# Conics – Chapter 10

## 10.4 Hyperbola

# Conics – Chapter 10

## 10.4 Homework

pg 760 7, 9, 13, 15, 21, 23, 25, 27, 31, 35

# Objectives:

Locate a hyperbola's vertices and foci.

Write equations of hyperbolas in standard form.

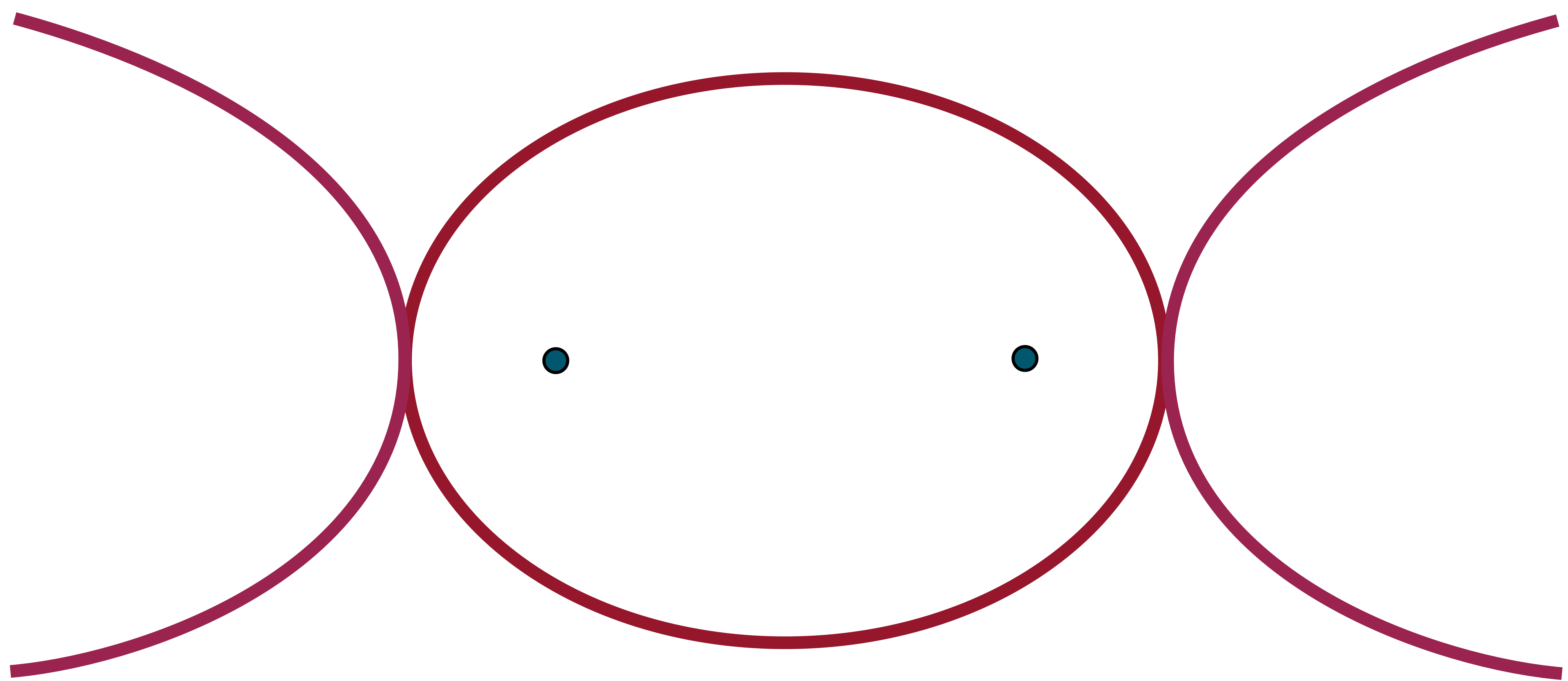
Graph hyperbolas centered at the origin.

Graph hyperbolas not centered at the origin.

Solve applied problems involving hyperbolas.

# Hyperbola

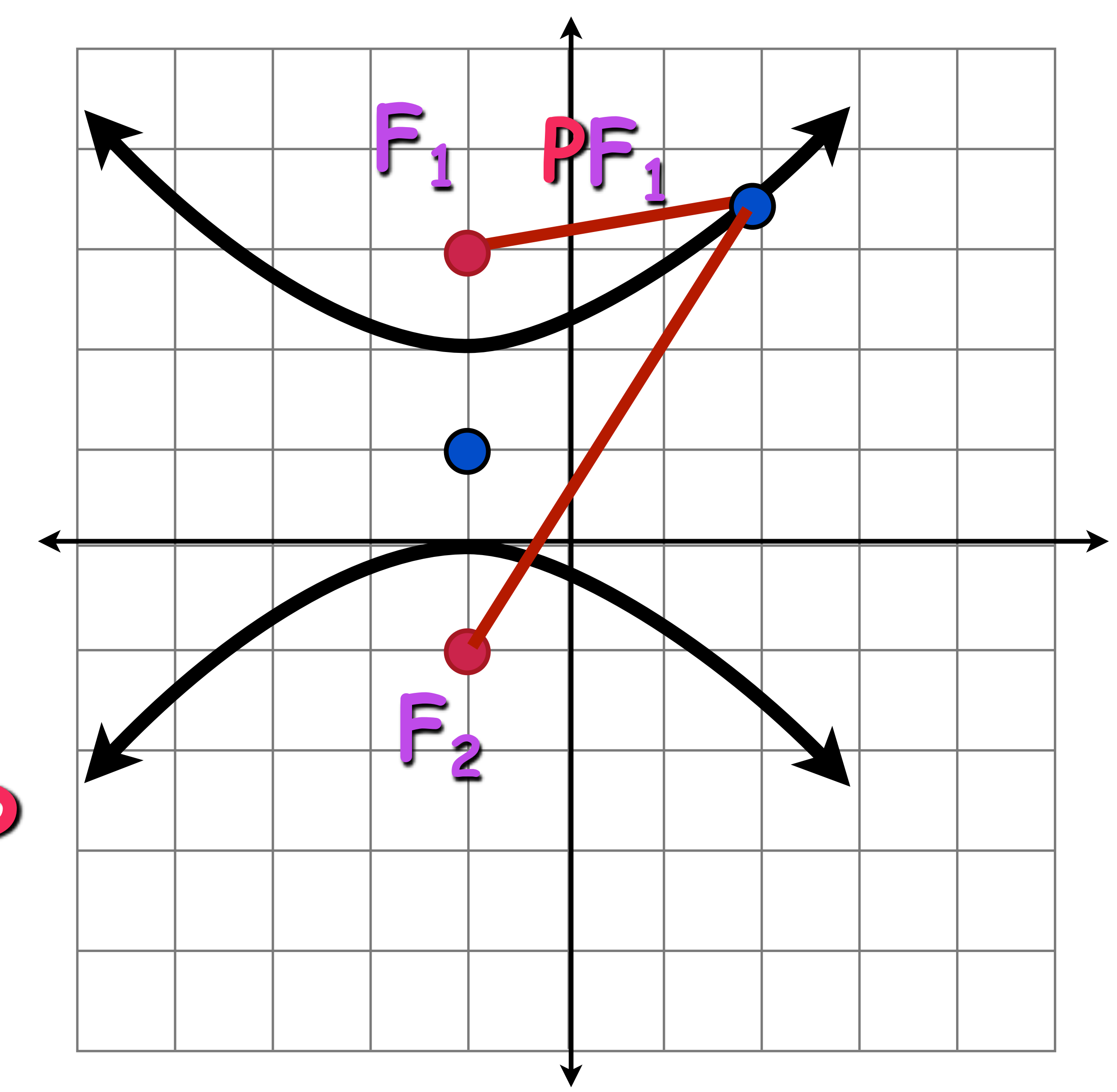
What would happen if you pulled the two foci of an ellipse so far apart that they moved **outside** the ellipse? The result would be a **hyperbola**, another conic section.



# Hyperbola

A **hyperbola** is a set of points **P(x, y)** in a plane such that the **difference** of the distances from P to fixed points $F_1$ and $F_2$, the **foci**, is constant. For a hyperbola, $d = |PF_1 - PF_2|$, where d is the constant difference. You can use the distance formula to find the equation of a hyperbola.

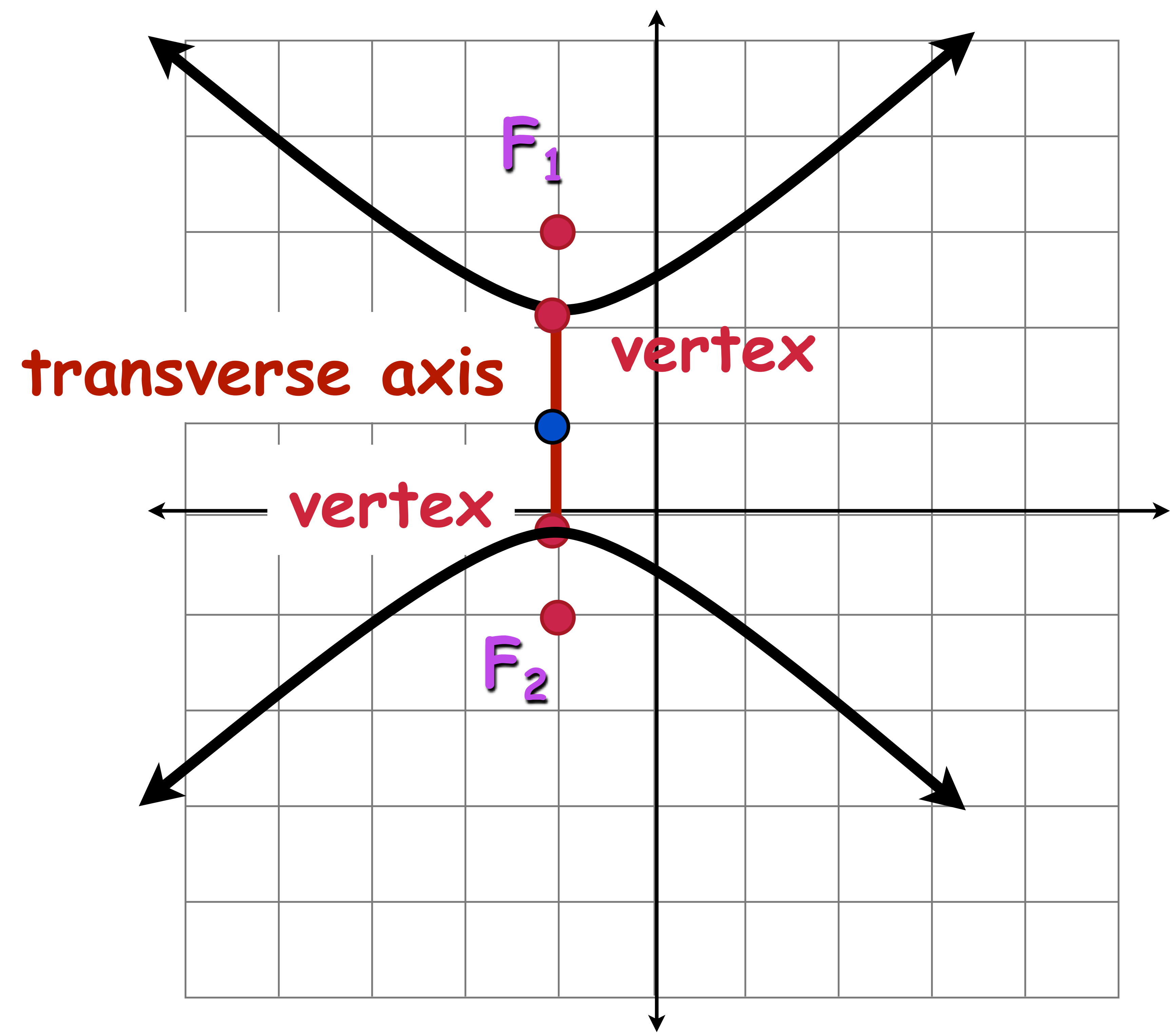
A hyperbola contains two symmetrical parts called **branches**.



