Chpt 3

- Exponential and Logarithmic Functions

  - 3.5 Exponential Growth and Decay, Modeling Data

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- 3.5 p447 1, 3, 5, 7, 10, 15, 17, 19, 33, 51, 53, 55, 57

Chpt 3.5

- Objectives

  - Model exponential growth and decay.
  - Choose an appropriate model for data.
  - Express an exponential model in base e .

Exponential Growth and Decay Models

- The mathematical model for exponential growth or decay is given by …

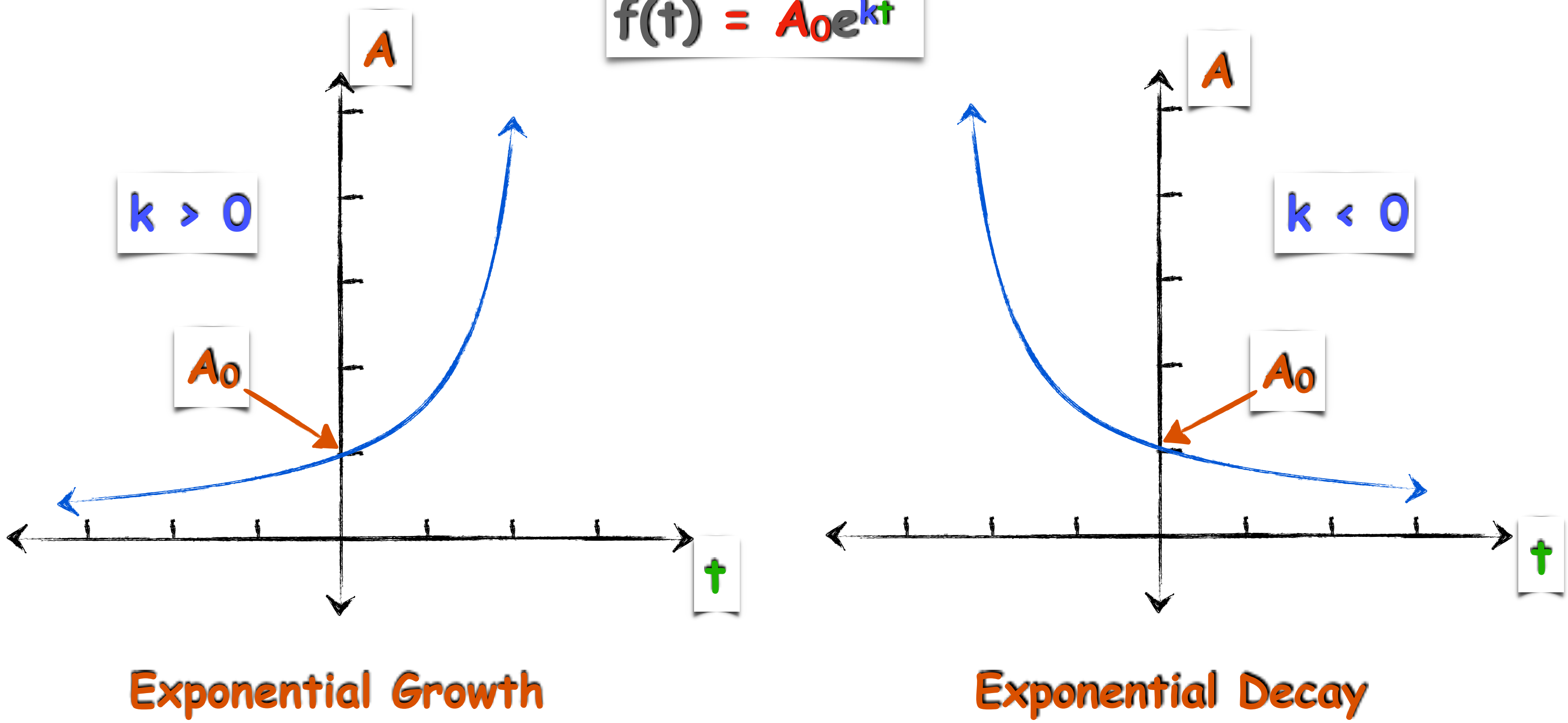
$$f(t) = A_0 e^{kt} \quad \text{or} \quad A = A_0 e^{kt} \quad \text{or} \quad B = Pe^{rt} \text{ (finance)}$$

