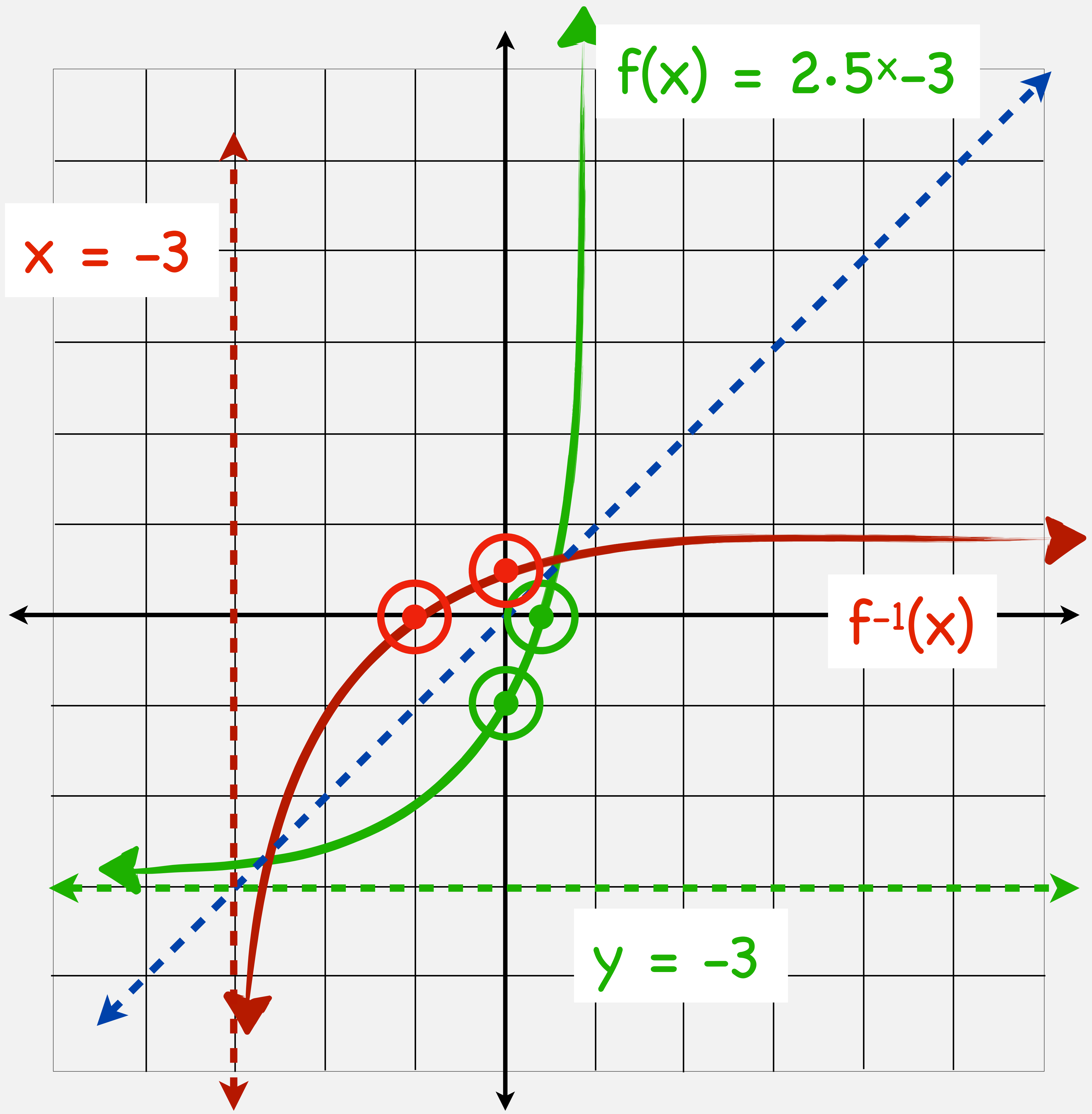
$0 = 2 \cdot 5^x - 3$ $\quad 3 = 2 \cdot 5^x$ $\quad \frac{3}{2} = 5^x$ $\quad \log_5 \frac{3}{2} = x \approx .25$

$y = 2 \cdot 5^0 - 3$ $\quad y = 2 - 3 = -1$

$f^{-1}(x) = \log_5\left(\frac{x+3}{2}\right)$

$0 = \log_5\left(\frac{x+3}{2}\right)$ $\quad \frac{x+3}{2} = 1$ $\quad x = -1$

$y = \log_5\left(\frac{0+3}{2}\right)$ $\quad y = \log_5\left(\frac{3}{2}\right) \approx .2519$



Finding the Inverse of an Exponential Equation.