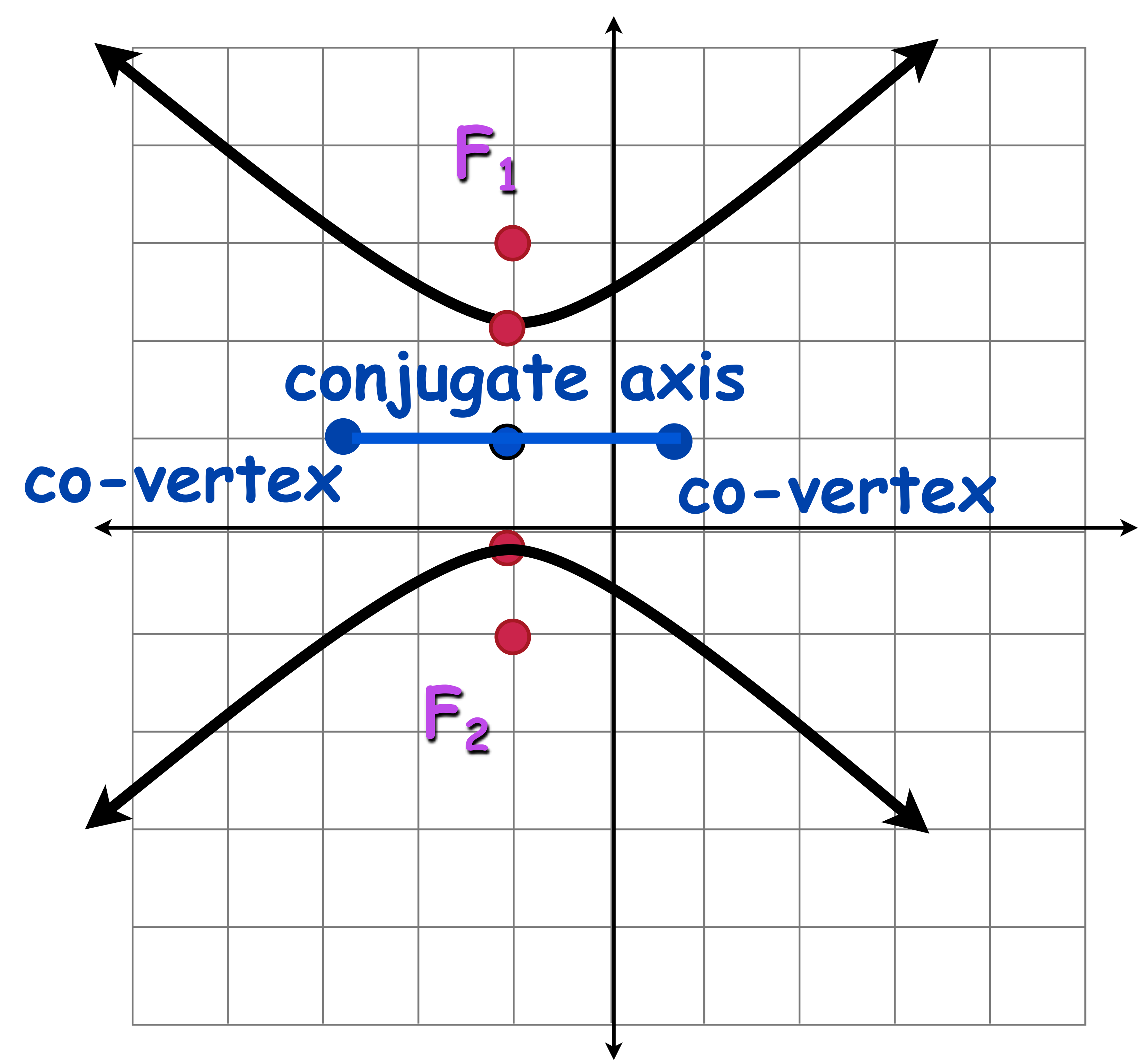

$d = |PF_1 - PF_2|$ = constant for all points **P**

# Hyperbola

A hyperbola has two axes of symmetry. The **transverse axis** of symmetry contains the vertices and, if it were extended, the foci of the hyperbola. The **vertices** of a hyperbola are the endpoints of the transverse axis.



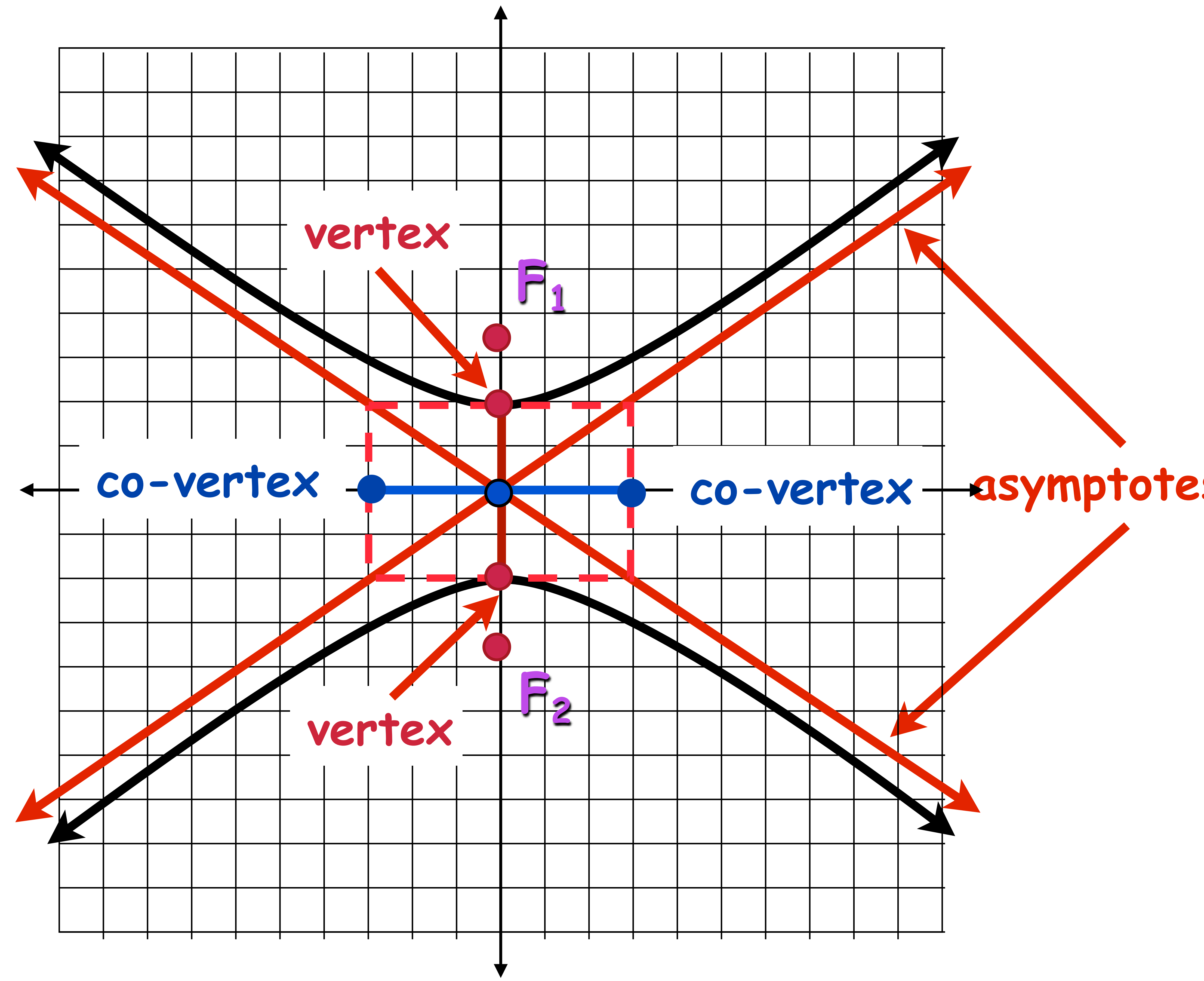
The **conjugate axis** of symmetry separates the two branches of the hyperbola. The **co-vertices** of a hyperbola are the endpoints of the conjugate axis.



# Eccentricity

All together now.

**The transverse axis is not always longer than the conjugate axis.**



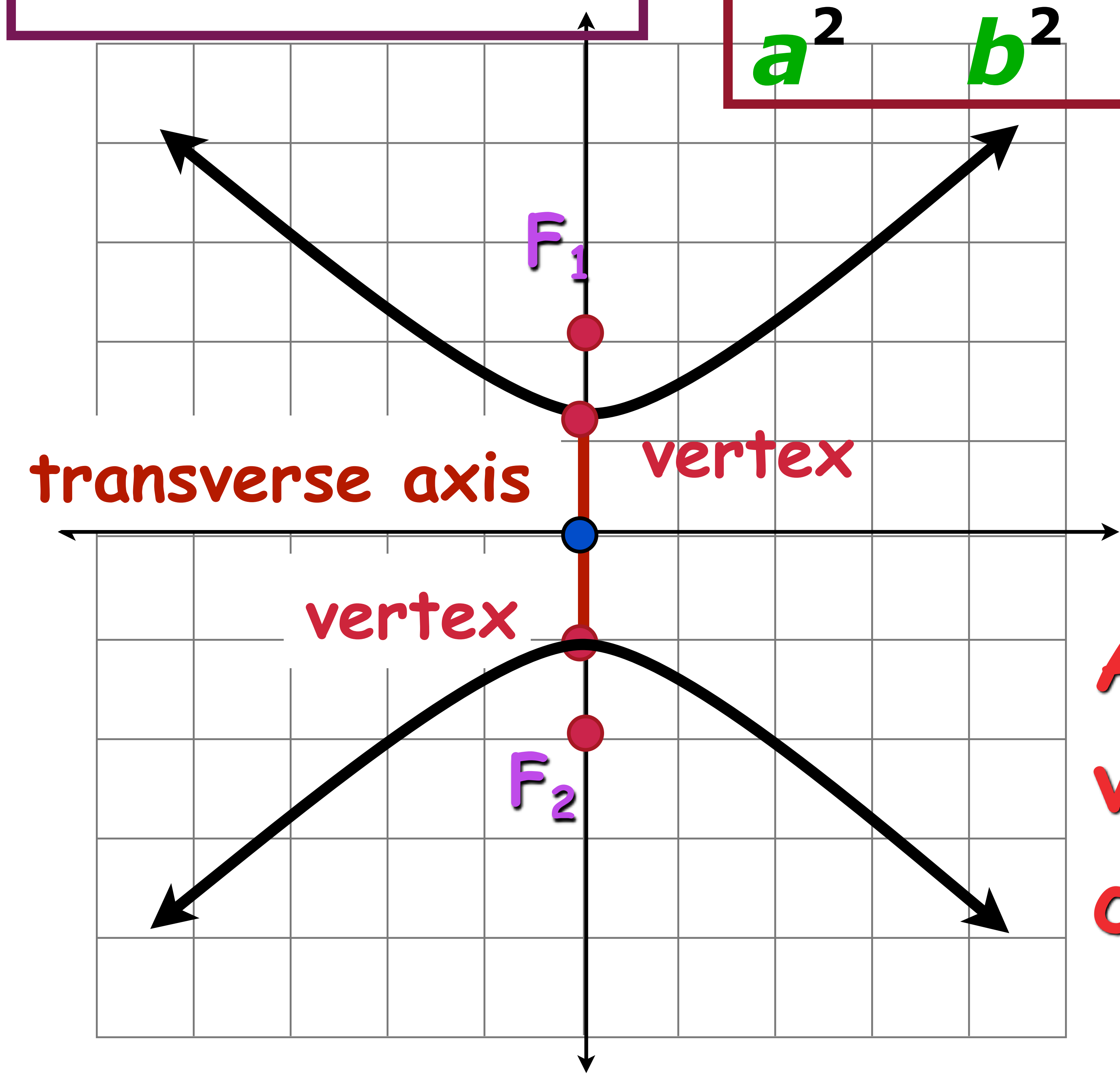

The **eccentricity** is $e = \frac{c}{a} > 1$.
Also, the larger the eccentricity is, the closer the branches of the hyperbola are to being lines.

