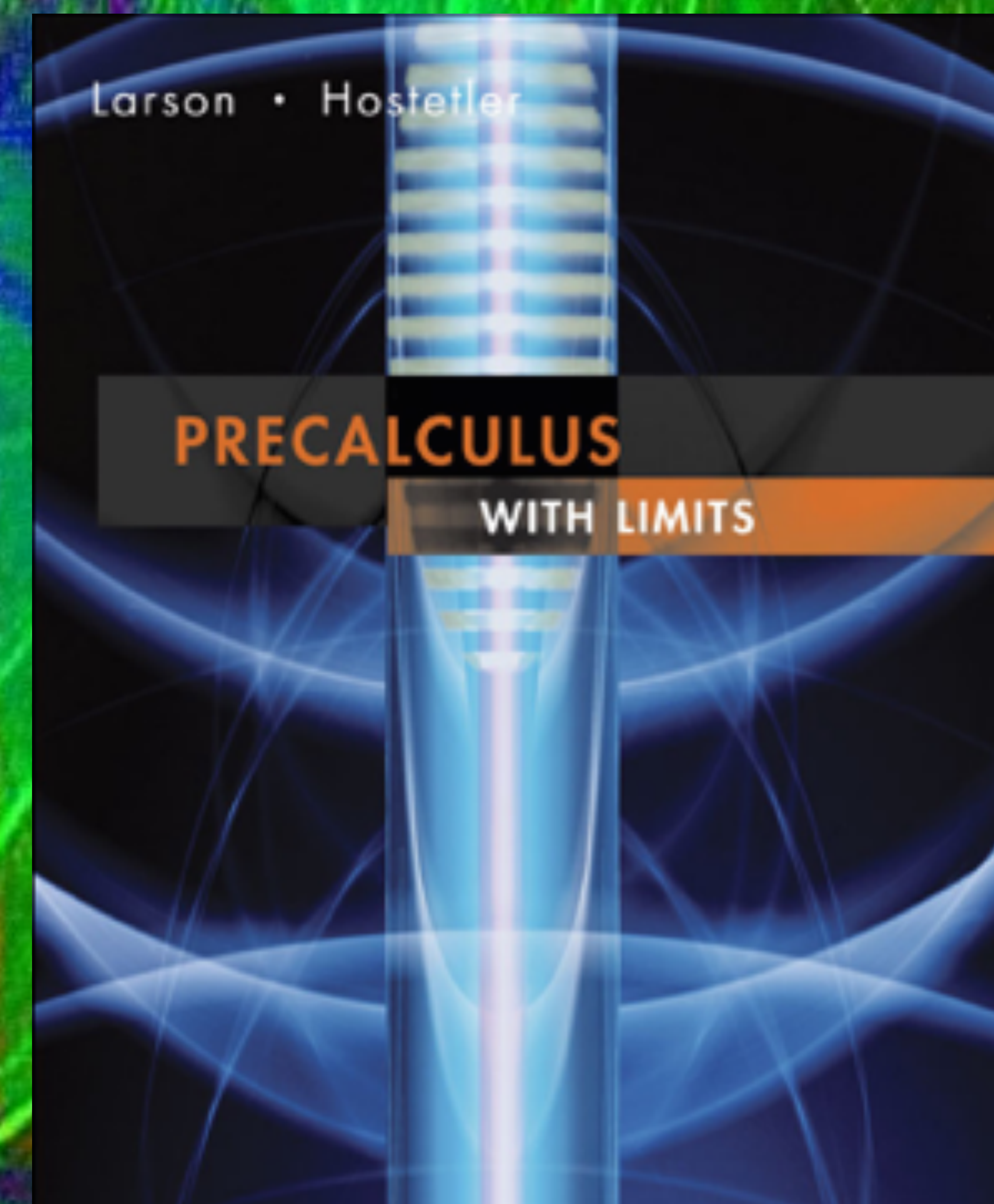
Chapter 10

10.3 Ellipse



Chapter 10

10.3 Homework

p750 7, 9, 21, 23, 37, 41, 43, 47, 51, 57

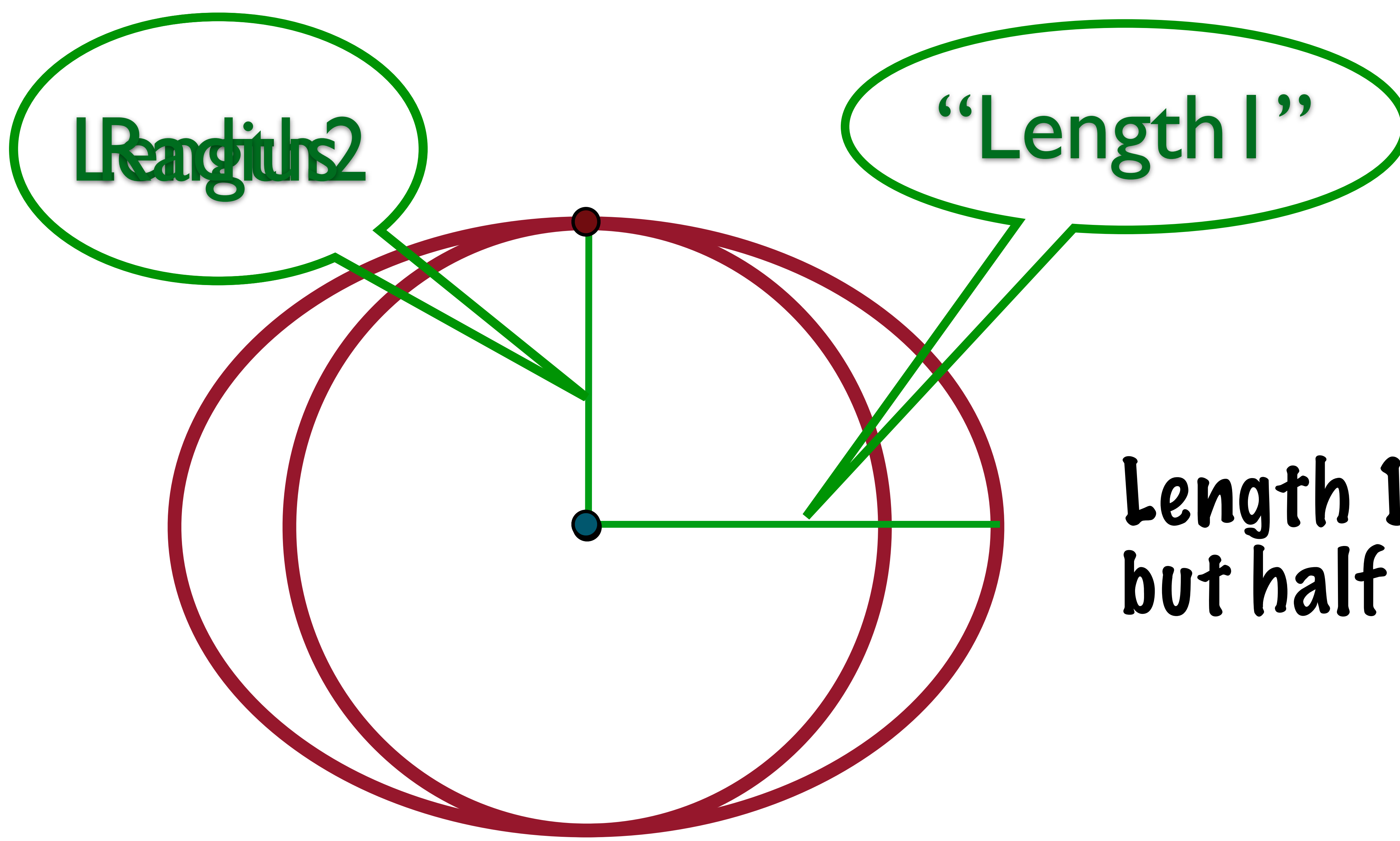
Chapter 10

10.3 Objectives

Graph ellipses centered at the origin.
Write equations of ellipses in standard form.
Graph ellipses not centered at the origin.
Solve applied problems involving ellipses.

Ellipse

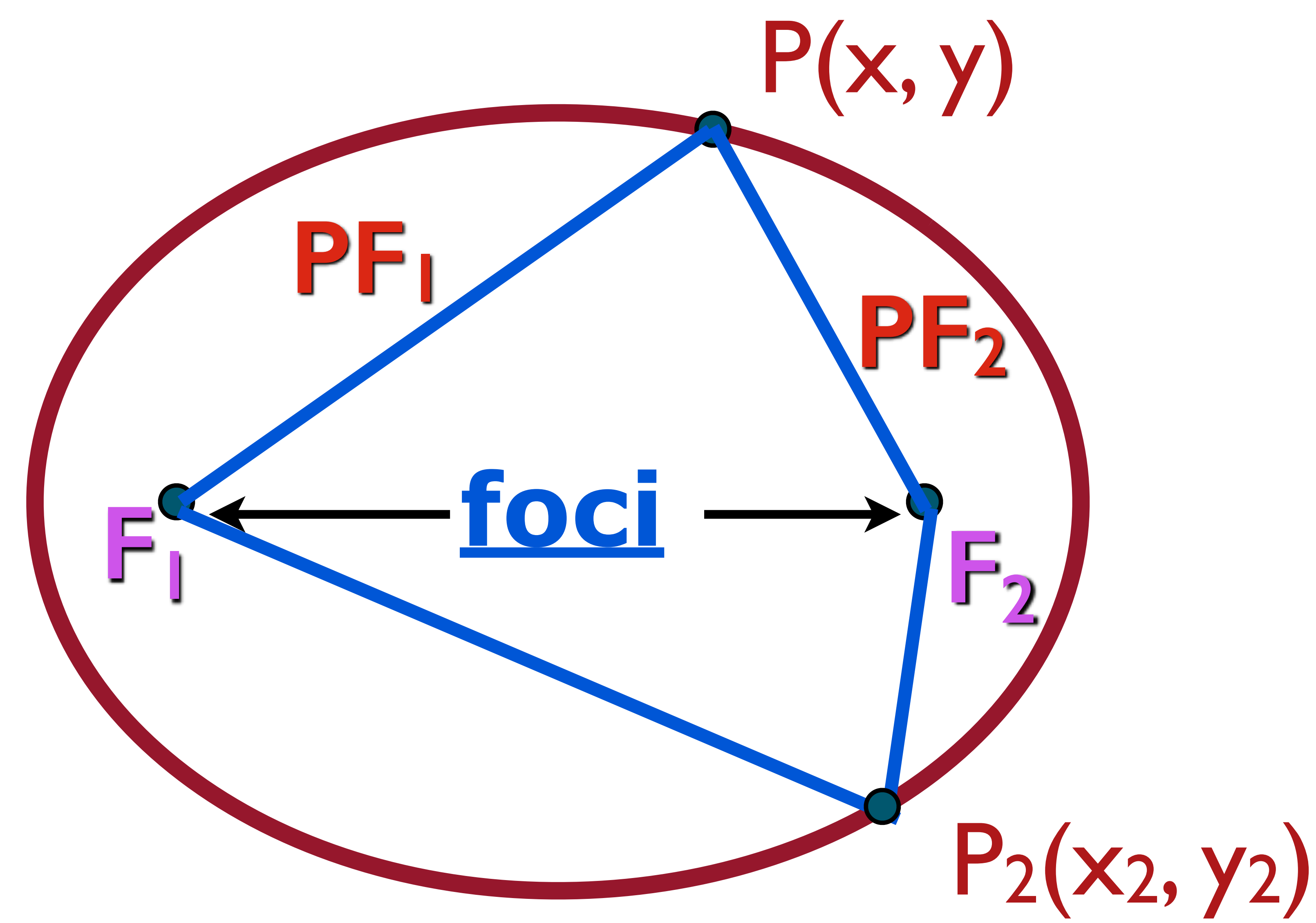
If you pulled the center of a circle apart into two points, it would stretch the circle into an ellipse.