  - If $k > 0$, the function models the amount, or size of a **growing** entity. A_0 is the **original** (beginning) amount, or size, of the growing entity at time $t = 0$. A is the amount at **time** t, and k is a constant representing the **growth rate**.

  - If $k < 0$, the function models the amount, or size of a **decaying** entity. A_0 is the **original** (beginning) amount, or size, of the decaying entity at time $t = 0$. A is the amount at **time** t, and k is a constant representing the **growth rate**.

Exponential Growth and Decay Models

$$f(t) = A_0 e^{kt}$$



Example: Application

In 2000, the population of Africa was 807 million and by 2011 it had grown to 1052 million. Use the exponential growth model **$A = A_0e^{kt}$**, in which **A** is the population (in millions), and **t** is the number of years **after 2000**, to find the exponential growth **function** that models the data.

$$A = A_0e^{kt}$$

$$A = 807e^{kt}$$

How do we find k?

$$1052 = 807e^{k11}$$

$$\frac{1052}{807} = e^{k11}$$

$$\ln\frac{1052}{807} = \ln e^{k11}$$

$$\ln\frac{1052}{807} = 11k$$

$$k = \frac{1}{11}\ln\frac{1052}{807}$$

$$k \approx .0241$$

$$A \approx 807e^{0.0241t}$$

Example: Application

In 2000, the population of Africa was 807 million and by 2011 it had grown to 1052 million. We use the exponential growth model $A = A_0e^{kt}$, in which A is the population (in millions), and t is the number of years **after 2000**, to find the year population is predicted to exceed **2 billion**.

$$A \approx 807e^{0.0241t}$$

2 billion = 2000 million

$$2000 = 807e^{0.0241t}$$

$$\frac{2000}{807} = e^{0.0241t}$$

$$\ln\frac{2000}{807} = \ln e^{0.0241t}$$

$$0.0241t = \ln\frac{2000}{807}$$

$$t = \frac{1}{0.0241}\ln\frac{2000}{807} \approx 37.6589$$

By the year 2038, the population is predicted to exceed 2 billion.

Half Life

The **half-life** of strontium-90 is 28 years, meaning that after 28 years a given amount of the substance will have decayed to half the original amount. **Find the exponential decay model for strontium-90.**

$$A = A_0 e^{kt}$$

Half in 28 yrs.

$$\frac{1}{2}A_0 = A_0 e^{k28}$$

$$\frac{1}{2} = e^{k28}$$

$$\ln\frac{1}{2} = \ln e^{28k}$$

$$28k = \ln\frac{1}{2}$$

$$k = \frac{1}{28}\ln\frac{1}{2} \approx -.0248$$

k < 0, decay

The exponential decay model is

$$\boxed{A = A_0 e^{-.0248t}}$$

Example: Application

We found the exponential decay model to be $A = A_0 e^{-.0248t}$ If there are originally 60 grams, how long will it take for strontium-90 to decay to a level of 10 grams?

$$\boxed{A = A_0 e^{-.0248t}}$$

$$10 = 60e^{-.0248t}$$

$$\frac{1}{6} = e^{-.0248t}$$

$$\ln\frac{1}{6} = \ln e^{-.0248t}$$

$$-0.0248t = \ln\frac{1}{6}$$

$$t = \frac{1}{-0.0248}\ln\frac{1}{6} \approx 72.2484$$

It will take about 72.25 years for 60 grams of strontium-90 to decay to a level of 10 grams.