# Standard Form of the Hyperbolic Equation

The standard form of the equation of a hyperbola **depends on whether the hyperbola's transverse axis is horizontal or vertical.**

**Vertical**

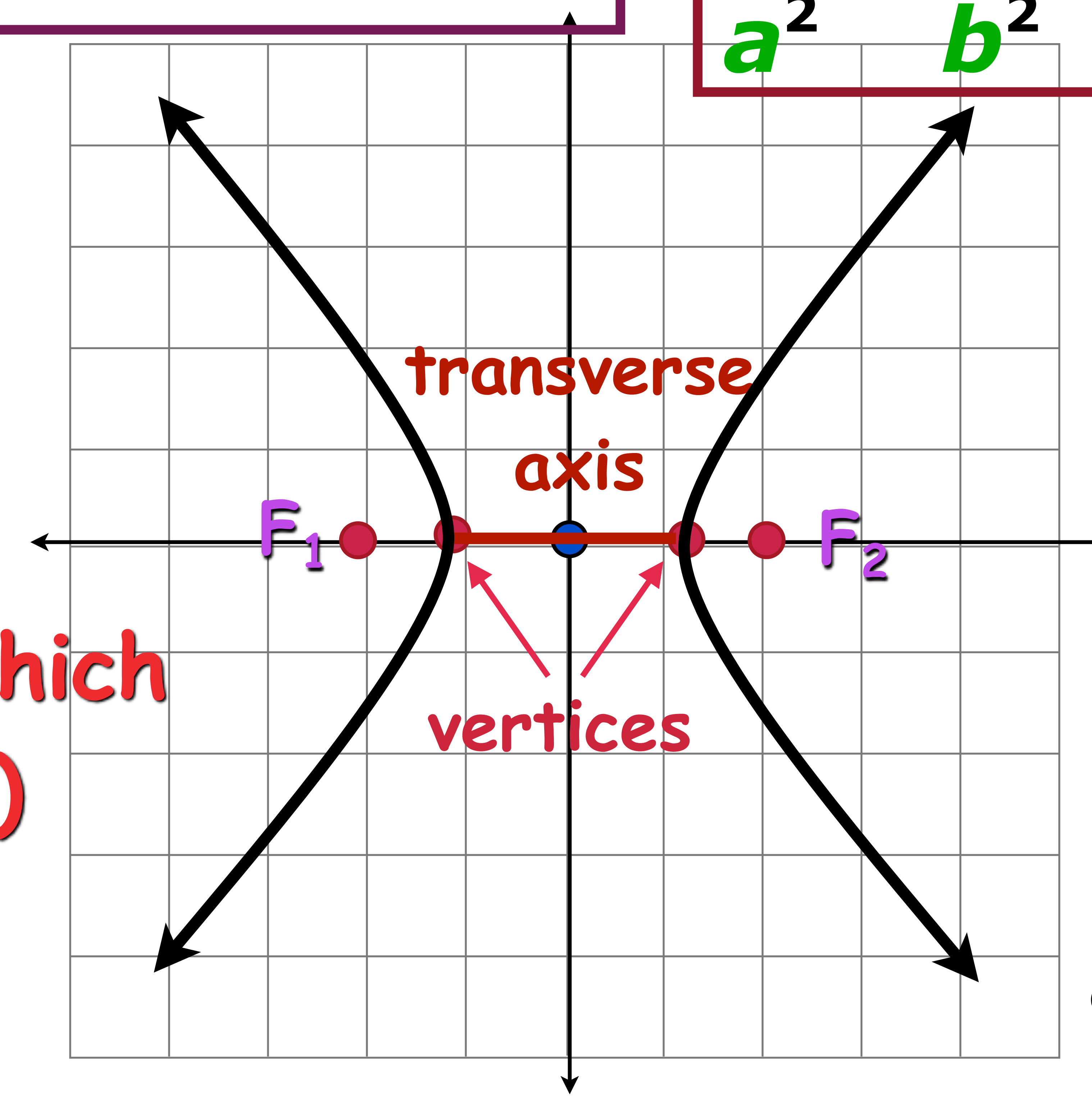
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$




Ask yourself, which variable (x or y) cannot be 0.

**Horizontal**

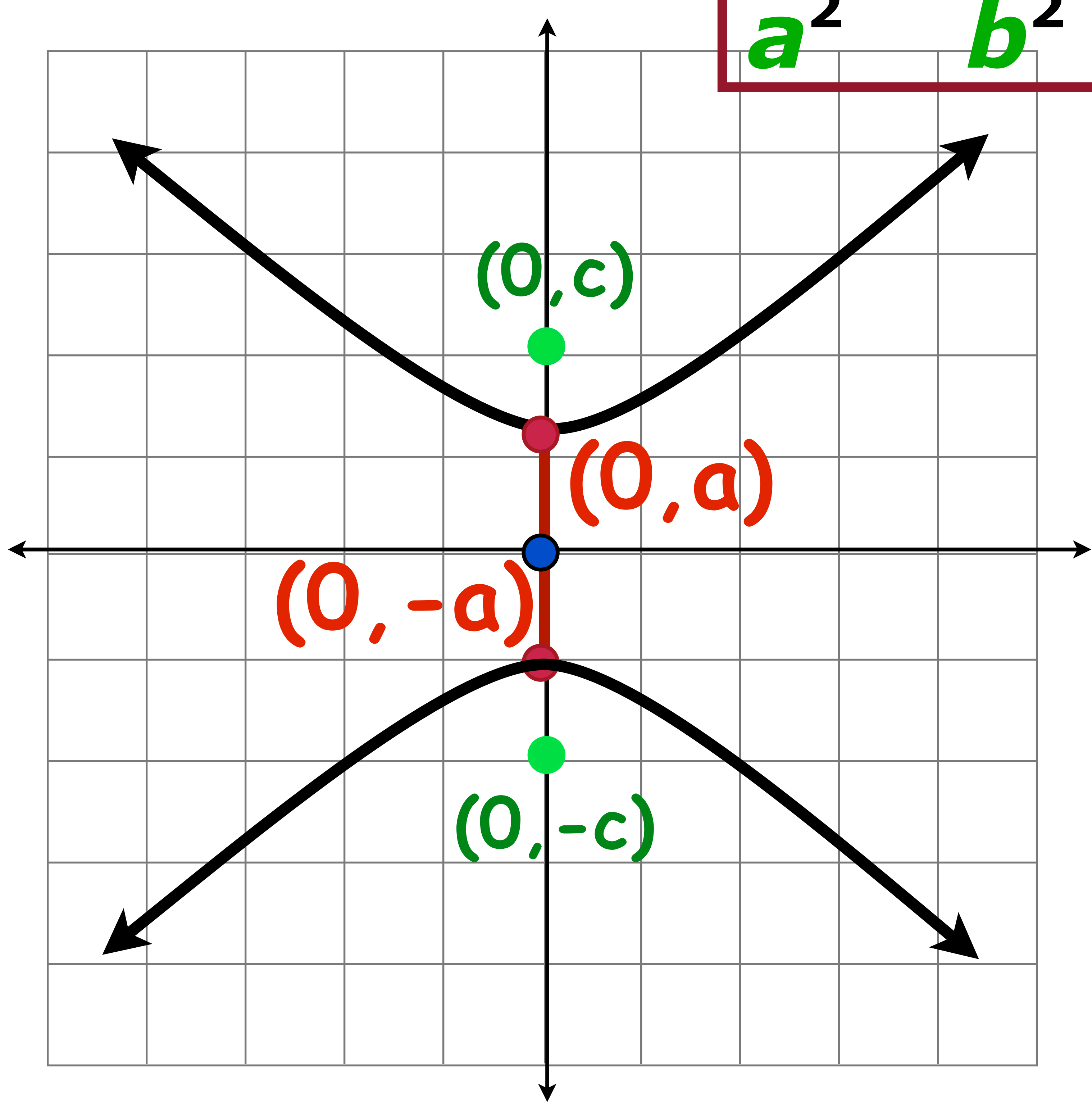
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



# Standard Form of the Hyperbolic Equation

The vertices are a units from the center and the foci are c units from the center.

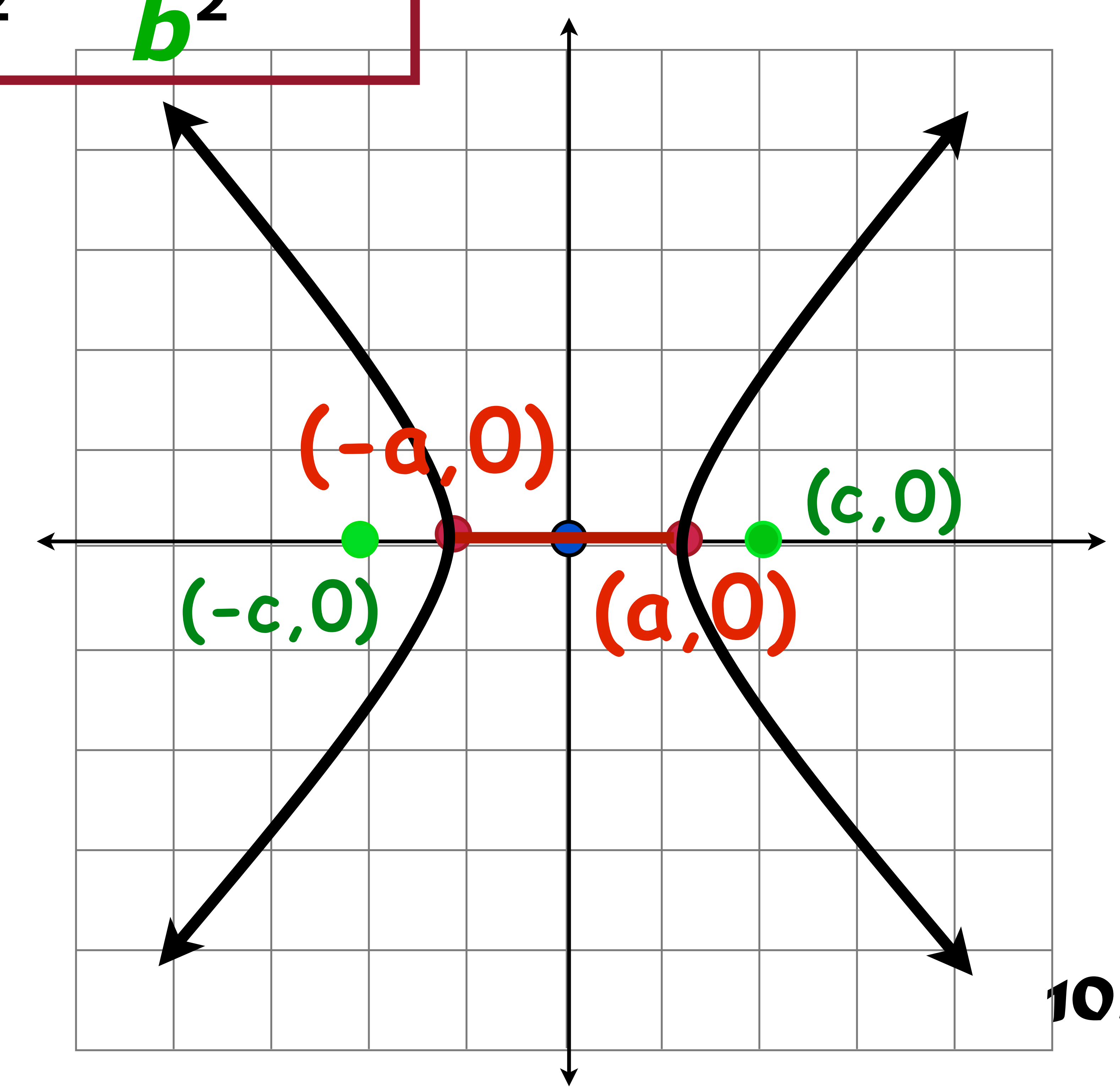
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$




$$b^2 = c^2 - a^2$$

$$c^2 = a^2 + b^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



# Standard Form of the Hyperbolic Equation

The values a, b, and c, are related by the equation $c^2 = a^2 + b^2$. Also note that **the length of the transverse axis is 2a and the length of the conjugate is 2b**.