- Find the inverse of $f(x)=\frac{1}{3}e^{x}+1$

1. Replace f(x) with y in the equation. $y=\frac{1}{3}e^{x}+1$
2. Exchange x and y in the equation. $x=\frac{1}{3}e^{y}+1$
3. Solve the equation for y in terms of x. $x-1=\frac{1}{3}e^{y}$ $3(x-1)=e^{y}$ $\ln e^{y}=\ln(3x-3)$

   $y=\ln(3x-3)$
4. If the inverse is a function, replace y with $f^{-1}(x)$. $f^{-1}(x)=\ln(3x-3)$

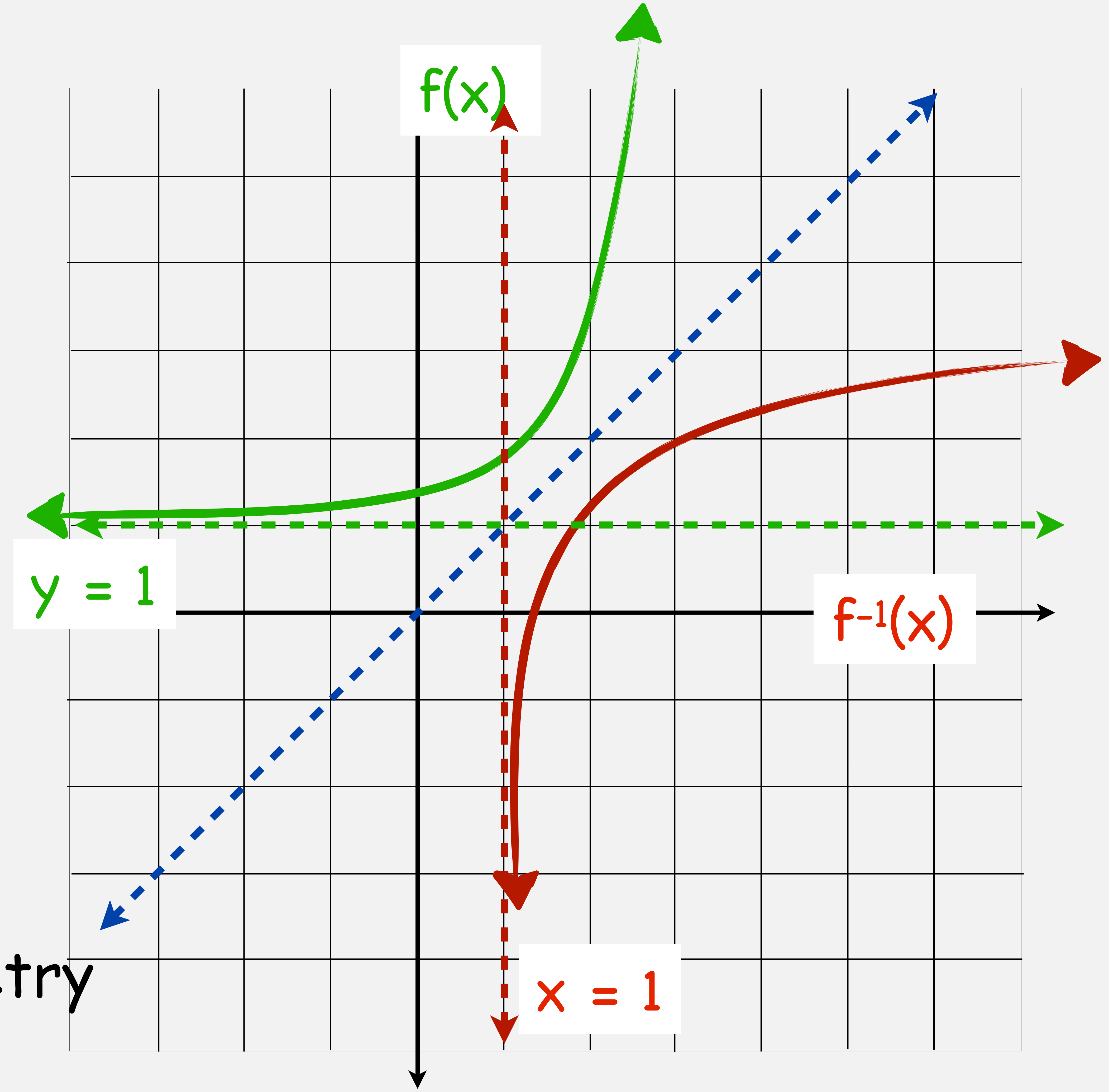
Finding the Inverse of an Exponential Equation.

- We can graph both functions

$$f(x) = \frac{1}{3}e^x + 1 \qquad f^{-1}(x) = \ln(3x - 3)$$

x	f(x)
-2	1.0451
-1	1.1226
0	1.333
1	1.9061
2	3.463

x	$f^{-1}(x)$
1	oops
2	1.0986
3	1.79
4	2.1972
6	2.7081



- Note asymptotes, intercepts, and axis of symmetry

Finding the Inverse of an Exponential Equation.