- The half-life of radioactive potassium is 1.3 billion years. If 20 grams are present now, how much will be present in 1000 years? When will there be 15 grams remaining?

$$A = A_0 e^{kt}$$

Half in 1.3 billion yrs.

$$\frac{1}{2}A_0 = A_0 e^{k1.3}$$

$$\frac{1}{2} = e^{k1.3}$$

$$\ln\frac{1}{2} = \ln e^{1.3k}$$

$$1.3k = \ln\frac{1}{2}$$

$$k = \frac{1}{1.3}\ln\frac{1}{2} \approx -.5332$$

$k < 0$, decay

The exponential decay model is

$$\boxed{A = A_0 e^{-.5332t}}$$

Where t is the number of **billions** of years.

The half-life of radioactive potassium is 1.3 billion years. If 20 grams are present now, how much will be present in 1000 years? When will there be 15 grams remaining?

$$A = A_0 e^{-.5332t}$$

Where t is the number of **billions** of years.

$$A = 20e^{-.5332\left(\frac{1000}{1000000000}\right)}$$

$$A = 19.99998934$$

In 1000 years there will be 19.9999 grams left for your great-great-great grandchildren to deal with.

$$15 = 20e^{-.5332\left(\frac{t}{1000000000}\right)}$$

$$.75 = e^{\left(\frac{-.5332t}{1000000000}\right)}$$

$$\ln(.75) = \ln e^{\left(\frac{-.5332t}{1000000000}\right)}$$

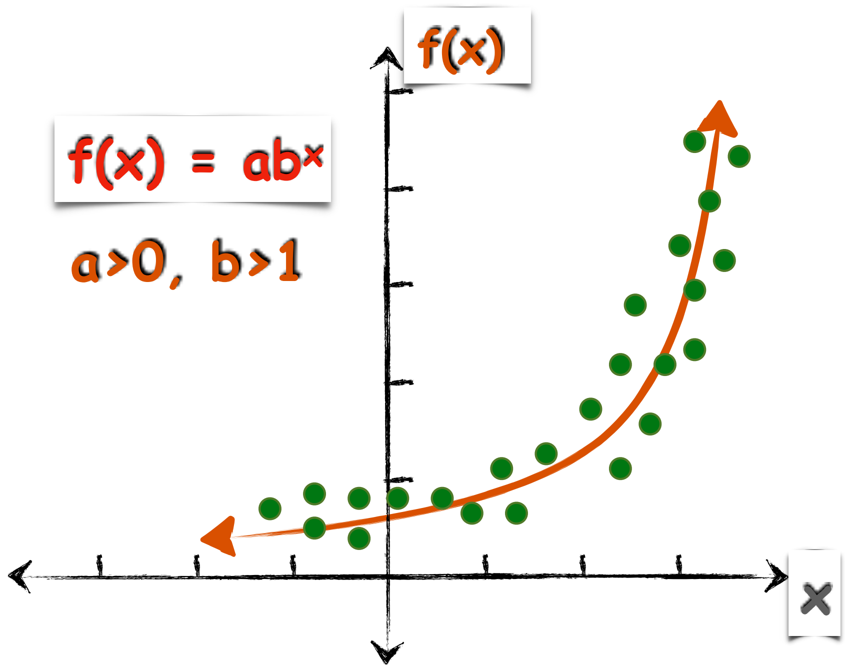
$$\ln(.75) = \frac{-.5332t}{1000000000}$$