**Standard Form for the Equation of a Hyperbola** — **Center at (0, 0)**

| TRANSVERSE AXIS | HORIZONTAL | VERTICAL |
|---|---|---|
| Equation | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ | $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ |
| Vertices | $(a, 0), (-a, 0)$ | $(0, a), (0, -a)$ |
| Foci | $(c, 0), (-c, 0)$ | $(0, c), (0, -c)$ |
| Co-vertices | $(0, b), (0, -b)$ | $(b, 0), (-b, 0)$ |
| Asymptotes | $y = \pm\frac{b}{a}x$ | $y = \pm\frac{a}{b}x$ |

# Asymptotes

We have yet to mention the final piece of the puzzle that is graphing a hyperbola.

Each hyperbola has two asymptotes that **intersect at the center of the hyperbola**.

A hyperbola with a horizontal transverse axis, the equations of the asymptotes are …

$$\boxed{y = \frac{b}{a}x}$$

A hyperbola with a vertical transverse axis, the equations of the asymptotes are …

$$\boxed{y = \frac{a}{b}x}$$

# Hyperbola at (0,0)

## Standard Form of the Equation of Hyperbola, center (0, 0)

| Horizontal | Vertical |
|---|---|
| $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ | $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ |
| **Asymptote** | **Asymptote** |
| $y = \pm \frac{b}{a} x$ | $y = \pm \frac{a}{b} x$ |

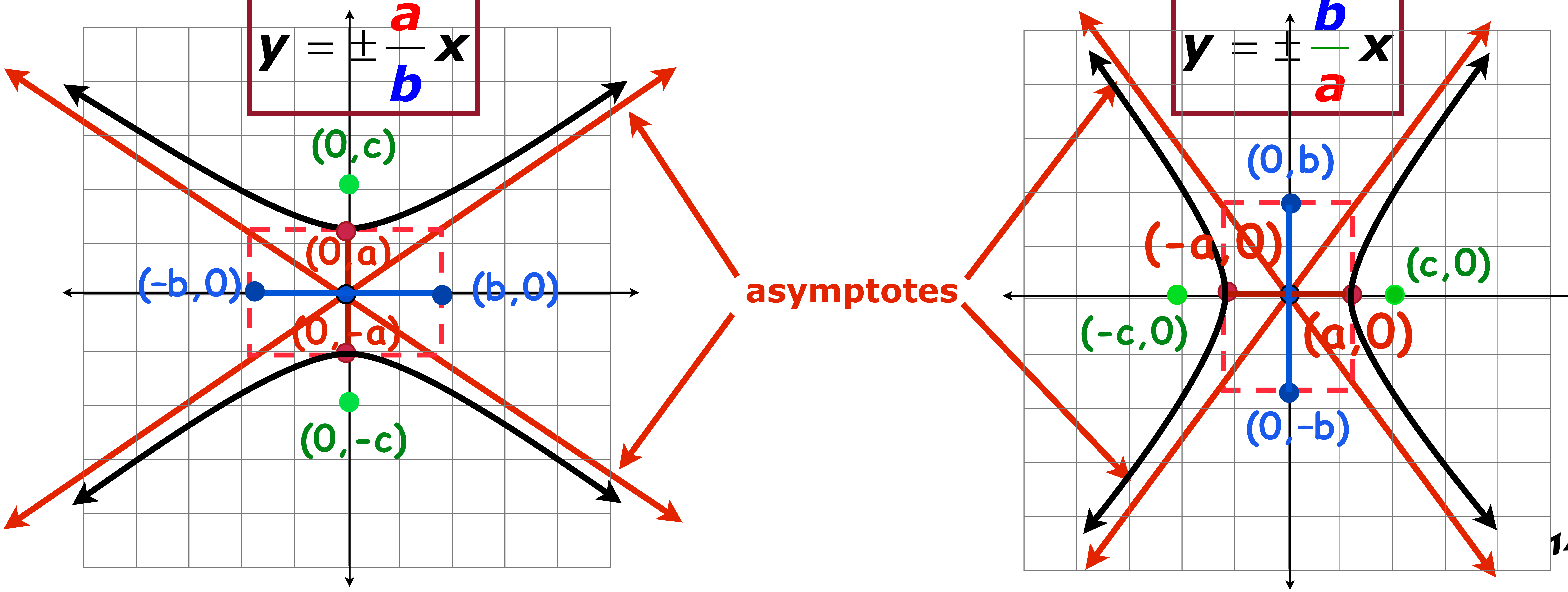
# Hyperbola at (0,0)

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

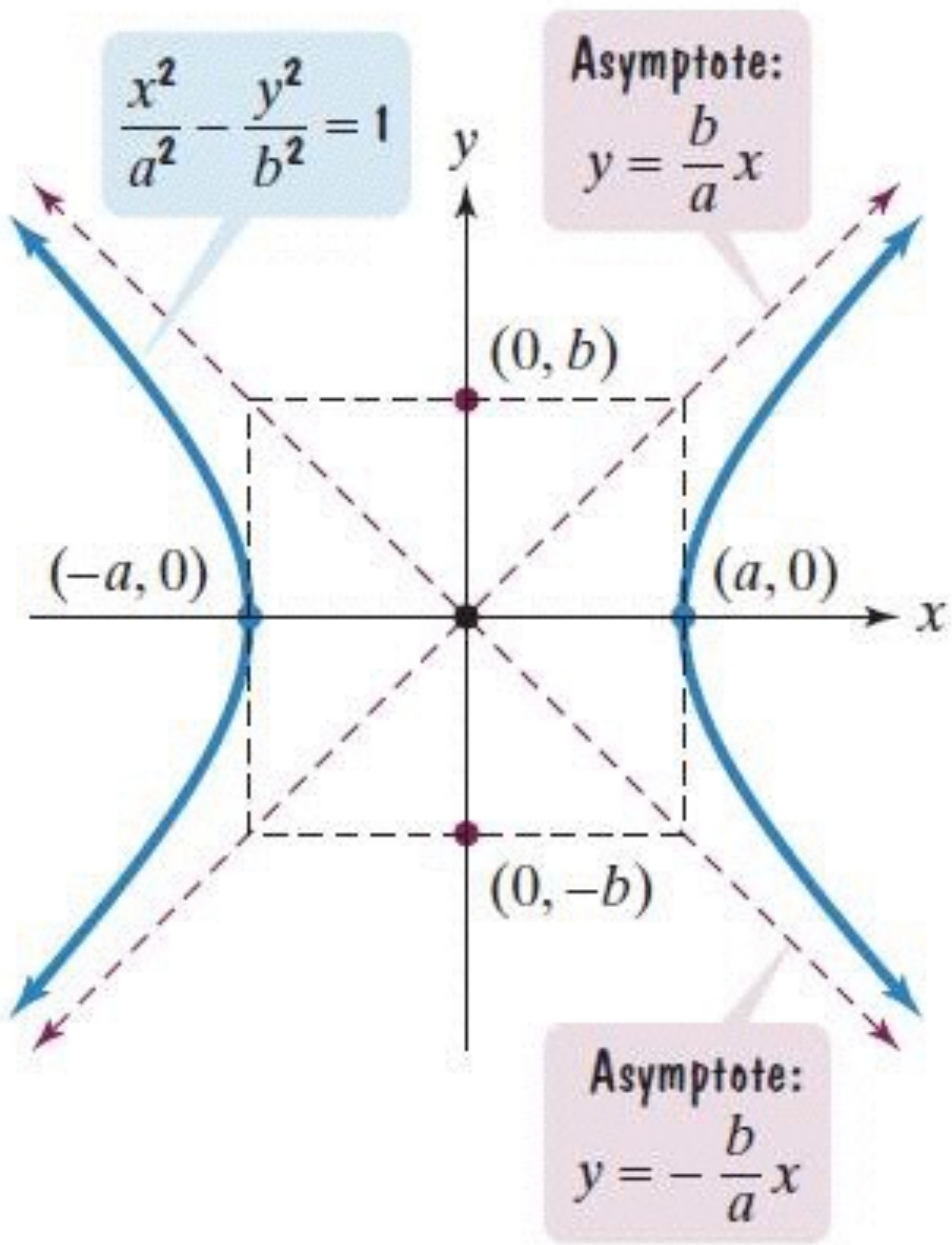
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