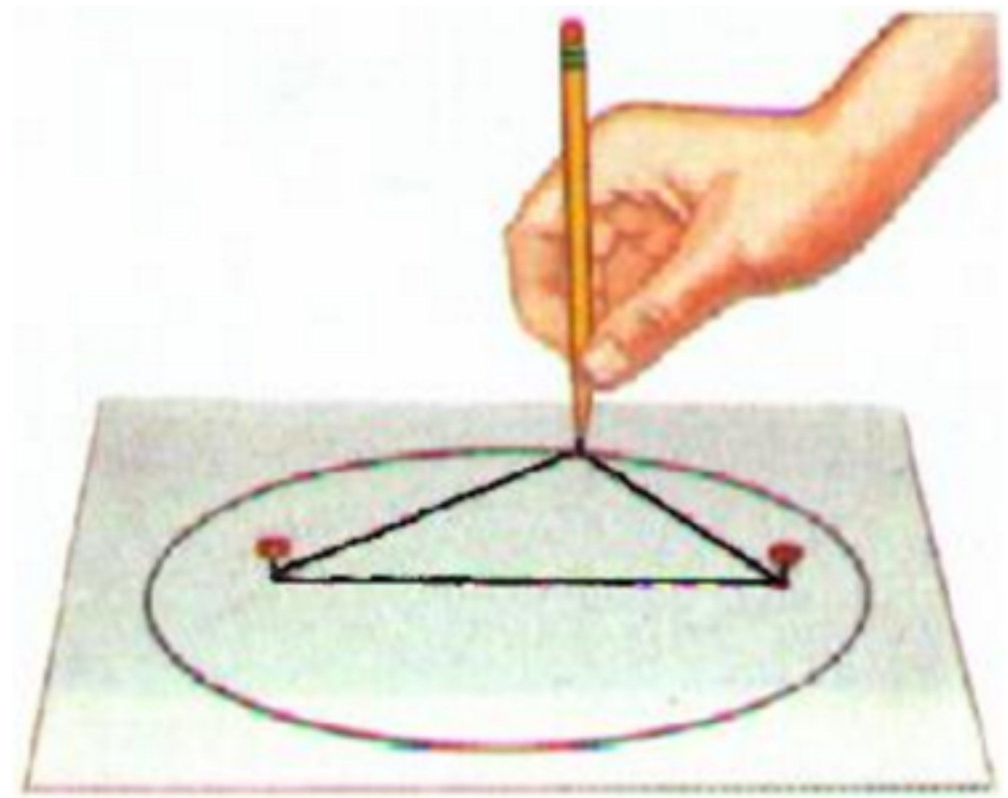


Length 1 and Length 2 are not radii, but half the axes of the ellipse.

Definition of an Ellipse

An ellipse is the set of all points, P, in a plane the sum of whose distances ($PF_1 + PF_2$) from two fixed points, F_1 and F_2, is constant. These two fixed points are called the foci (plural of focus). The midpoint of the segment connecting the foci is the **center** of the ellipse.



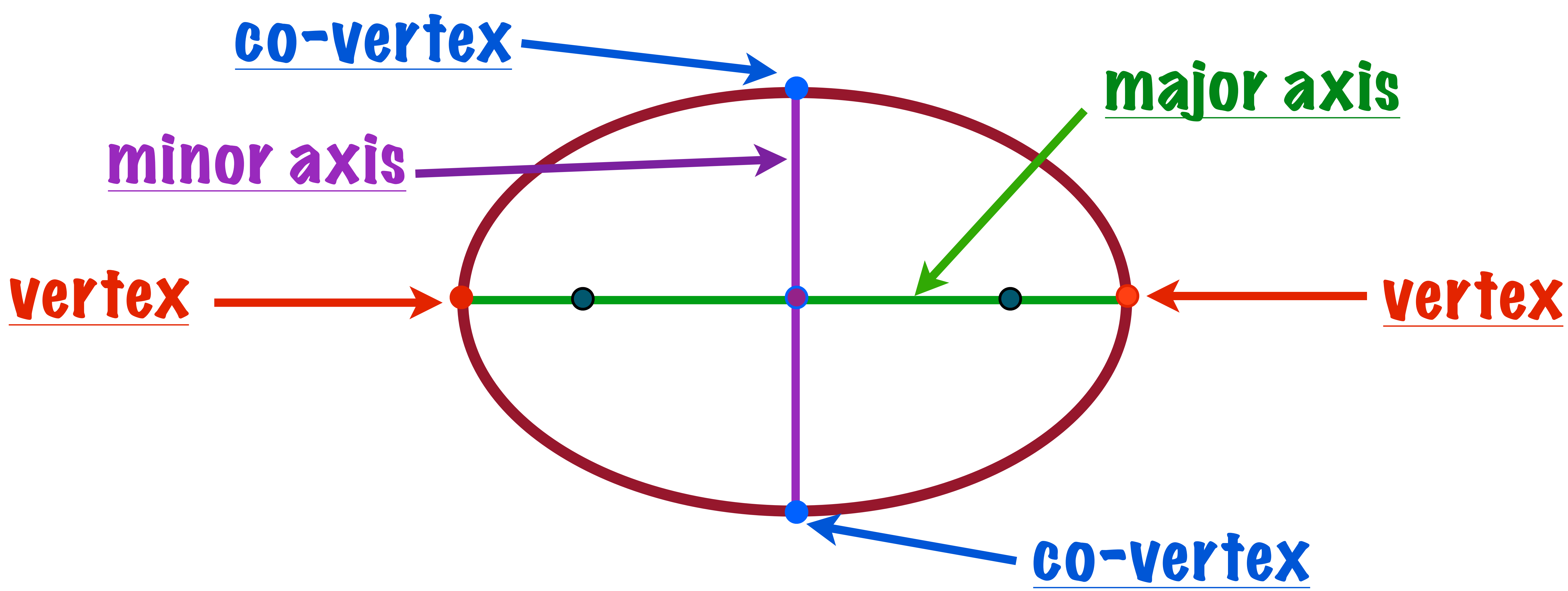


This distance d can be represented by the length of a piece of string connecting two pushpins located at the foci.

Ellipse

Instead of a single radius, an ellipse has two axes. The long axis is the major axis and passes through both foci. The endpoints of the major axis are the vertices of the ellipse.

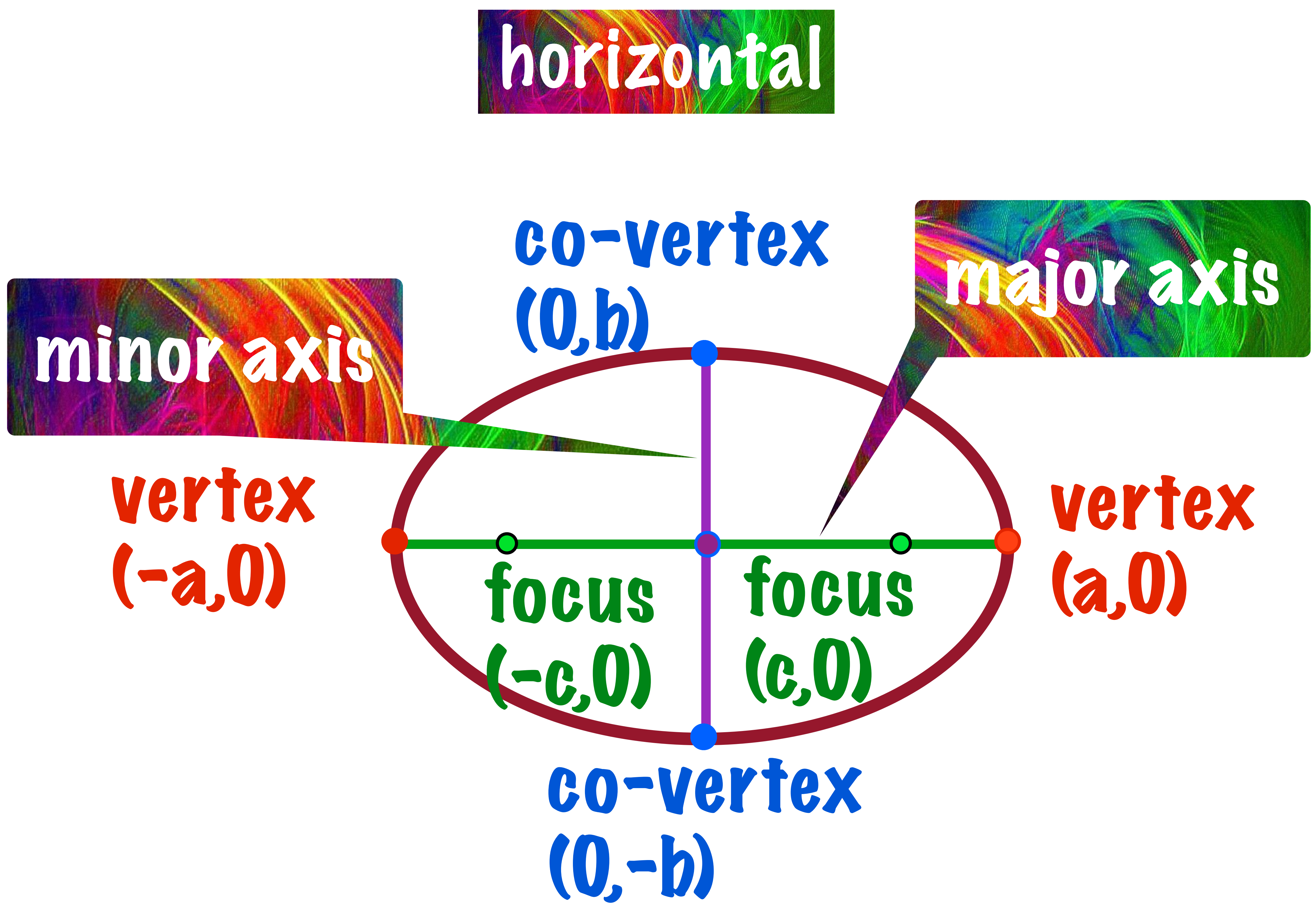
The short axis is the minor axis. The endpoints of the minor axis are the co-vertices of the ellipse.