$$\ln(.75)\frac{1000000000}{-.5332} = t = 539{,}538{,}770.5$$

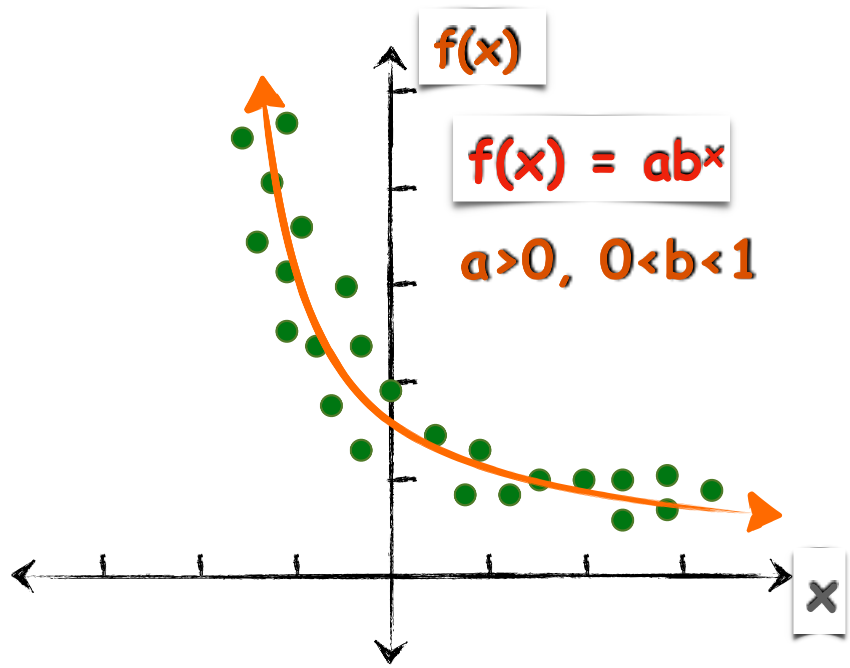
It will take about 540 million years for 20 grams of radioactive potassium to decay to a level of 15 grams.

The Art of Modeling

- Scatterplots of data give an indication of what model might be appropriate.