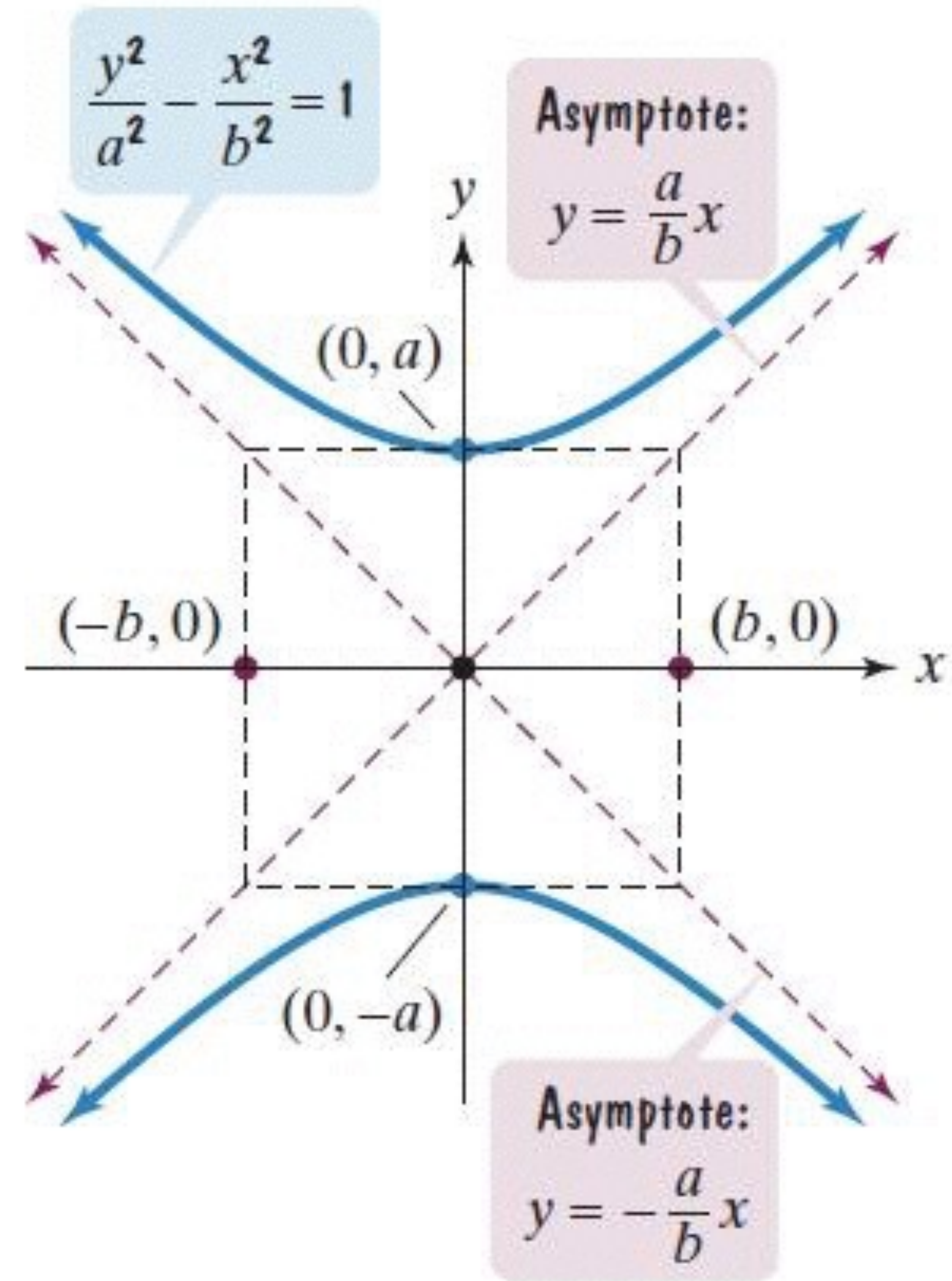
$$y = \pm \frac{a}{b} x$$

$$y = \pm \frac{b}{a} x$$



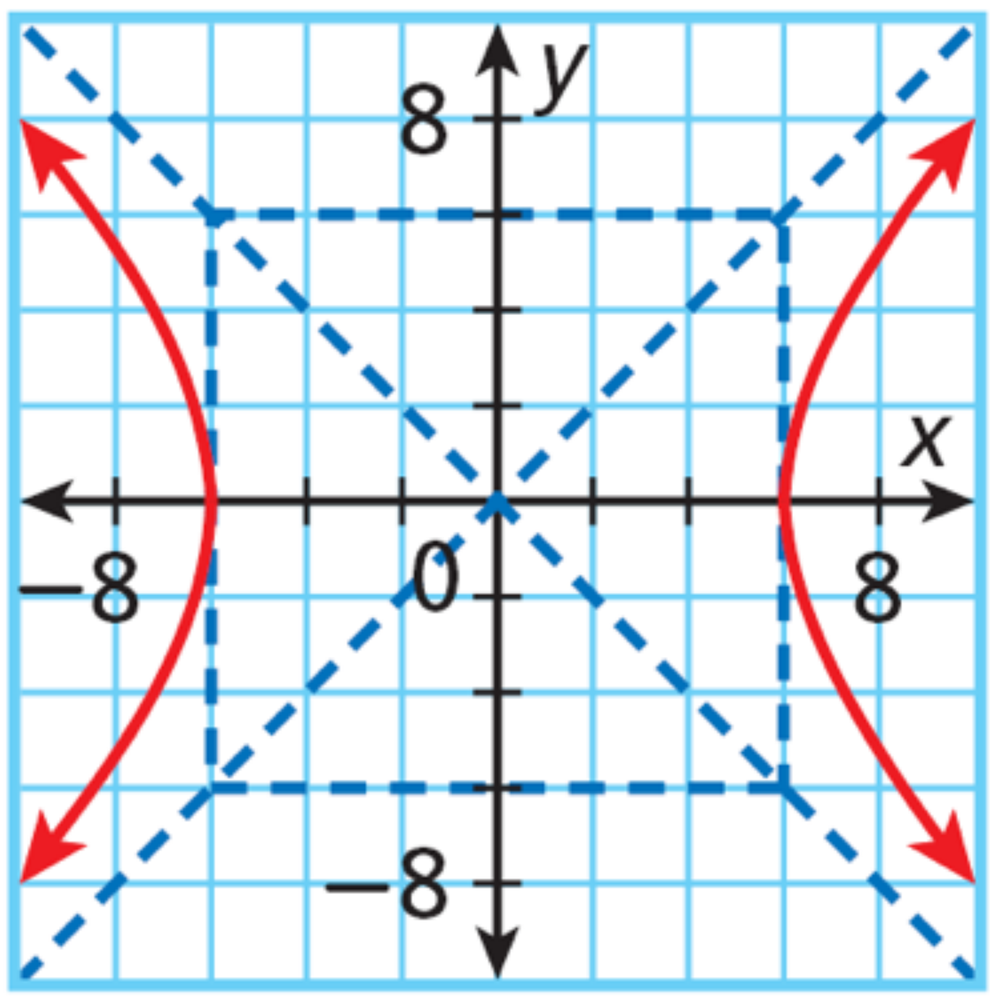
# Hyperbola at (0,0)



# Example

**Write an equation in standard form for the hyperbola.**




**Step 1** Identify the form of the equation.

The graph opens horizontally, so the equation will be in the form of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

**Step 2** Identify the center and the vertices.

Center (0, 0), vertices (±6, 0) and, co-vertices (0, ±6).

So **a = 6**, and **b = 6**.

**Step 3** Because a = 6 and b = 6, the equation of the graph is $\dfrac{x^2}{6^2} - \dfrac{y^2}{6^2} = 1$ or $\dfrac{x^2}{36} - \dfrac{y^2}{36} = 1$

# Graphing a Hyperbola center (0,0)

The steps to graphing a hyperbola are as follows:

1. Determine the orientation of the hyperbola.
2. Find the vertices and co-vertices.
3. Using the vertices and co-vertices draw a rectangle centered at the origin, parallel to axes, with intercepts $\pm a$, and $\pm b$.
4. Draw dashed lines through corners of rectangle as asymptotes.
5. Draw branches of hyperbola through vertices and approaching asymptotes.

# Example

Find the vertices, foci, asymptotes and graph the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$

**Step 1** The equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ so the transverse axis is horizontal with center (0, 0).

**Step 2** $a^2 = 25$, and $b^2 = 16$, $a = 5$ and $b = 4$, the vertices are $(\pm 5, 0)$ and the co-vertices are $(0, \pm 4)$

**Step 3** The equations of the asymptotes are $y = \pm \frac{4}{5}x$

$c^2 = a^2 + b^2 = 5^2 + 4^2 = 41$ $c = \sqrt{41}$ foci: $\left(\pm\sqrt{41}, 0\right)$

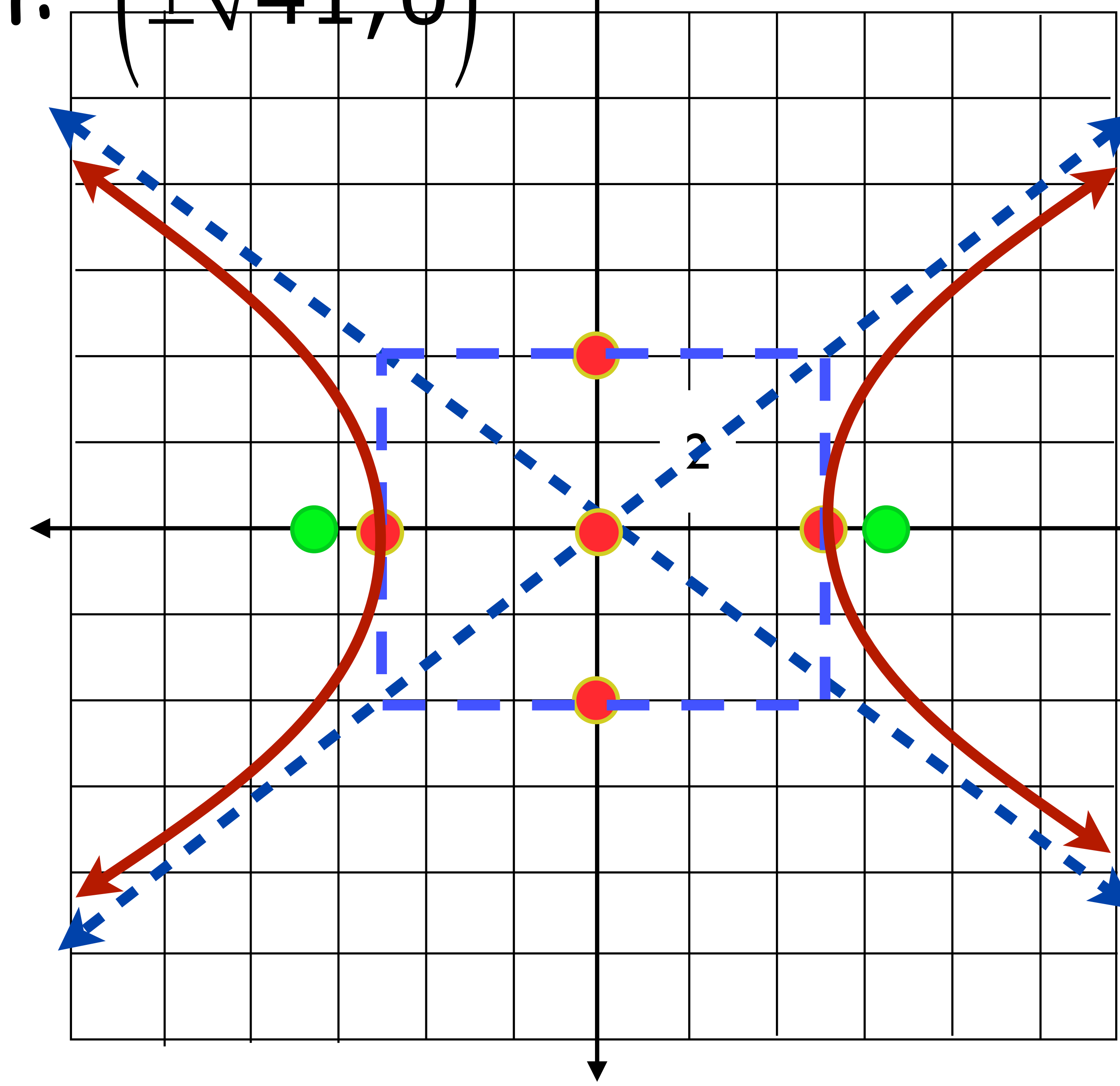
# Example

Find the vertices, foci, asymptotes and graph the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$

the vertices are (±5, 0) and
the co-vertices are (0, ±4)

$y = \pm\frac{4}{5}x$

foci: $\left(\pm\sqrt{41}, 0\right)$




**Step 4** Draw a box by using the vertices and co-vertices. Draw the asymptotes through the corners of the box.

**Step 5** Draw the hyperbola by using the vertices and the asymptotes.

# Example

Find the vertices, foci, asymptotes and graph the hyperbola $\frac{y^2}{25} - \frac{x^2}{16} = 1$

**Step 1** The equation is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ so the transverse axis is vertical with center (0, 0).

**Step 2** $a^2 = 25$, and $b^2 = 16$, $a = 5$ and $b = 4$, the vertices are (0, ±5) co-vertices are (±4, 0).

**Step 3** The equations of the asymptotes are $y = \pm\frac{5}{4}x$

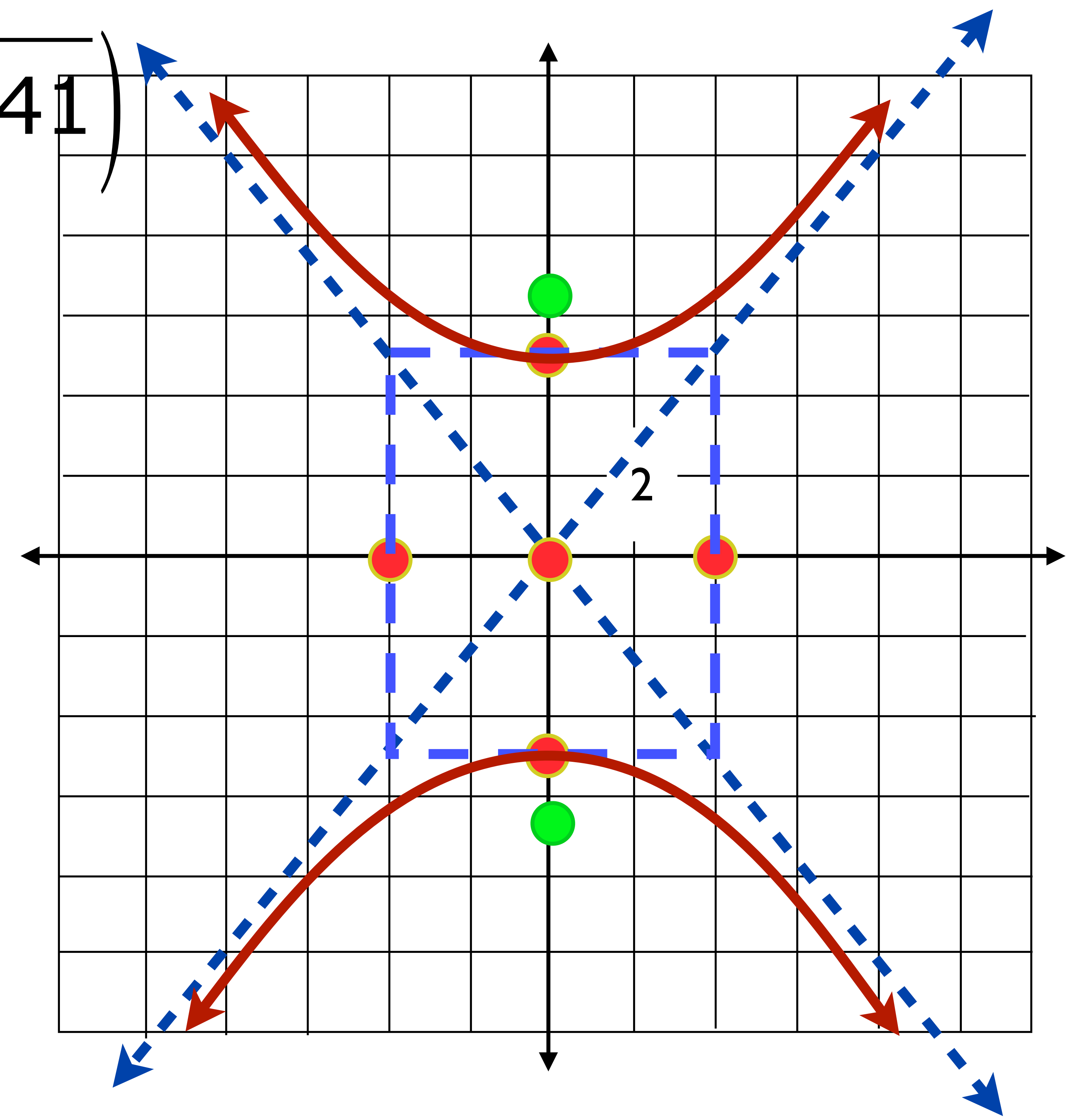
$c^2 = a^2 + b^2 = 5^2 + 4^2 = 41$ $\quad c = \sqrt{41}$ $\quad$ foci: $\left(0, \pm\sqrt{41}\right)$