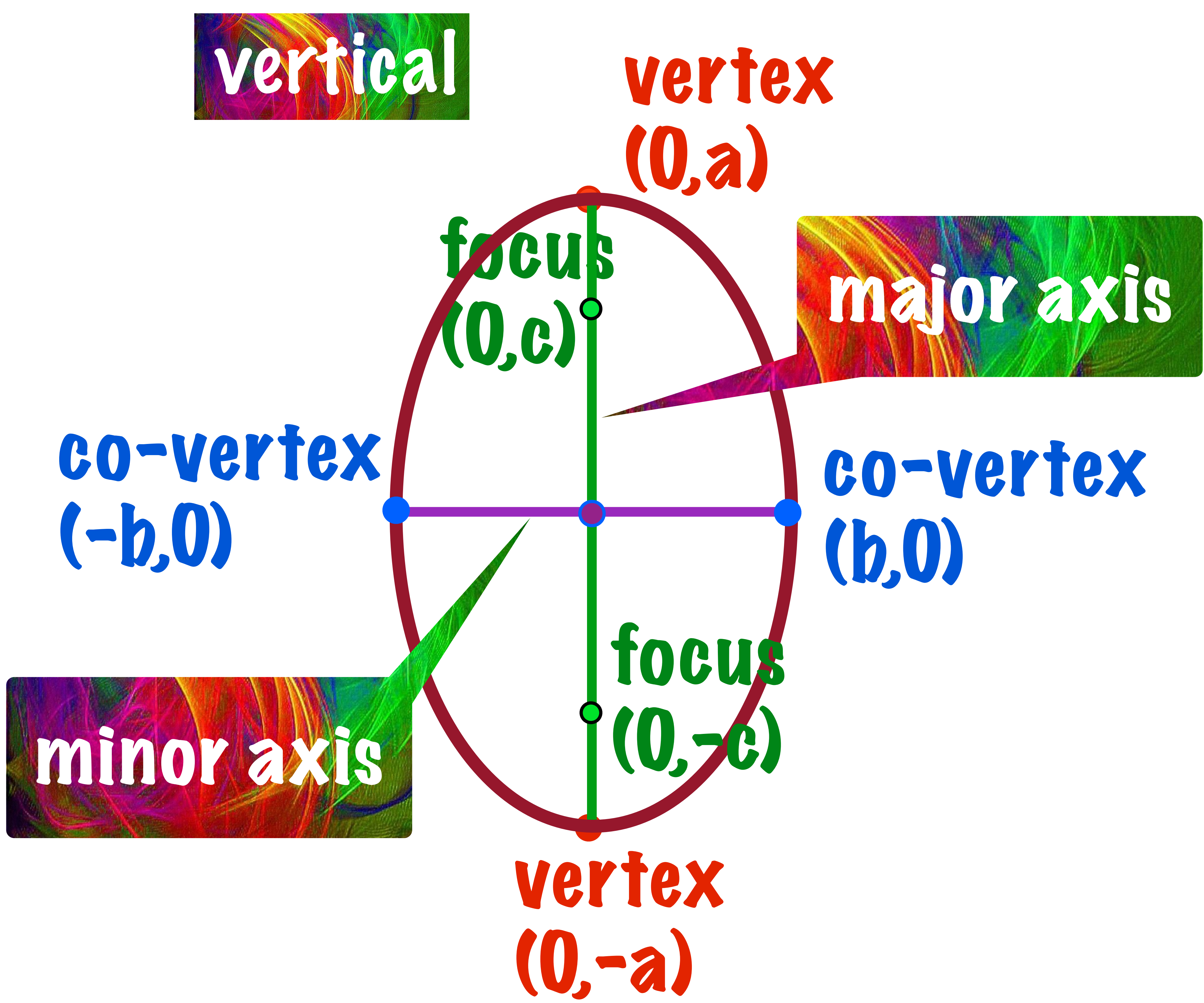


The major axis and minor axis are perpendicular and intersect at the center of the ellipse.

Horizontal and Vertical Stretch

The standard form of an ellipse centered at (0, 0) depends on whether the major axis is horizontal or vertical.