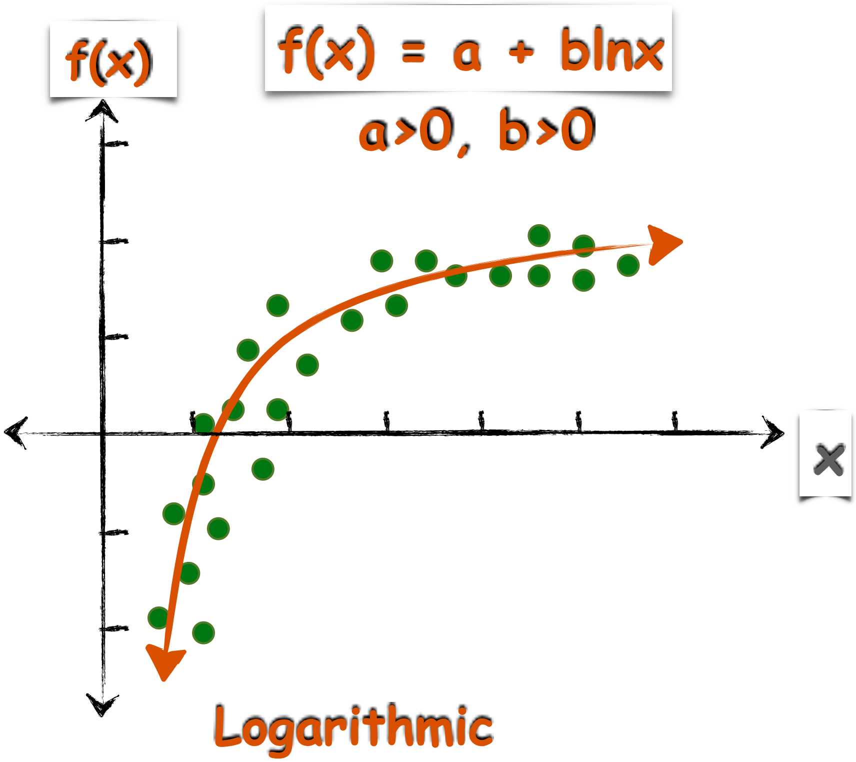
Exponential Growth

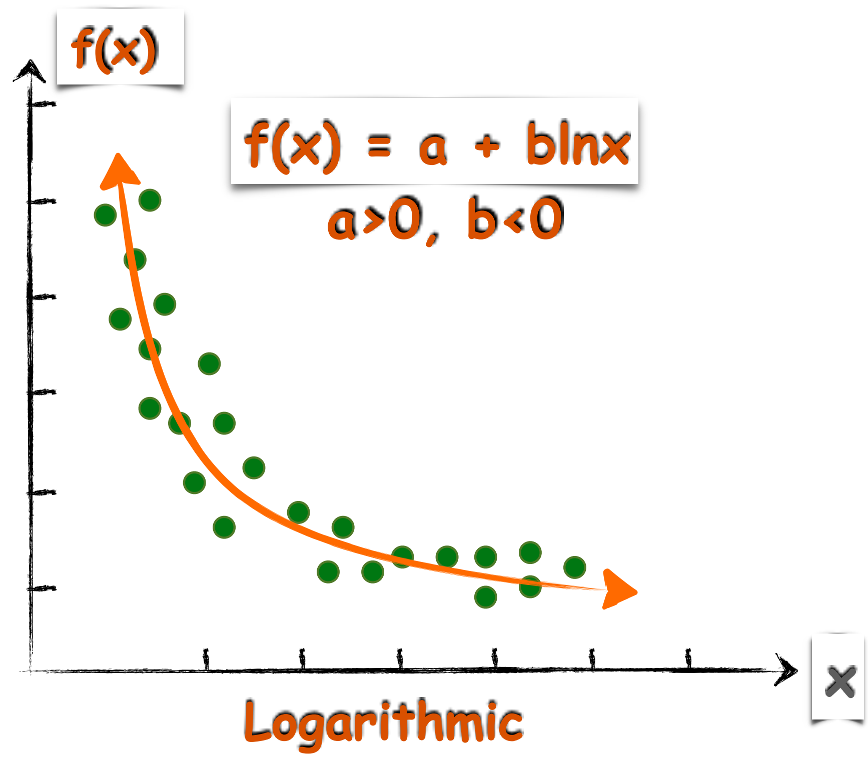


Exponential Decay

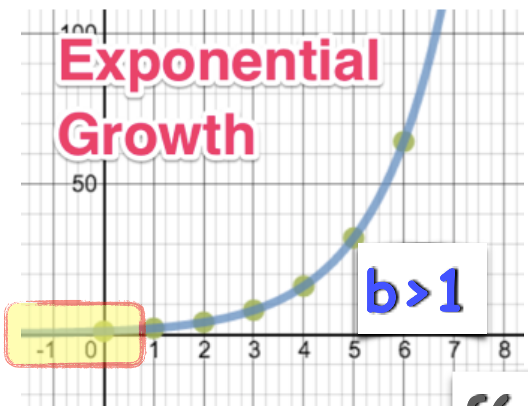
The Art of Modeling

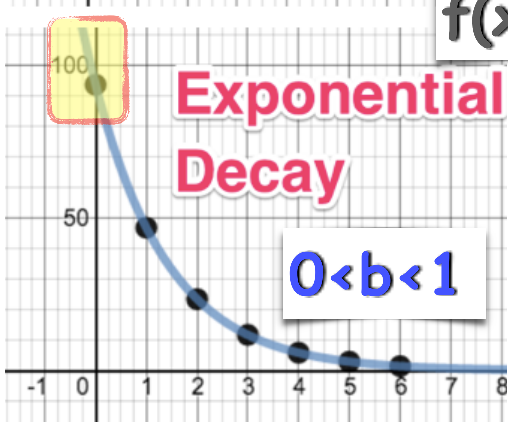
- Scatterplots of data give an indication of what model is appropriate.