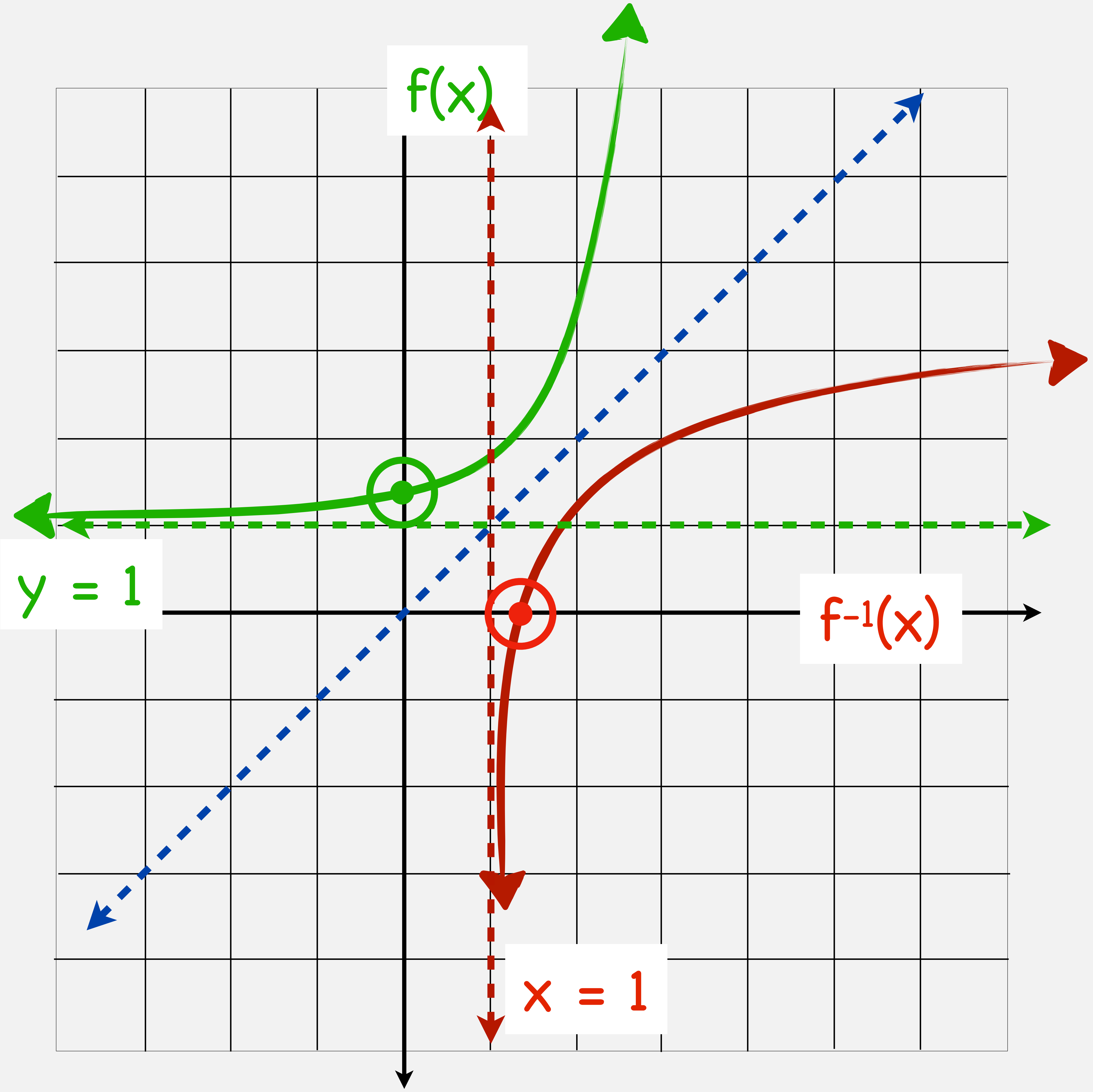
$$f(x) = \frac{1}{3}e^{x} + 1$$

$$0 = \frac{1}{3}e^{x} + 1 \qquad \frac{1}{3}e^{x} = -1 \qquad y = \frac{1}{3}e^{0} + 1 \qquad y = \frac{4}{3}$$

$$f^{-1}(x) = \ln(3x - 3)$$

$$0 = \ln(3x - 3) \qquad 3x - 3 = 1 \qquad x = \frac{4}{3}$$

$$y = \ln(0 - 3)$$



Finding the Inverse of a Logarithm Function

- Find the inverse of $f(x) = 2\log_3(x-2) - 1$

1. Replace f(x) with y in the equation. $y = 2\log_3(x-2) - 1$
2. Exchange x and y in the equation. $x = 2\log_3(y-2) - 1$
3. Solve the equation for y in terms of x. $\frac{x+1}{2} = \log_3(y-2)$ $\quad 3^{\left(\frac{x+1}{2}\right)} = y-2$ $\quad y = 3^{\left(\frac{1}{2}(x+1)\right)} + 2$
4. If the inverse is a function, replace y with $f^{-1}(x)$. $f^{-1}(x) = 3^{\left(\frac{1}{2}(x+1)\right)} + 2$