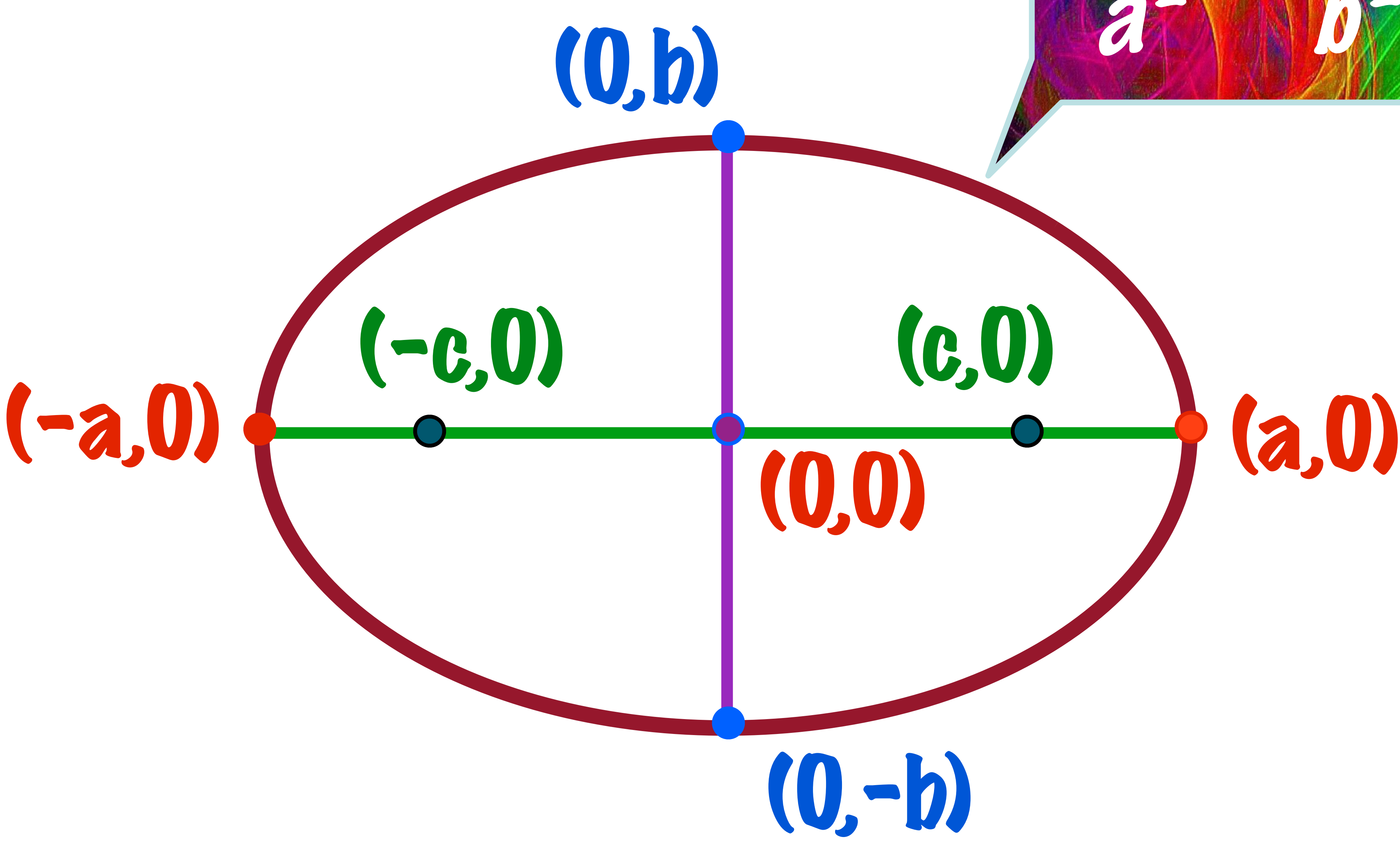




Standard Form of Equation for an Ellipse

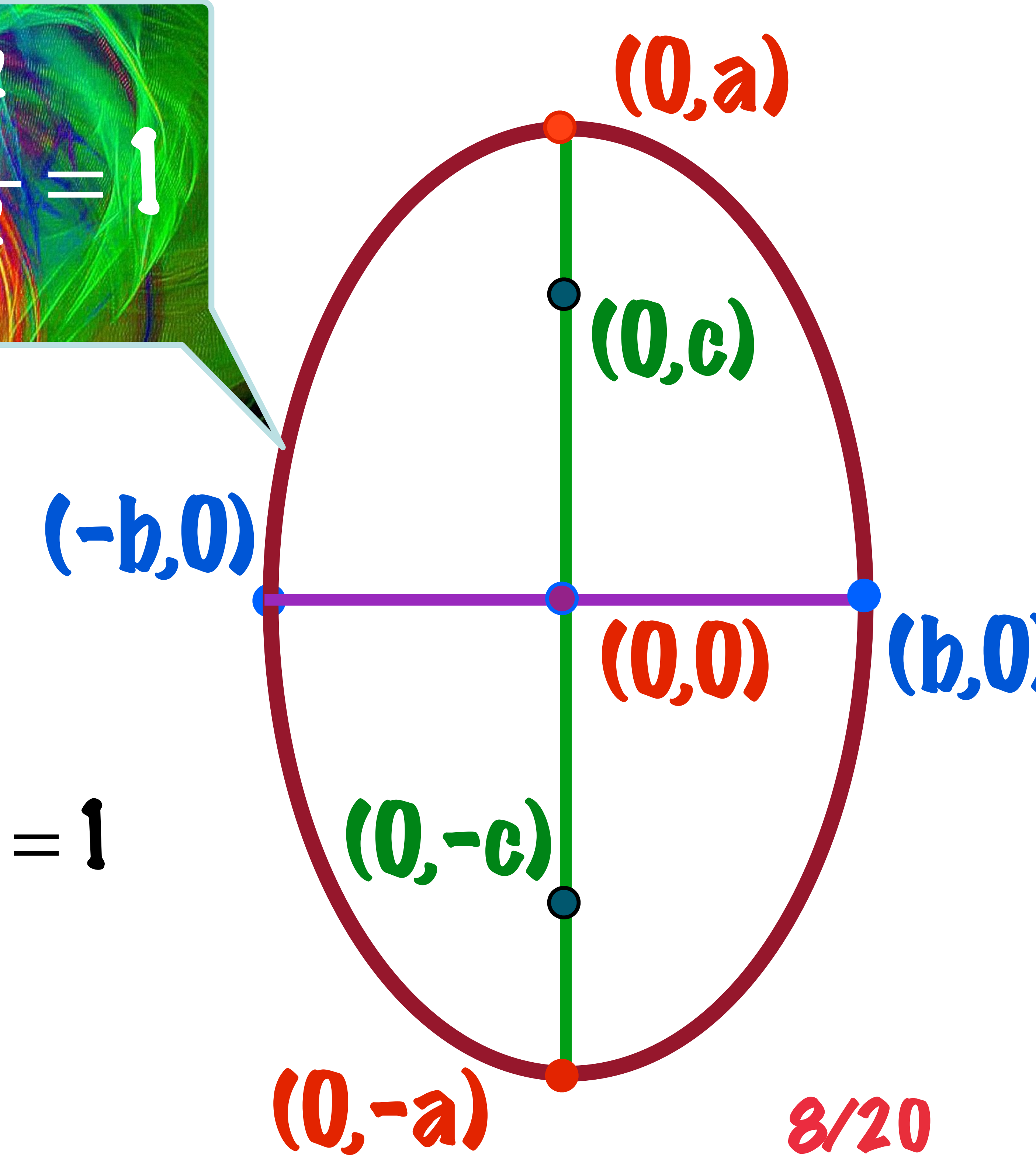
The standard form of the equation of an ellipse with center at the origin, and major and minor axes of lengths 2a and 2b (where a and b are positive, and $a^2 > b^2$) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{or} \quad \frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$



Note: when a = b, the equation can be written $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$

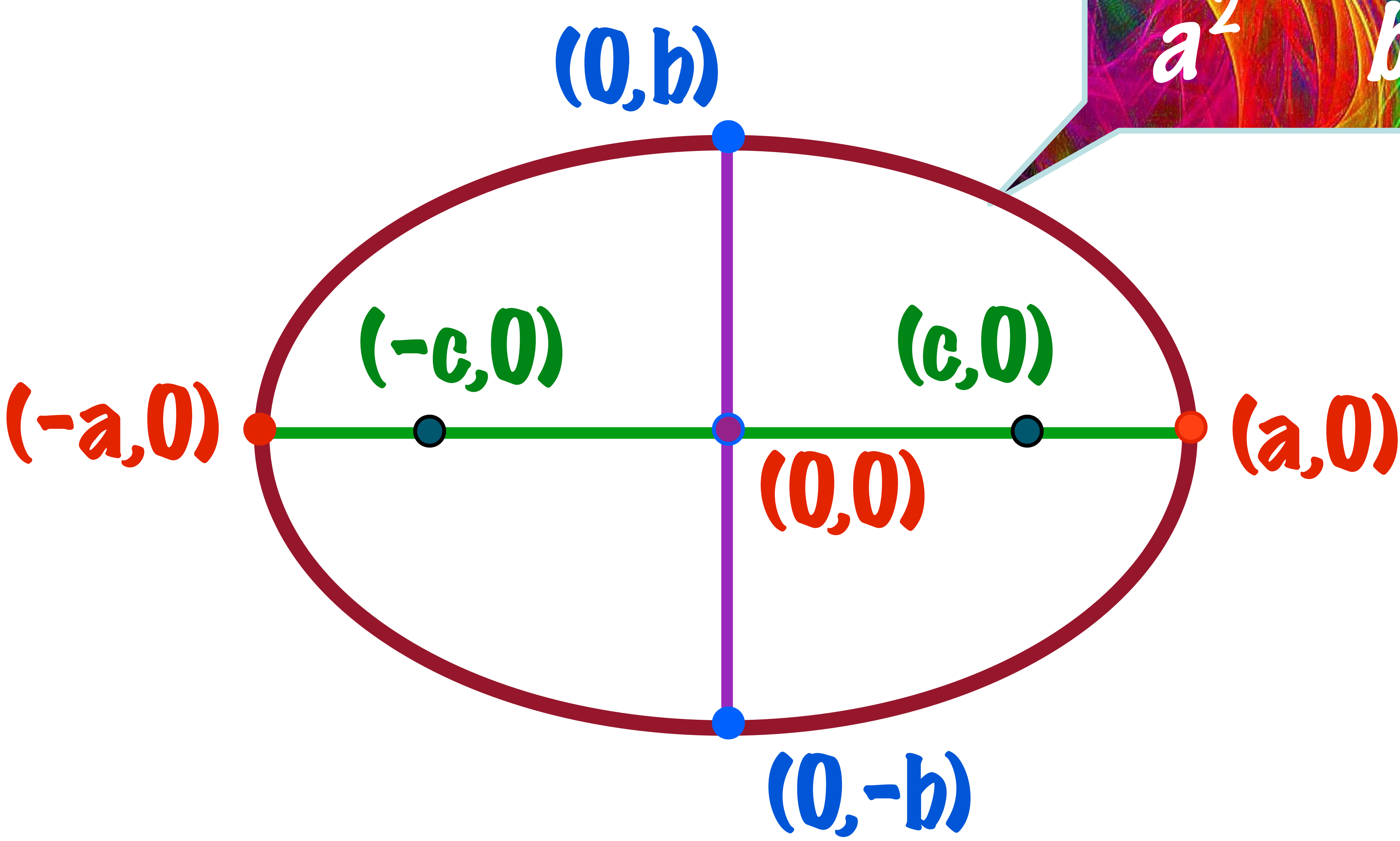
or $x^2+y^2=a^2$



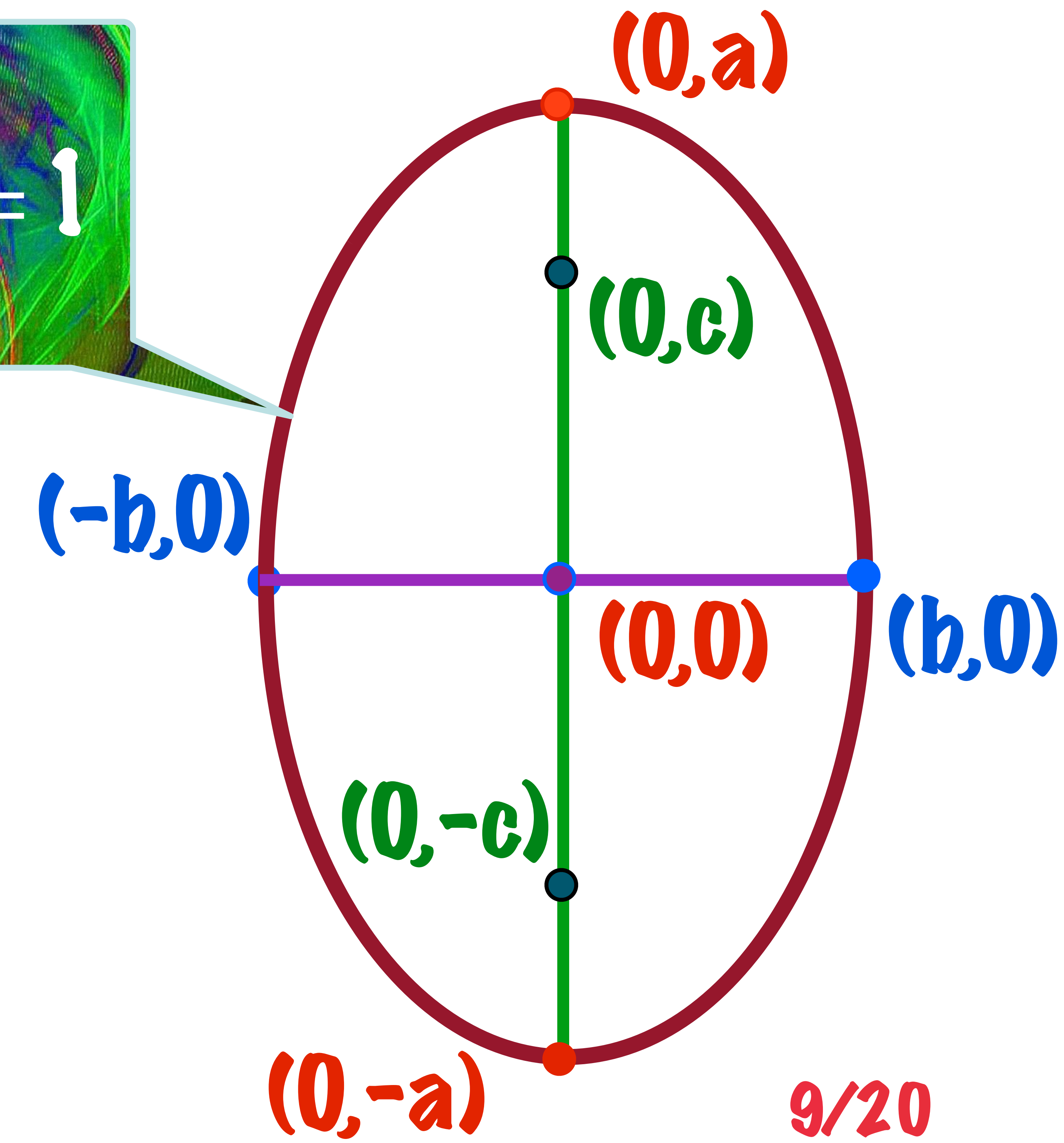
Standard Form for an Ellipse

The vertices are on the major axis, a units from the center. The foci are on the major axis, c units from the center. For both equations, $b^2 = a^2 - c^2$. Equivalently, $c^2 = a^2 - b^2$. Easily verified using the distance formula.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