All Together

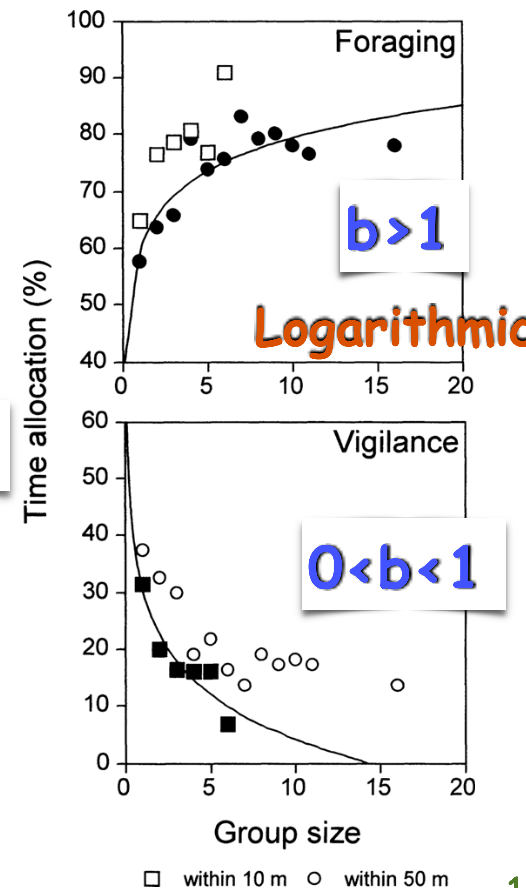




- Scatterplots of data give an indication of what model is appropriate.

$f(x)=ab^x$ $\qquad$ $f(x)=\log_b x$

- What is the dead giveaway differentiating between logarthmic and exponential?

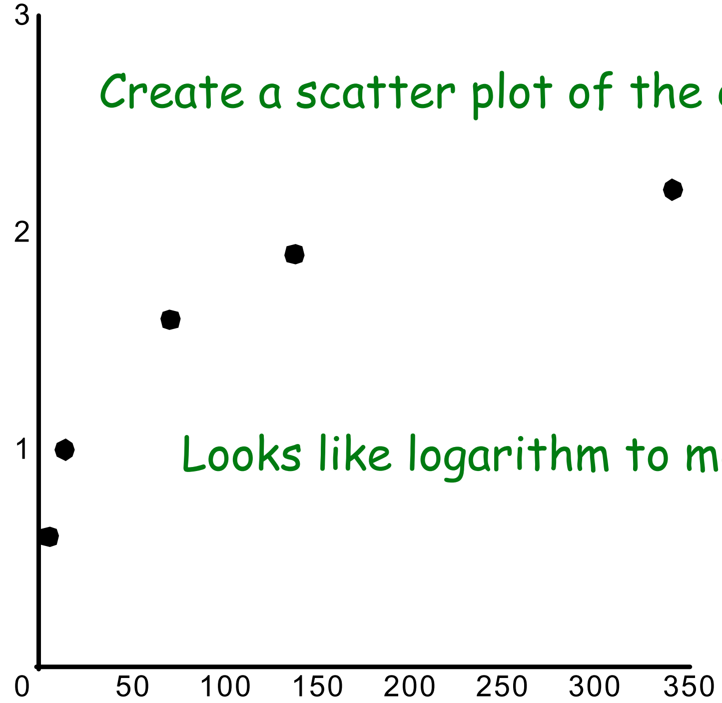


The Art of Modeling

The table shows the populations of various cities, in thousands, and the average walking speed, in feet per second, of a person living in the city.

Population (thousands)	Speed (ft/sec)
5.5	0.6
14	1
71	1.6
138	1.9
342	2.2

Create a scatter plot of the data.



Looks like logarithm to me.

The Art of Modeling

Population (thousands)	Speed (ft/sec)
5.5	0.6
14	1
71	1.6
138	1.9
342	2.2

Looks like a logarithm to me.