# Example

Find the vertices, foci, asymptotes and graph the hyperbola $\dfrac{y^2}{25} - \dfrac{x^2}{16} = 1$

the vertices are (0, ±5)
co-vertices are (±4, 0).

$y = \pm\dfrac{5}{4}x$ foci: $\left(0, \pm\sqrt{41}\right)$




**Step 4** Draw a box by using the vertices and co-vertices. Draw the asymptotes through the corners of the box.

**Step 5** Draw the hyperbola by using the vertices and the asymptotes.

# Example

Find the standard form of the equation of a hyperbola with foci at (0, –5) and (0, 5) and vertices (0, –3) and (0, 3).

Because the foci are located at (0, –5) and (0, 5), the transverse axis lies on the y-axis. Thus, the form of the equation is $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

The distance from the center to either vertex is 3, so **$a = 3$**.

The distance from the center to either foci is 5, so **$c = 5$**.

$b^2 = c^2 - a^2 = 5^2 - 3^2 = 16, \; b = 4$ The co-vertices are (0, –4) and (0, 4)

The equation is $\dfrac{y^2}{5^2} - \dfrac{x^2}{4^2} = 1$ or $\dfrac{y^2}{25} - \dfrac{x^2}{16} = 1$

# Standard Form of Hyperbola, center (h,k)

As with all conics, hyperbolas do not have to be centered at the origin.

**Standard Form of the Equation of Hyperbola, Center (h, k)**

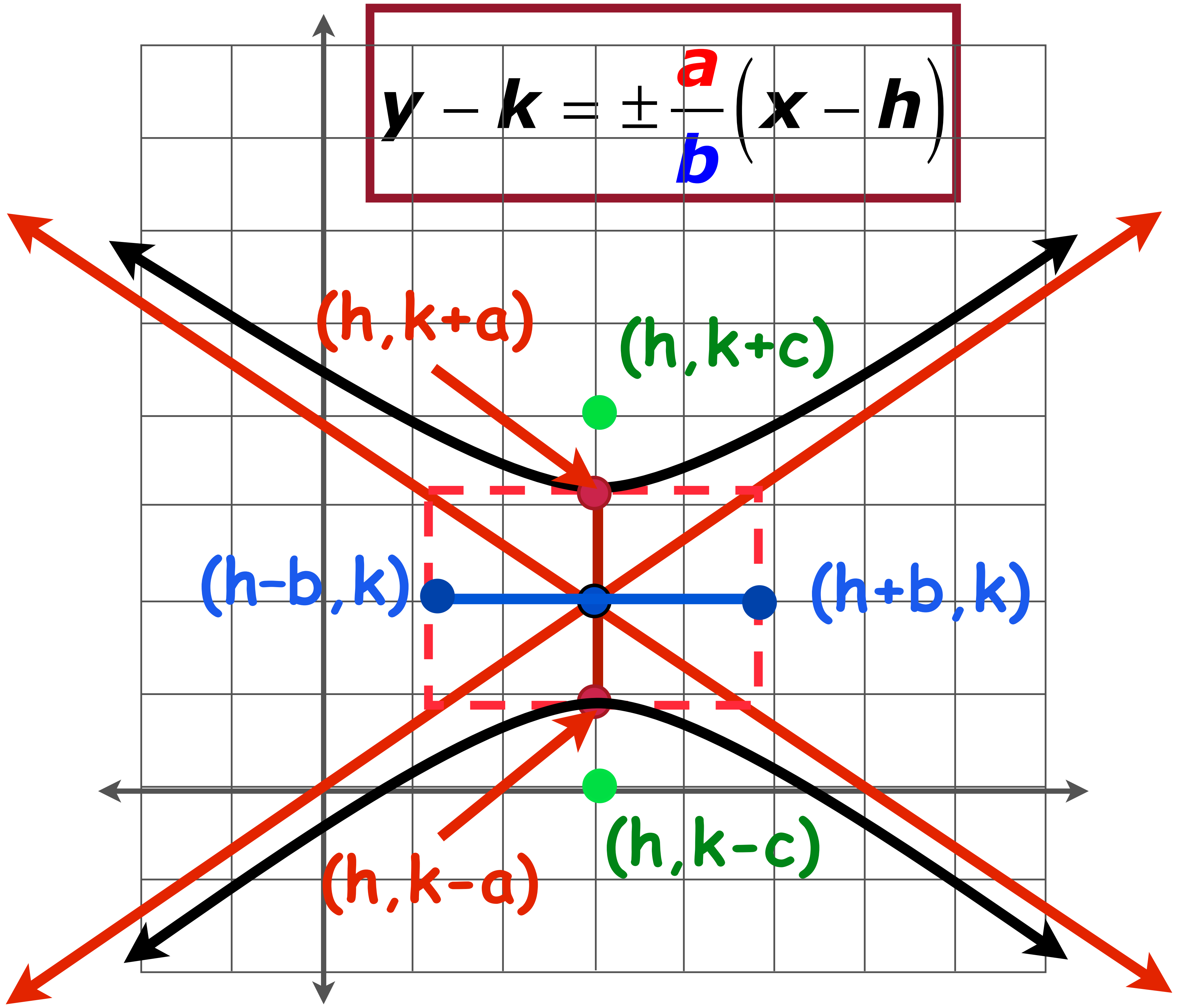
| Horizontal | Vertical |
|---|---|
| $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ | $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ |
| Asymptote | Asymptote |
| $y - k = \pm\frac{b}{a}(x-h)$ | $y - k = \pm\frac{a}{b}(x-h)$ |

Recognize the form?

# Hyperbola at (h,k)

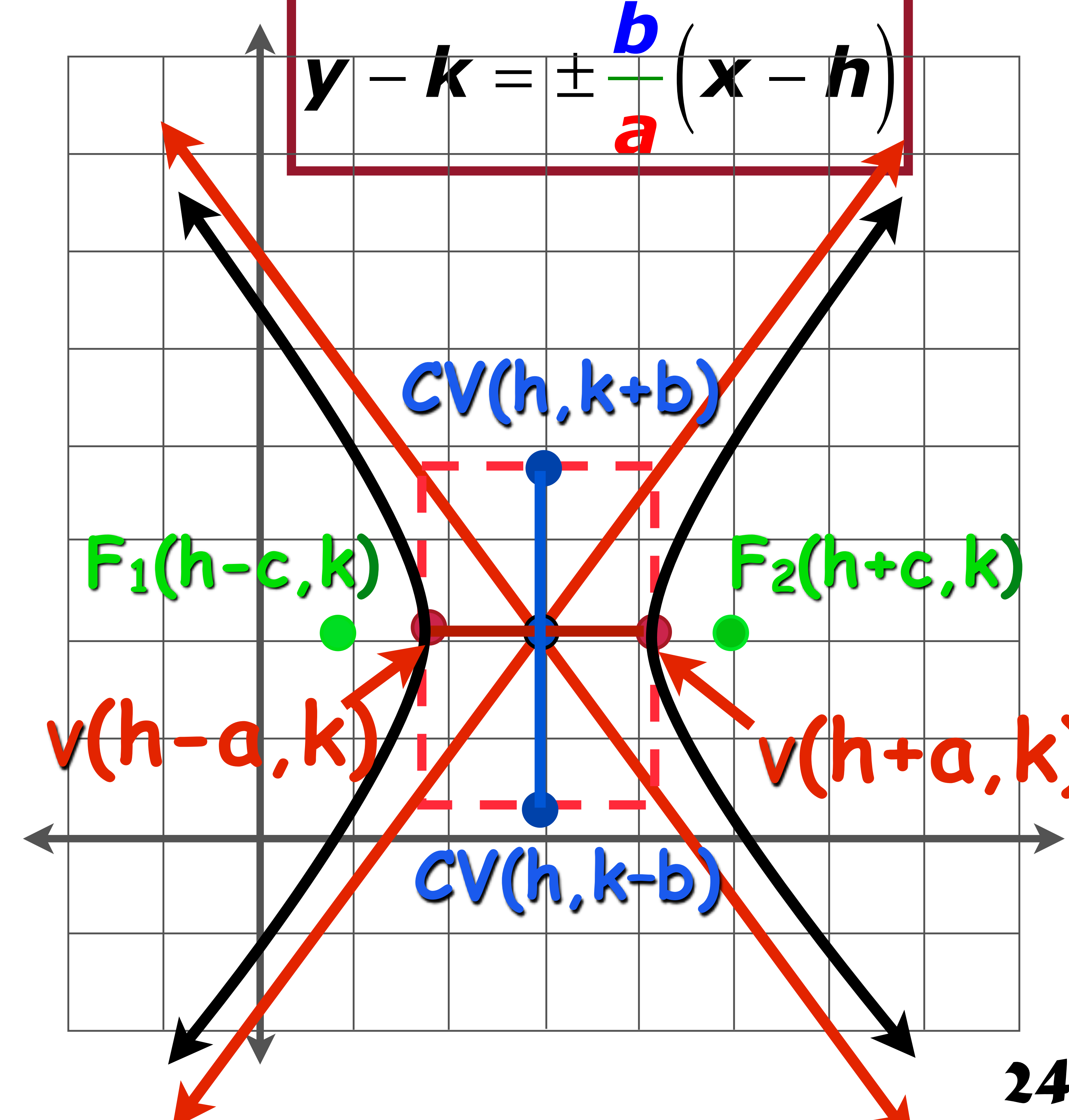
$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