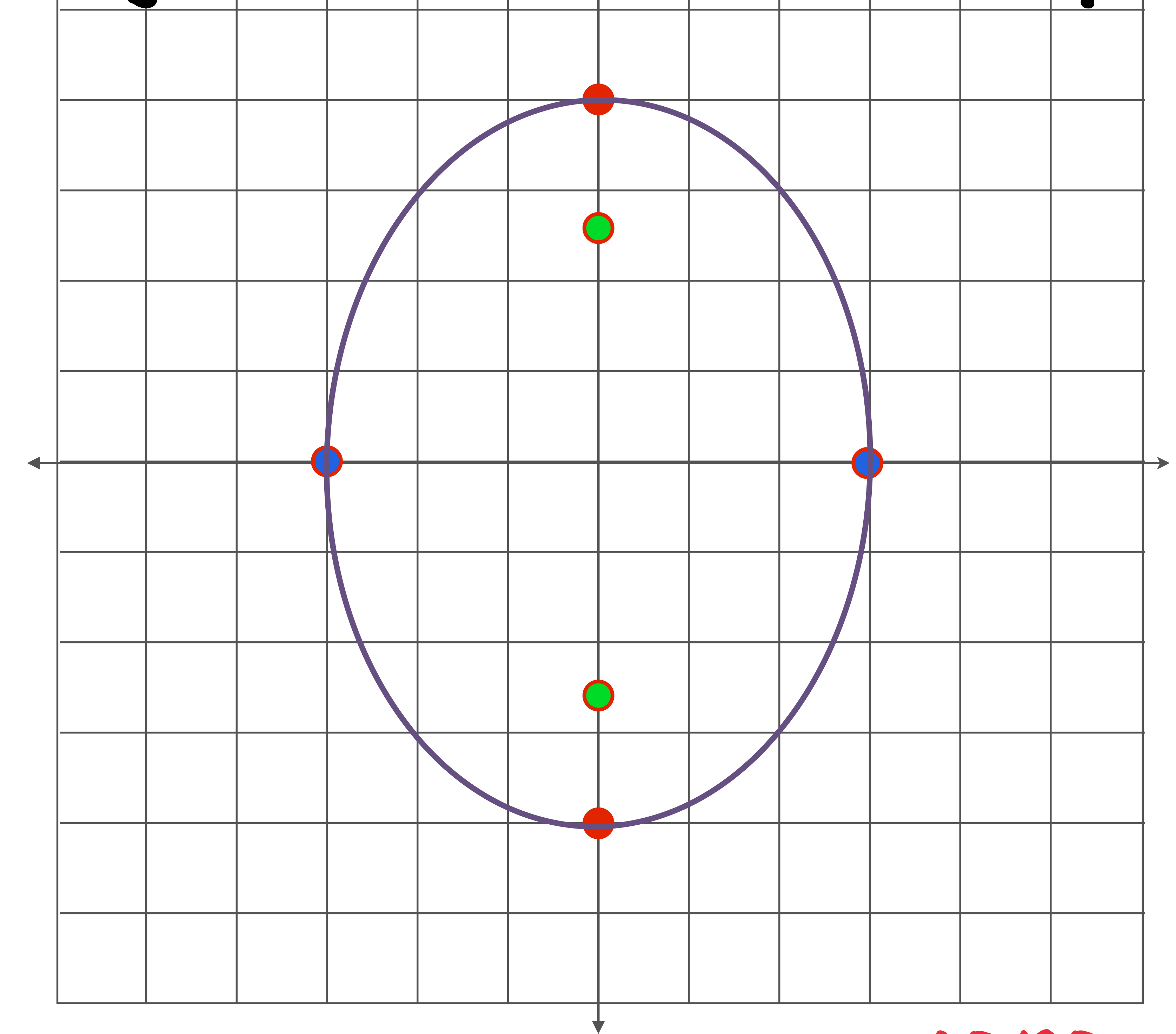


Example

Graph and locate the foci: $16x^2 + 9y^2 = 144$

Re-write in standard form: $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

Larger Denominator under y^2



Major Axis: Vertical $a^2 = 16$, $b^2 = 9$

$c^2 = a^2 - b^2 = 16 - 9 = 7$ $c = \pm\sqrt{7}$

Foci: $(0, \sqrt{7})$ and $(0, -\sqrt{7})$

Vertices: $a^2 = 16$, $a = \pm 4$ $(0,4)$ and $(0,-4)$

Co-vertices: $b^2 = 9$, $b = \pm 3$ $(3,0)$ and $(-3,0)$

Example

Find the standard form of the equation of an ellipse with foci at (-2, 0) and (2, 0) and vertices at (-3, 0) and (3, 0).

Foci: $(0,2)$ *and* $(0,-2)$

Vertices: (-3,0) and (3,0)

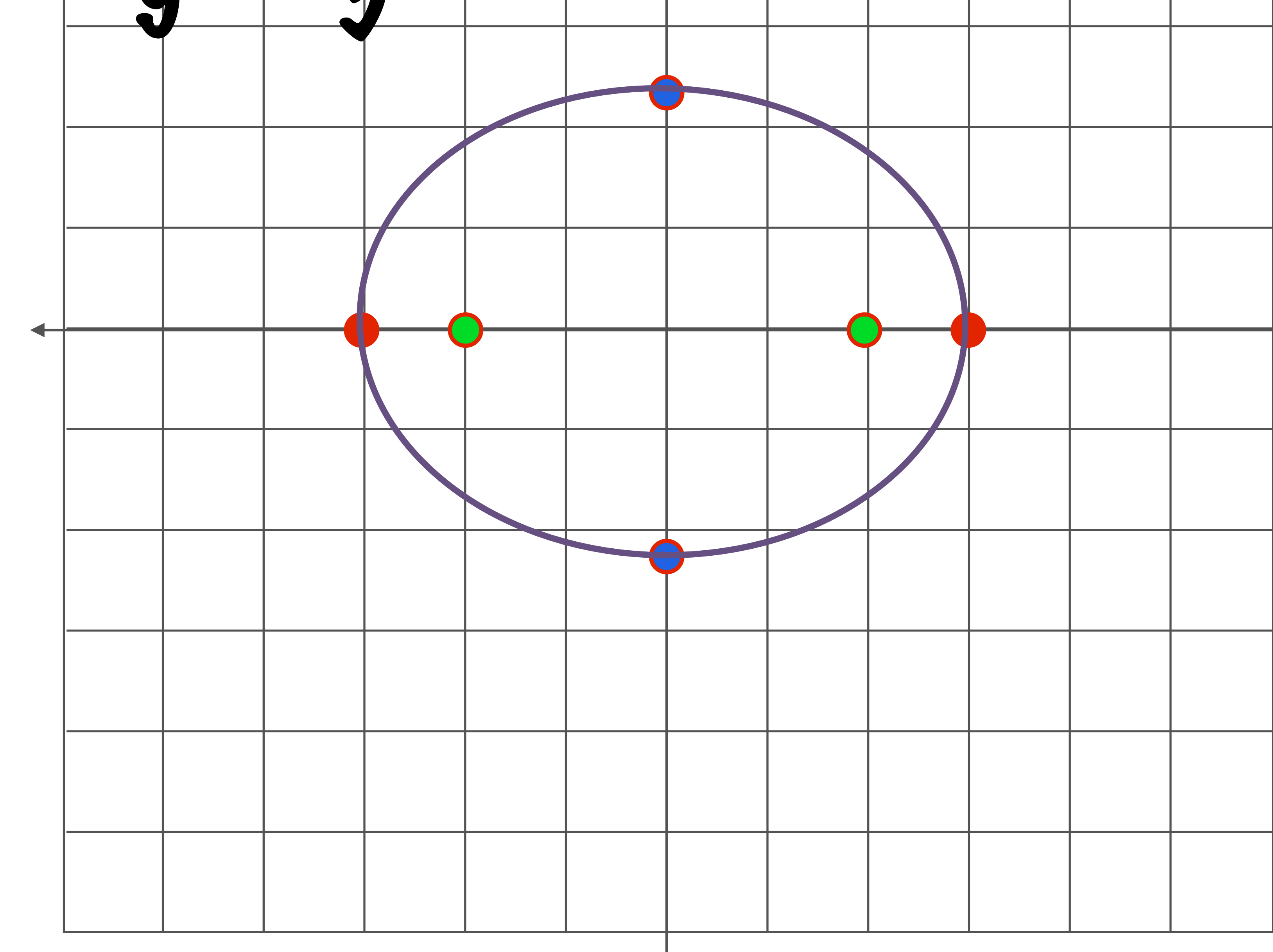
Major Axis: Horizontal   a = ±3, c = ±2

Center: (0,0)