$\boxed{f(x) = a + b\ln x}$ $a > 0,\ b > 0$

$1.0 = a + b\ln 14$

$2.2 = a + b\ln 342$

$1.2 = b\ln 342 - b\ln 14$

$1.2 = b(\ln 342 - \ln 14)$

$$1.2 = b\ln\frac{342}{14}$$

$$b = \frac{1.2}{\ln\frac{342}{14}} \approx 0.3755$$

$1 = a + 0.3755\ln 14$

$a \approx .009$

$f(x) = .009 + 0.3755\ln x$

Try a few values.

TI-84

- Now let's get real and let the TI-84 do the grunt work.
  - Enter the data into two lists in the calculator

    STAT ➢ EDIT ⋎ 1:Edit ENTER

  - Enter population into List 1 (L_1) and the speed into List 2 (L_2)

Pop	S
5.5	0.6
14	1
71	1.6
138	1.9
342	2.2

$f(x) = .001 + 0.3755\ln x$

  - First we create the scatterplot.

    STAT PLOT
    2nd y= Enter ON ⋎ 1:Plot1 Enter ⋎ Type: ➢ ⋎ XList: 2nd 1 (L_1) 2nd 2 (L_2) Zoom 9

  - To see the point values… trace

TI-84

- Now we can let the calculator find the model.

$f(x) = .001 + 0.3755\ln x$

STAT ➢ CALC ∨ 9:LnReg

XList: L_1
YList: L_2
FreqList:
StoreRegEq: Y_1
Calculate

ENTER

VARS ➢ Y-VARS ➢ 1:Function ∨ 1:Y_1 **ENTER** (1)

y= a + blnx

a=–.0444127476
b=.3883648413

$f(x) = -.044 + 0.3884\ln x$

- Just for snicks and giggles, let us try a power model.

STAT ➢ CALC ∨ A:PwrReg

XList: L_1
YList: L_2
FreqList:
StoreRegEq: Y_2
Calculate

ENTER

VARS ➢ Y-VARS ➢ 1:Function ∨ 1:Y_1 **ENTER** (2)

y= a*x^b

a=.3974657321
b=.3099210898

$f(x) = .3975x^{.3099}$

Expressing $y=ab^x$ in Base e

We can change an exponential equation with base b into an exponential equation in base e.

Start with **$f(x)=ab^x$**.

Then recall that $e^{\ln b} = b$

$$y = ab^x \qquad y = a\left(e^{\ln b}\right)^x \qquad y = ae^{x\ln b}$$

$y = ab^x$ can be written $y = ae^{x\ln b}$

The reason one might want to to this is to convert a rate of change per time (say 1 year) to a continuous rate of change (k).

Expressing $y=ab^x$ in Base e

Rewrite $y = 4(7.8)^x$ in terms of base e. Express the answer in terms of a natural logarithm and then round to three decimal places.

Start with $y = ab^x$.

$y = 4(7.8)^x$.

$e^{\ln 7.8} = 7.8$

$y = 4(7.8)^x$ can be written $y = 4e^{x \ln 7.8}$

$y \approx 4e^{2.054x}$

- Let us say we have deposited \$1500 at an annual interest rate of 7% compounded quarterly. The resulting equation is $A = 1500\left(1 + \frac{.07}{4}\right)^{4t}$

- So the quarterly interest rate is .0175. What would be the rate for continuous compounding? $A = Pe^{kt}$ In this case k would be the continuous rate.

$$\boxed{A = 1500(1.0175)^{4t}} \qquad A = 1500e^{\ln(1.0175)^{4t}} \qquad A = 1500e^{4t\ln(1.0175)} \qquad A = 1500e^{4\ln(1.0175)t}$$

$$\boxed{A = 1500e^{.0694t}}$$

- So the continuous interest rate is would be approximately 6.94%