$$y-k=\pm\frac{a}{b}(x-h)$$

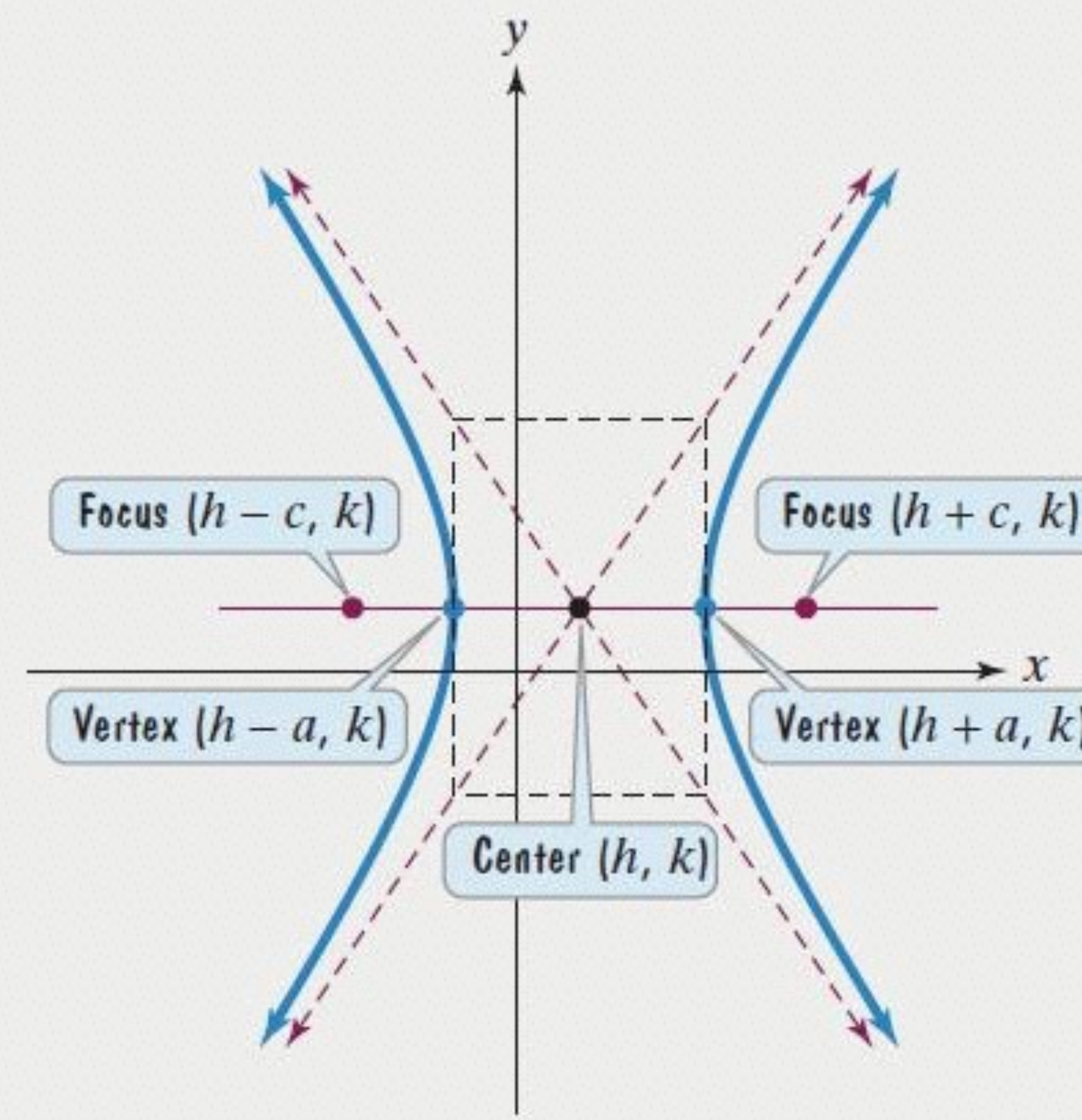



$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

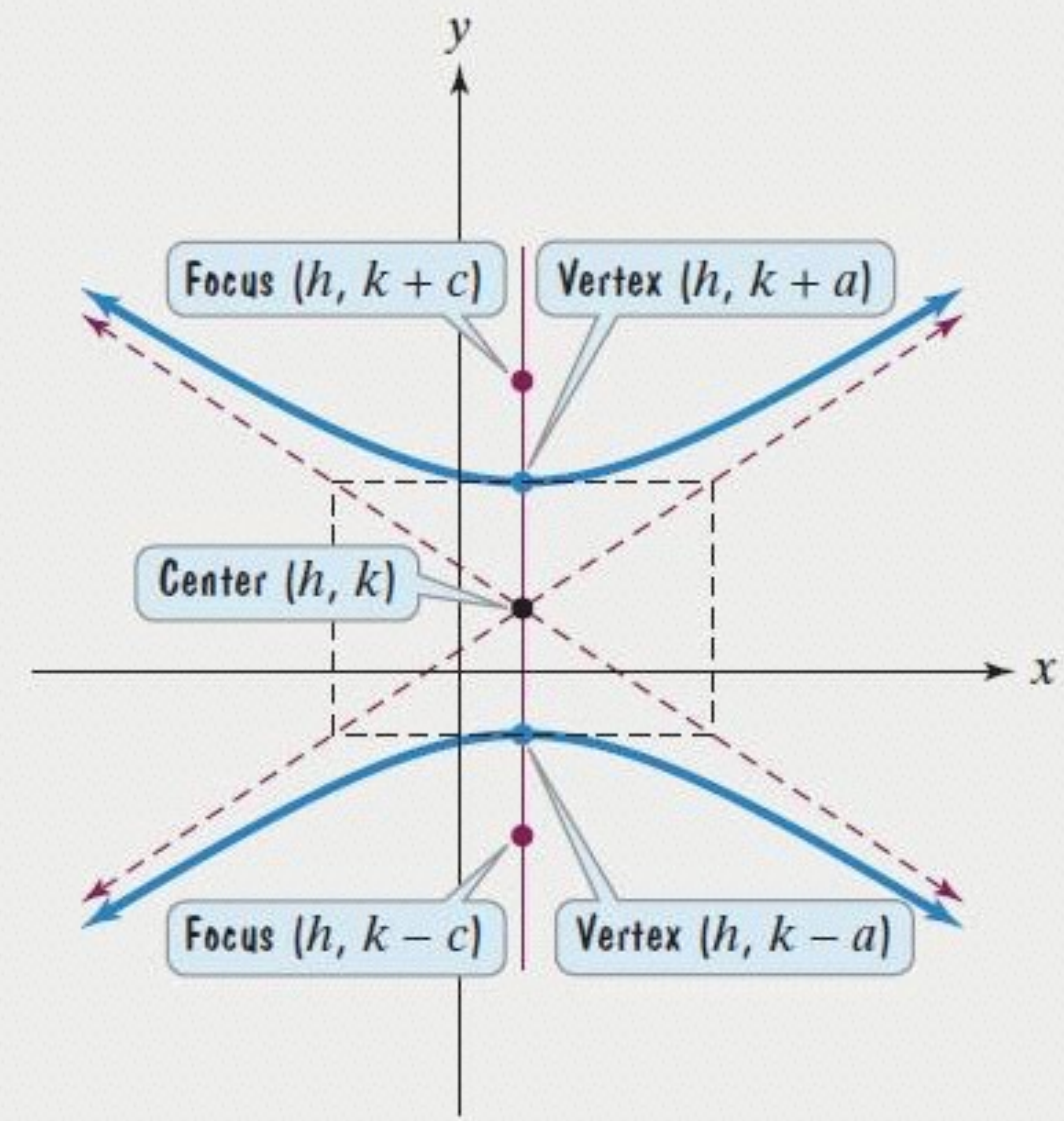
$$y-k=\pm\frac{b}{a}(x-h)$$



# Hyperbola at (h,k)

| Equation | Center | Transverse Axis | Vertices | Graph |
|---|---|---|---|---|
| $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$<br>Vertices are $a$ units right and $a$ units left of center.<br>Foci are $c$ units right and $c$ units left of center, where $c^2 = a^2 + b^2$. | $(h, k)$ | Parallel to the $x$-axis; horizontal | $(h - a, k)$<br>$(h + a, k)$ |  |

# Hyperbola at (h,k)

| | | | | |
|---|---|---|---|---|
| $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ <br> Vertices are $a$ units above and $a$ units below the center. <br> Foci are $c$ units above and $c$ units below the center, where $c^2 = a^2 + b^2$. | $(h, k)$ | Parallel to the $y$-axis; vertical | $(h, k-a)$ <br> $(h, k+a)$ |   |

# Example

Find the vertices, co-vertices, foci, and asymptotes of the hyperbola, and then graph. $\frac{(x-3)^2}{9} - \frac{(y+5)^2}{49} = 1$

**Step 1** The equation is in the form $y-k = \pm\frac{b}{a}(x-h)$, the transverse axis is horizontal with $C(3, -5)$.

**Step 2** Since **a** = **3** and **b** =**7**, the vertices are (0, –5) and (6, –5), the co-vertices are (3, –12) and (3, 2).

**Step 3** The equations of the asymptotes are $y+5 = \pm\frac{7}{3}(x-3)$

$c^2 = a^2 + b^2 = 9 + 49 = 58$ $\quad c = \sqrt{58}$ $\quad$ foci: $\left(3 \pm \sqrt{58}, -5\right)$

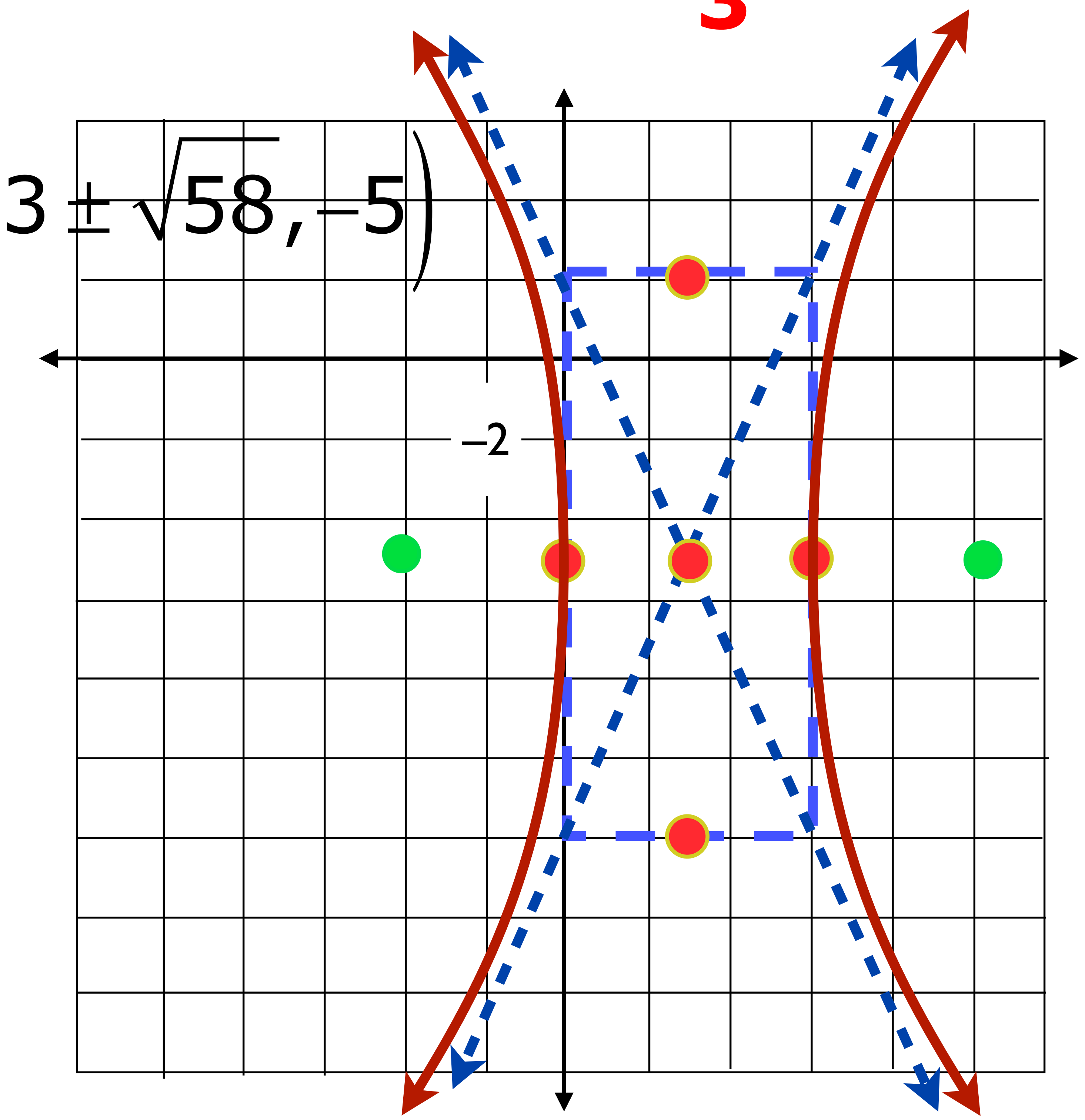
# Example

**Find the vertices, co-vertices, foci, and asymptotes of the hyperbola, and then graph.** $\dfrac{(x-3)^2}{9}-\dfrac{(y+5)^2}{49}=1$

$$y+5=\pm\frac{7}{3}(x-3)$$

the vertices are (0, –5) and (6, –5),
the co-vertices are (3, –12) and (3, 2).

foci: $\left(3\pm\sqrt{58},-5\right)$




**Step 4** Draw a box by using the vertices and co-vertices. Draw the asymptotes through the corners of the box.

**Step 5** Draw the hyperbola by using the vertices and the asymptotes.

# Example

Find the vertices, co-vertices, foci, and asymptotes of each hyperbola, and then graph. $\frac{(y+5)^2}{1} - \frac{(x-1)^2}{9} = 1$

**Step 1** The equation is in the form $\frac{(y+5)^2}{1^2} - \frac{(x-1)^2}{3^2} = 1$ so the transverse axis is vertical with $C(1, -5)$.