$b^2 = a^2 - c^2 = 9 - 4 = 5$ $b = \pm\sqrt{5}$

Co-vertices: $(0,\sqrt{5})$ *and* $(0,-\sqrt{5})$

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$



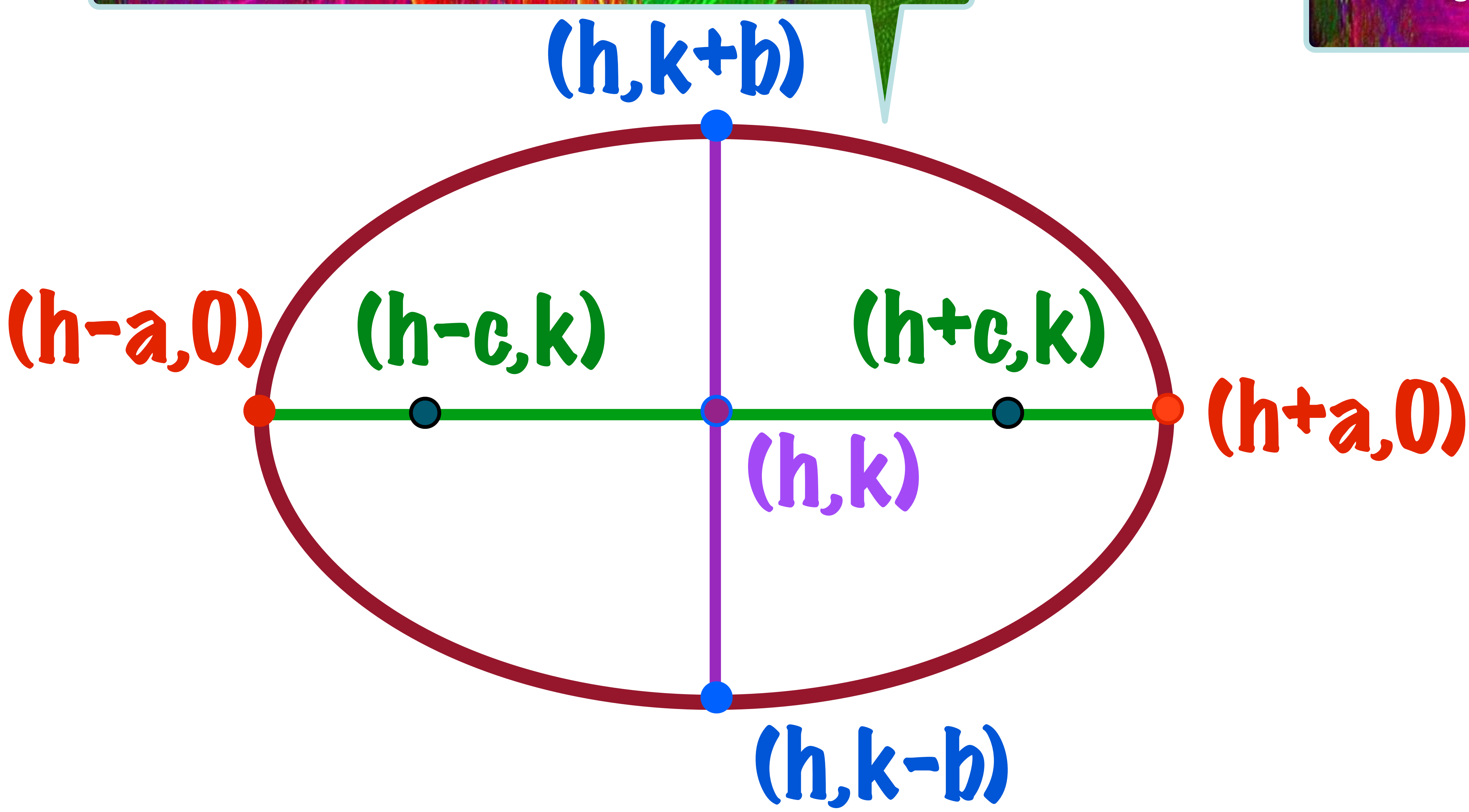
Standard Form with Center (h,k)

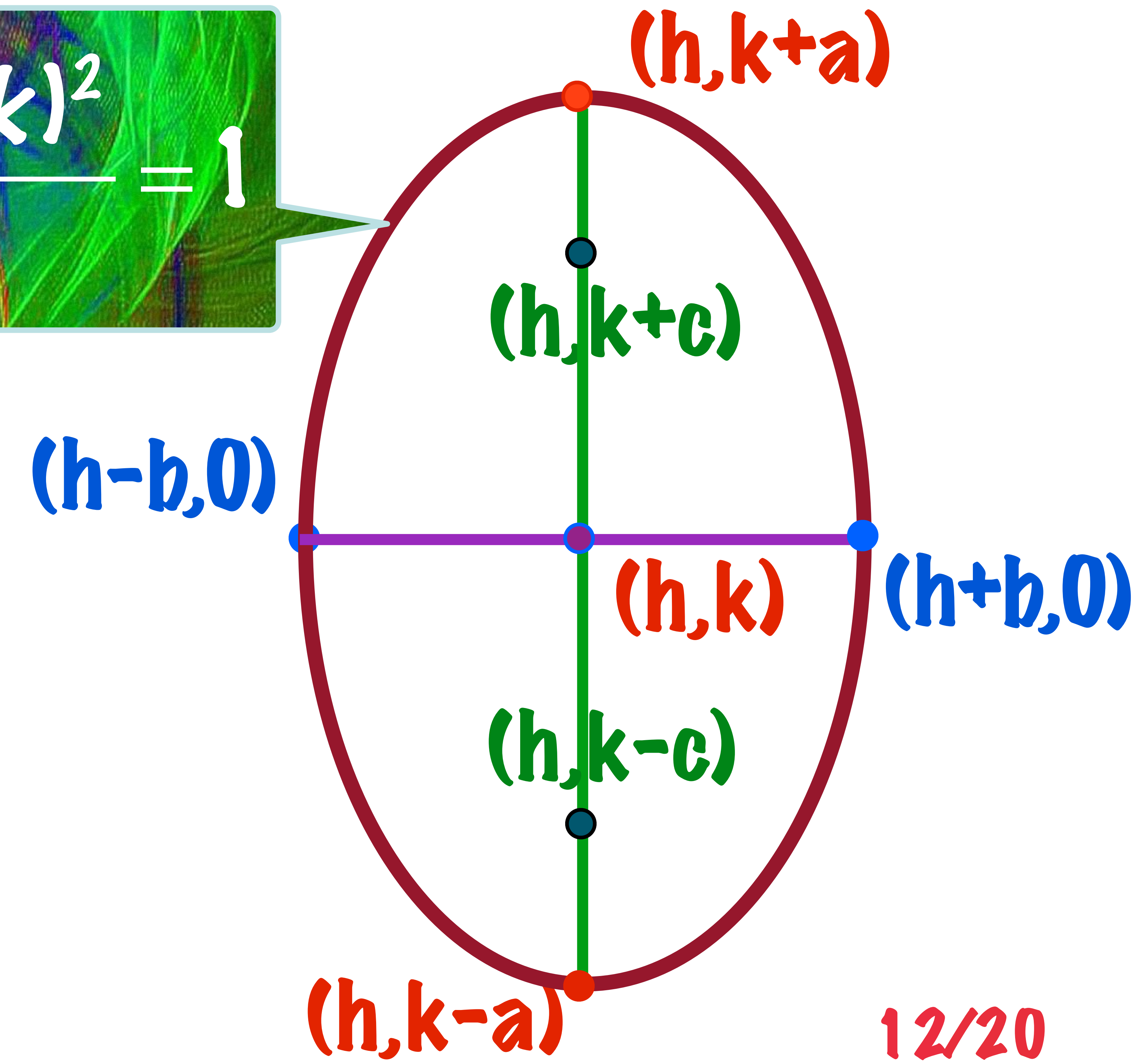
Once again, let us get off the center at the origin. The standard form of the equation of an ellipse with center at (h,k), and major and minor axes of lengths 2a and 2b (where a and b are positive, and $a^2 > b^2$) is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

or

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$





Example

Graph $\frac{(x+1)^2}{9}+\frac{(y-2)^2}{4}=1$

Major Axis: Horizontal $a = \pm 3$, $b = \pm 2$

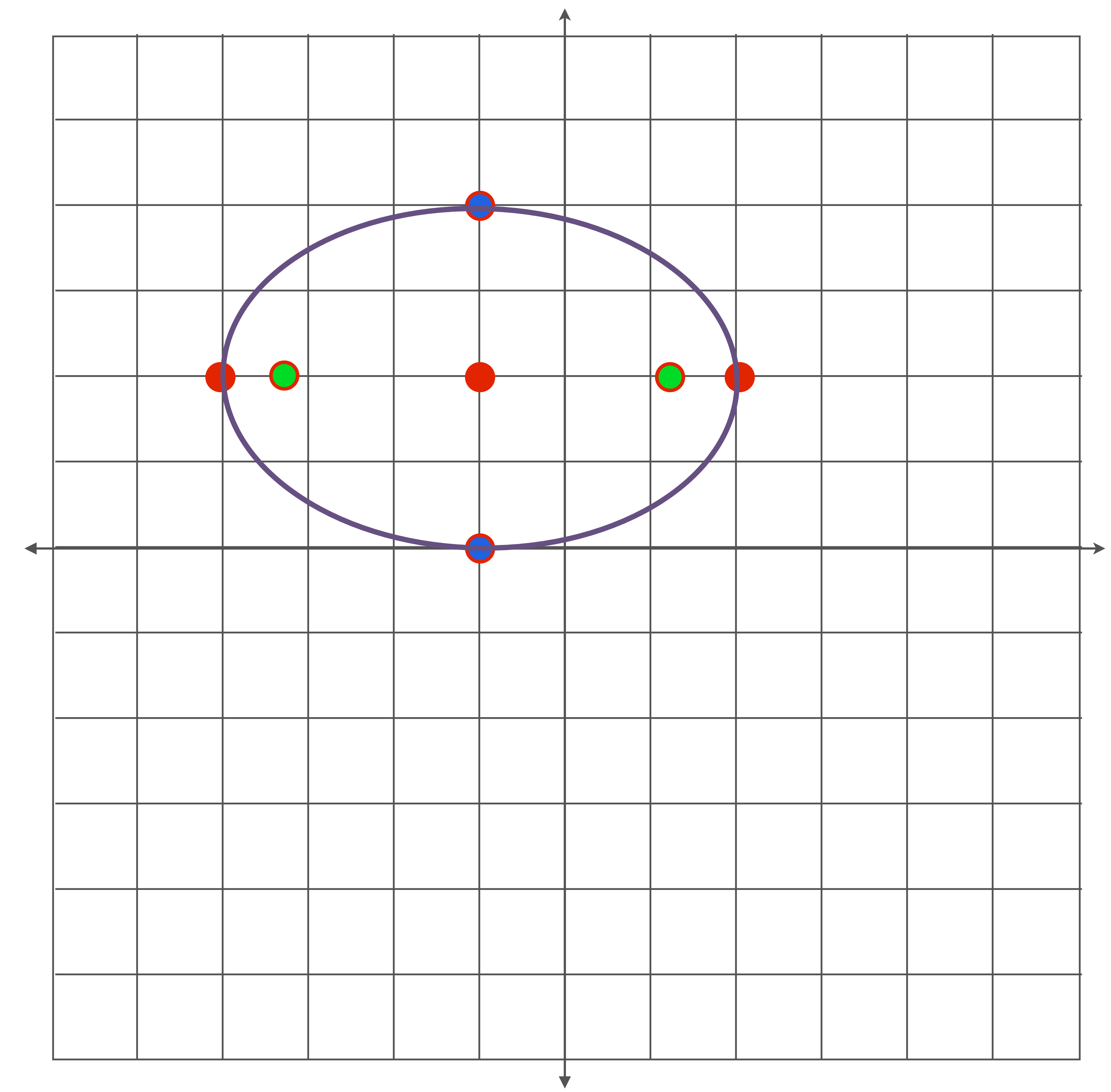
Center: (−1,2)

Vertices: (-1-3,2) and (-1+3,2) (-4,2) and (2,2)

Co-vertices: (-1,2-2) and (-1,2+2) (-1,0) and (-1,4)

$c^2 = a^2 - b^2 = 9 - 4 = 5$ $c = \pm\sqrt{5}$

Foci: $(-1+\sqrt{5},0)$ *and* $(-1-\sqrt{5},0)$



Example

Graph the ellipse $16x^2 - 160x + 25y^2 + 100y + 100 = 0$.