Finding the Inverse of a Logarithm Function

- We have graphed enough, so let us find the intercepts.

$$f(x) = 2\log_3(x-2) - 1$$

$$0 = 2\log_3(x-2) - 1 \qquad \frac{1}{2} = \log_3(x-2) \qquad x = 3^{\frac{1}{2}} + 2 = 2 + \sqrt{3} \approx 3.732$$

$$y = 2\log_3(0-2) - 1$$

$$f^{-1}(x) = 3^{\left(\frac{1}{2}(x+1)\right)} + 2$$

$$0 = 3^{\left(\frac{1}{2}(x+1)\right)} + 2 \qquad -2 = 3^{\left(\frac{1}{2}(x+1)\right)}$$

$$y = 3^{\left(\frac{1}{2}(0+1)\right)} + 2 \qquad y = 3^{\left(\frac{1}{2}(0+1)\right)} + 2 = 3^{\left(\frac{1}{2}\right)} + 2$$

Finding the Inverse of a Logarithmic Function

- Find the inverse of $f(x) = 2\log(2x+3) - 5$

1. Replace f(x) with y in the equation. $y = 2\log(2x+3) - 5$
2. Exchange x and y in the equation. $x = 2\log(2y+3) - 5$
3. Solve the equation for y in terms of x. $\frac{x+5}{2} = \log(2y+3)$ $\quad 10^{\left(\frac{x+5}{2}\right)} = 2y+3$ $\quad y = \frac{10^{\left(\frac{x+5}{2}\right)} - 3}{2}$
4. If the inverse is a function, replace y with $f^{-1}(x)$. $f^{-1}(x) = \frac{10^{\left(\frac{x+5}{2}\right)} - 3}{2}$

Finding Inverse of Logarithm

- Again, find the intercepts.

$$f(x) = 2\log(2x+3) - 5$$

$$0 = 2\log(2x+3) - 5 \qquad \frac{5}{2} = \log(2x+3) \qquad 10^{\frac{5}{2}} = 2x+3 \qquad x = \frac{10^{\frac{5}{2}} - 3}{2} \approx 156.61$$

$$y = 2\log(2 \cdot 0 + 3) - 5 \qquad y = 2\log(3) - 5 \approx -4.0458$$

$$f^{-1}(x) = \frac{10^{\left(\frac{x+5}{2}\right)} - 3}{2}$$

$$0 = \frac{10^{\left(\frac{x+5}{2}\right)} - 3}{2} \qquad 0 = 10^{\left(\frac{x+5}{2}\right)} - 3 \qquad 10^{\left(\frac{x+5}{2}\right)} = 3 \qquad \frac{x+5}{2} = \log 3 \qquad x = 2\log 3 - 5 \approx -4.0458$$

$$y = \frac{10^{\left(\frac{0+5}{2}\right)} - 3}{2} \qquad y = \frac{10^{\left(\frac{5}{2}\right)} - 3}{2} \approx 156.61$$

Measuring Earthquakes

- On the Richter scale, the magnitude R of an earthquake of intensity I is given by $R = \frac{\ln I - \ln I_0}{\ln 10}$ where I_0 is the minimum intensity used for comparison.
  - Write this as a single common logarithmic expression.
  - Using the change of base formula.

$$R = \frac{\ln I - \ln I_0}{\ln 10} = \frac{\frac{\log I}{\log e} - \frac{\log I_0}{\log e}}{\frac{\log 10}{\log e}} = \frac{\log I - \log I_0}{\log 10} = \frac{\log I - \log I_0}{1} = \log \frac{I}{I_0}$$

Another Look

- On the Richter scale, the magnitude R of an earthquake of intensity I is given by $R = \frac{\ln I - \ln I_0}{\ln 10}$ where I_0 is the minimum intensity used for comparison.
  - Write this as a single common logarithmic expression.
  - Using the converse of the change of base formula.

$$R = \frac{\ln I - \ln I_0}{\ln 10} = \frac{\ln I}{\ln 10} - \frac{\ln I_0}{\ln 10} = \log I - \log I_0 = \log \frac{I}{I_0}$$