1. Change to standard form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

Complete the square for both x and y.

$$(16x^2 - 160x + \square) + (25y^2 + 100y + \square) = -100 + \square + \square$$

$$16(x^2 - 10x + \square) + 25(y^2 + 4y + \square) = -100 + \square + \square$$

$$16(x^2 - 10x + 25) + 25(y^2 + 4y + 4) = -100 + 400 + 100$$

$$16(x-5)^2 + 25(y+2)^2 = 400$$

$$\frac{(x-5)^2}{25} + \frac{(y+2)^2}{16} = 1$$

Example

Graph the ellipse $16x^2 - 160x + 25y^2 + 100y + 100 = 0.$ $\frac{(x-5)^2}{25} + \frac{(y+2)^2}{16} = 1$

The center is (5, -2)

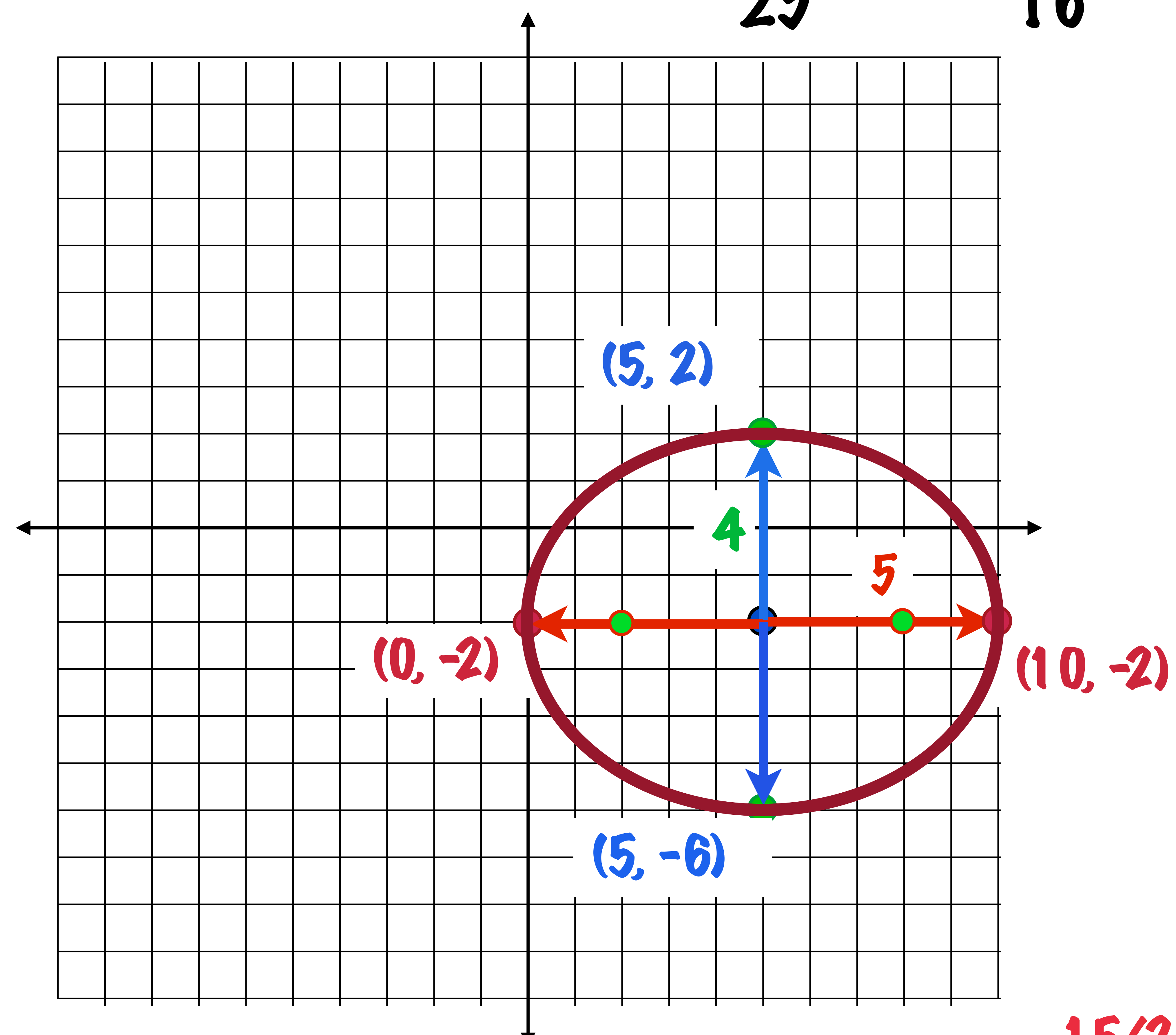
The vertices are (5 ± 5, -2) or (10, -2) and (0, -2)

The co-vertices are (5, -2 ± 4), or (5, 2) and (5, -6).

$c^2 = a^2 - b^2 = 25 - 16 = 9 \quad c = \pm 3$

Foci: $(5+3,-2)$ *and* $(5-3,-2)$

$(8,-2)$ *and* $(2,-2)$



Example

Graph the ellipse $24x^2 + 48x + 10y^2 - 60y - 126 = 0.$

1. Change to standard form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

Complete the square for both x and y.

$$(24x^2 + 48x + \square) + (10y^2 - 60y + \square) = 126 + \square + \square$$

$$24(x^2 + 2x + \square) + 10(y^2 - 6y + \square) = 126 + \square + \square$$

$$24(x^2 + 2x + 1) + 10(y^2 - 6y + 9) = 126 + 24 + 90$$

$$24(x + 1)^2 + 10(y - 3)^2 = 240$$

$$\frac{(x+1)^2}{10} + \frac{(y-3)^2}{24} = 1$$

Example

Graph the ellipse $24x^2 + 48x + 10y^2 - 60y - 126 = 0$.

$$\frac{(x+1)^2}{10} + \frac{(y-3)^2}{24} = 1$$

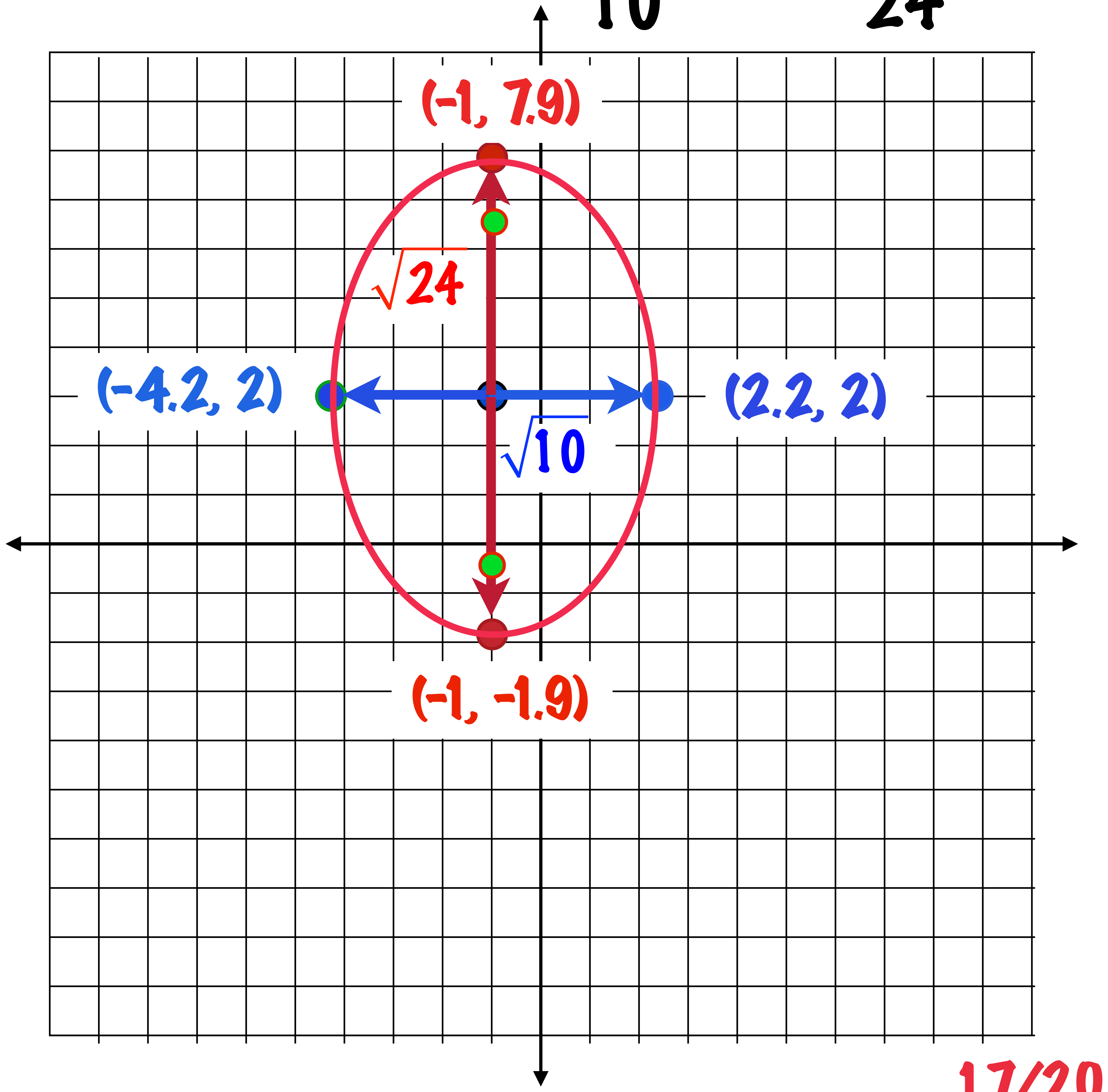
The center is $(-1,3)$

The vertices are $(-1, 3 \pm \sqrt{24})$

The co-vertices are $(-1 \pm \sqrt{10}, 3)$

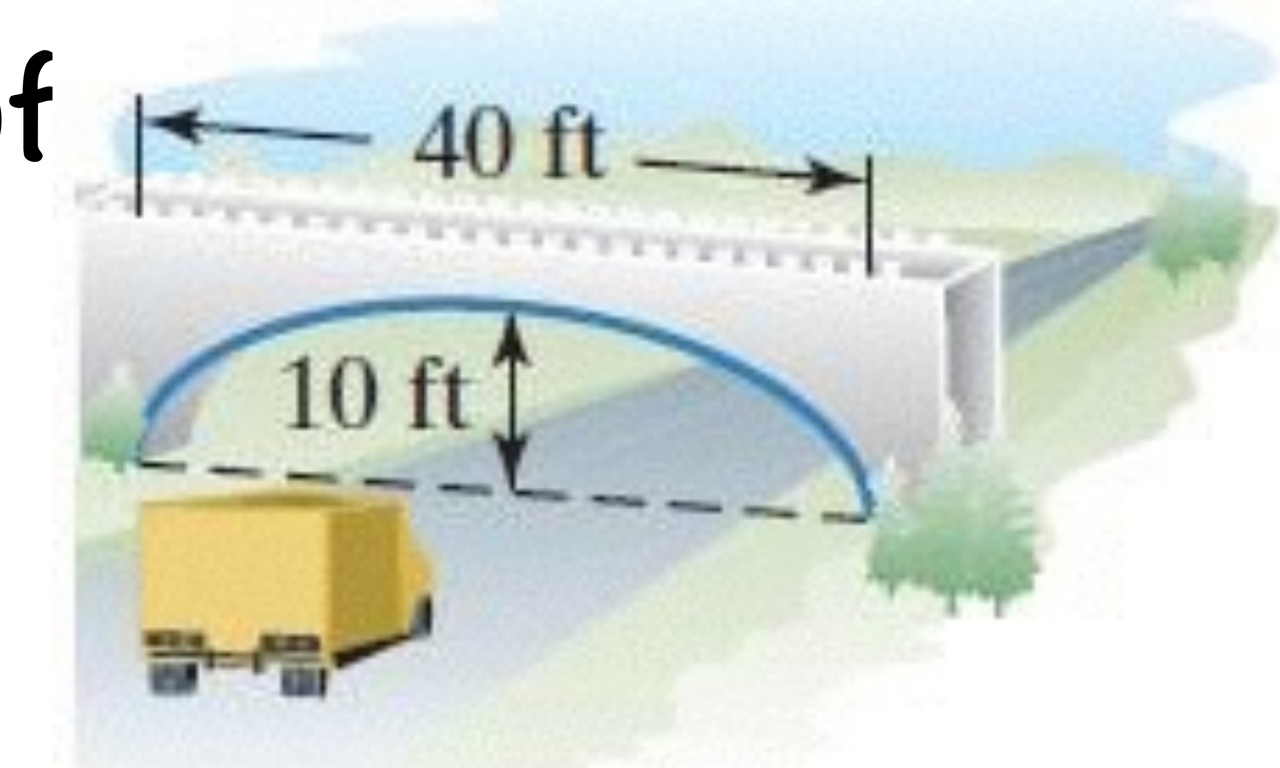
$c^2 = a^2 - b^2 = 24 - 10 = 14 \quad c = \pm\sqrt{14}$

Foci: $(-1, 3 \pm \sqrt{14})$

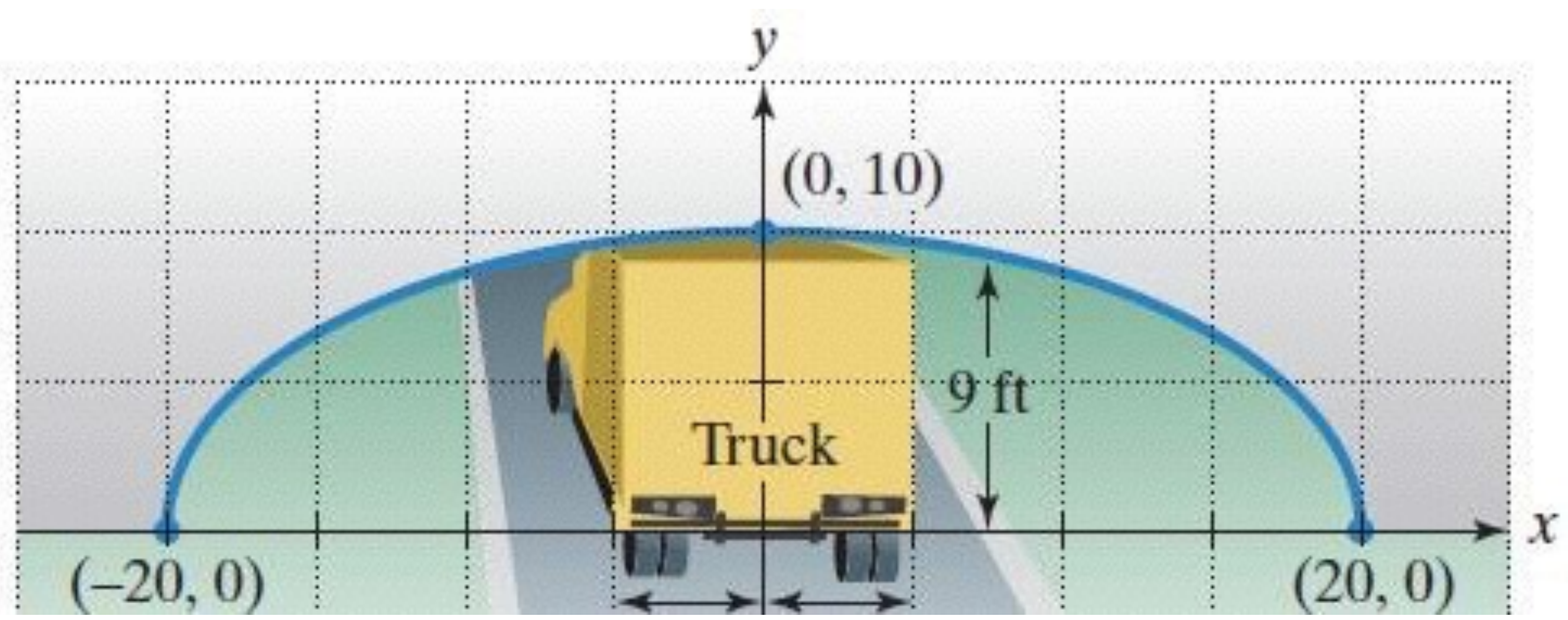


Application

A semielliptical archway over a one-way road has a height of 10 feet and a width of 40 feet.



Will a truck that is 12 feet wide and has a height of 9 feet clear the opening of the archway?



We construct a coordinate system with the x-axis on the ground and the origin below the center of the archway under the truck.

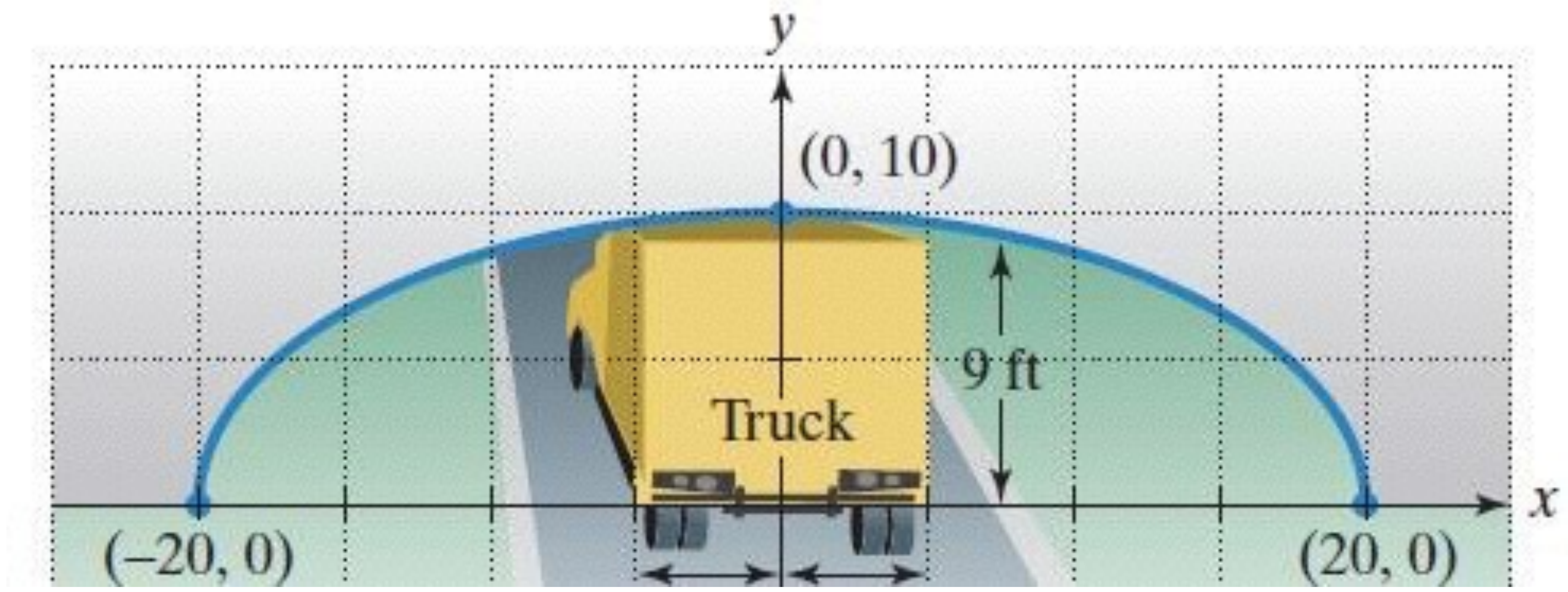
With the center of an ellipse established at the origin, we can express the equation of the archway as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \frac{x^2}{20^2}+\frac{y^2}{10^2}=1 \qquad \frac{x^2}{400}+\frac{y^2}{100}=1$$

Application

A semielliptical archway over a one-way road has a height of 10 feet and a width of 40 feet.

$$\frac{x^2}{400}+\frac{y^2}{100}=1$$



The edge of the 12-foot-wide truck corresponds to x = 6. We find the height of the archway 6 feet from the center by substituting 6 for x and solving for y.

$$\frac{6^2}{400}+\frac{y^2}{100}=1 \qquad 36+4y^2=400 \qquad y^2=91 \qquad y=\pm\sqrt{91} \qquad y\approx 9.54$$

The edge of the 12-foot-wide truck will clear the overpass by just under 6" as long as the driver is dead center of the archway.

Eccentricity

Ellipses come in wide and narrow versions. A measure of the ratio of length to width is known as the **eccentricity** of the ellipse.

A numeric measure of the eccentricity of an ellipse is $e = c/a$, where $0 < e < 1$.

An eccentricity of near 1 indicates a very narrow ellipse. (c approaches a.)

An eccentricity of near 0 indicates an ellipse very close to being a circle. ($c \rightarrow 0$.)