**Step 2** $a = 1$, $b = 3$

the vertices are (1, –4) and (1, –6) and the co-vertices are (4, –5) and (–2, –5).

**Step 3** The equations of the asymptotes are $y + 5 = \pm\frac{1}{3}(x-1)$

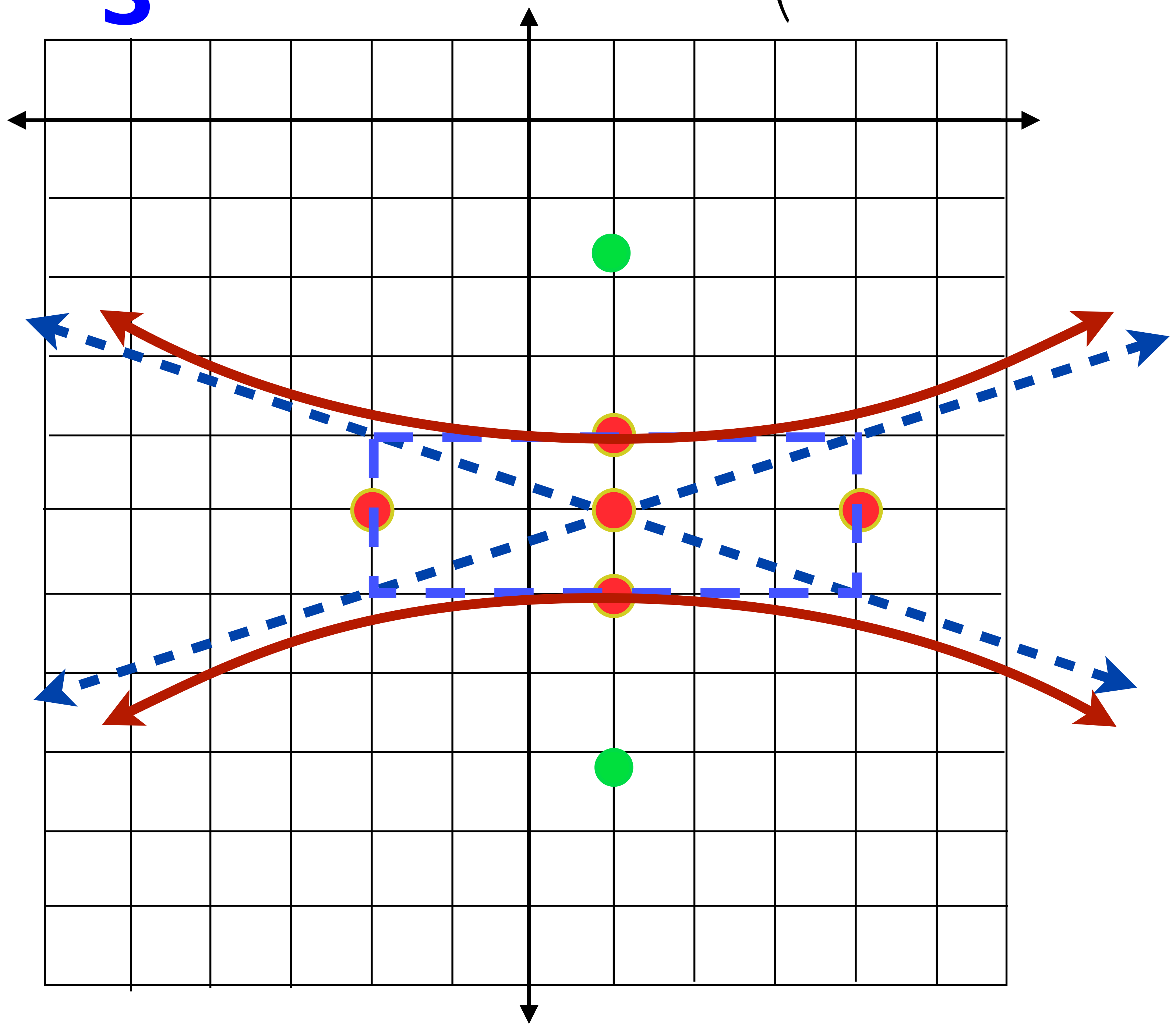
$c^2 = a^2 + b^2 = 1 + 9 = 10$ $c = \sqrt{10}$ foci: $\left(1, -5 \pm \sqrt{10}\right)$

# Example

**Find the vertices, co-vertices, foci, and asymptotes of each hyperbola, and then graph.**

$$y+5=\pm\frac{1}{3}(x-1) \quad \text{foci: } \left(1,-5\pm\sqrt{10}\right)$$

the vertices are (1, –4) and (1, –6) and the co-vertices are (4, –5) and (–2, –5).



**Step 4** Draw a box by using the vertices and co-vertices. Draw the asymptotes through the corners of the box.

**Step 5** Draw the hyperbola by using the vertices and the asymptotes.

Note the changes when the center is translated.

| Parameter | Transformation |
|---|---|
| $h$ | Translates the graph left for $h > 0$ and right for $h < 0$ |
| $k$ | Translates the graph up for $k > 0$ and down for $k < 0$ |
| $a$ | Stretches the graph in the direction of the transverse axis; as $a$ increases, the vertices move farther apart. |
| $b$ | Stretches the graph in the direction of the conjugate axis; as $b$ increases, the co-vertices move farther apart. |

**Find the vertices, co-vertices, foci, and asymptotes of the hyperbola, then graph $9x^2 - 54x - 16y^2 - 64y - 127$**

**Step 1** Write in standard form by completing the square.

$$9x^2 - 16y^2 - 54x - 64y - 127 = 0$$

$$9x^2 - 54x + \quad - 16y^2 - 64y + \quad = 127$$

$$9(x^2 - 6x + \square) - 16(y^2 + 4y + \square) = 127 + \square + \square$$

$$9(x^2 - 6x + 9) - 16(y^2 + 4y + 4) = 127 + 81 - 64$$

$$9(x-3)^2 - 16(y+2)^2 = 144$$

$$\frac{(x-3)^2}{16} - \frac{(y+2)^2}{9} = 1$$

**Find the vertices, co-vertices, foci, and asymptotes of the hyperbola, then graph** $9x^2 - 54x - 16y^2 - 64y - 127$

$$\frac{(x-3)^2}{16} - \frac{(y+2)^2}{9} = 1$$

The transverse axis is horizontal with **C(3,−2)**.

The center is (3, -2)

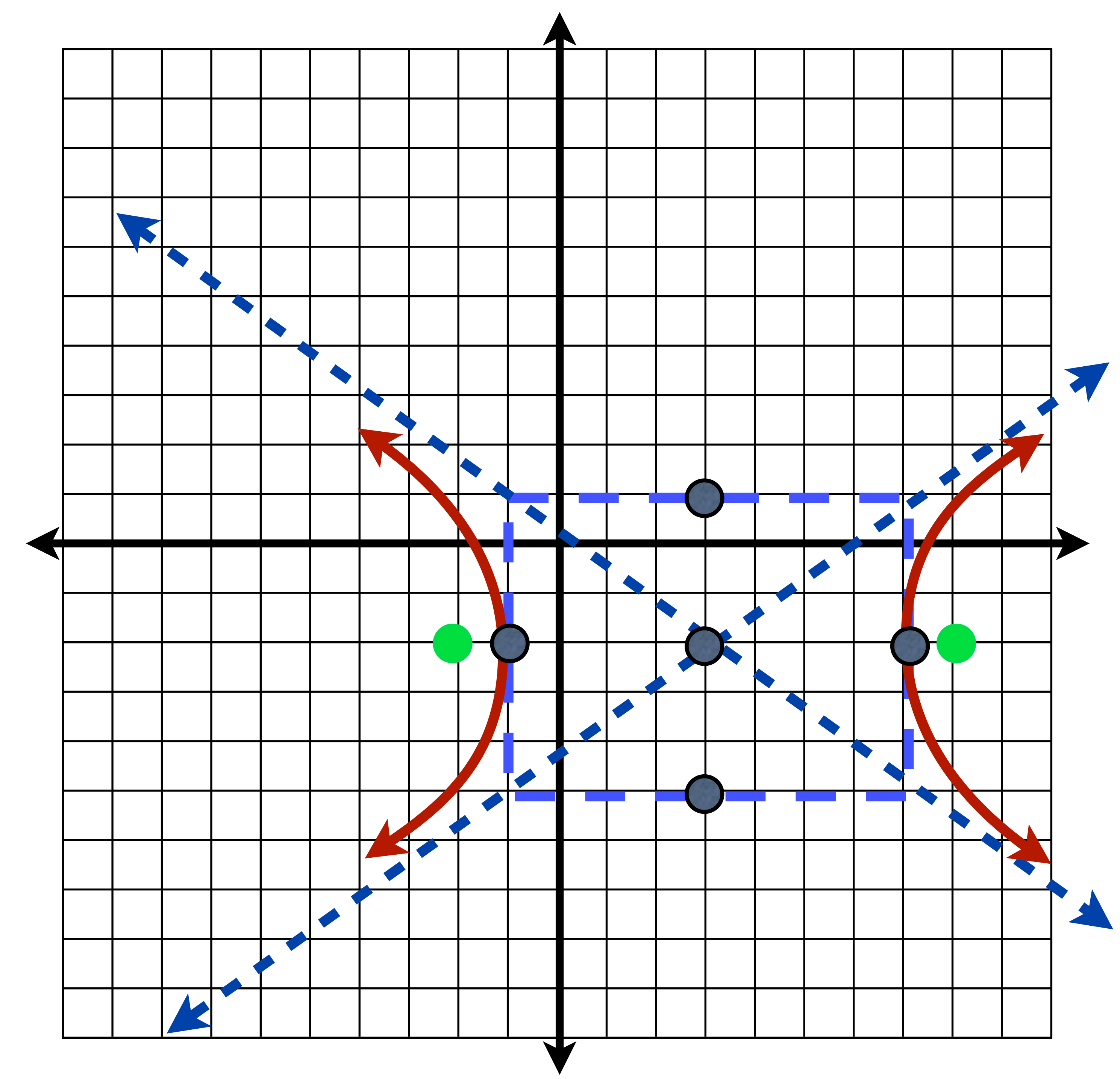
The vertices are (3±4, -2)

The co-vertices are (3, -2±3)

$c^2 = a^2 + b^2 = 16 + 9 = 25 \quad c = 5$

The foci are (3±5, -2)

The asymptotes are $y + 2 = \pm\frac{3}{4}(x-3)$



**Find the vertices, co-vertices, foci, and asymptotes of the hyperbola, then graph** $\mathbf{49y^2 - 4x^2 - 294y - 48x + 111 = 0}$

**Step 1** Write in standard form by completing the square.

$$49y^2 - 4x^2 - 294y - 48x + 111 = 0$$

$$49y^2 - 294y - 4x^2 - 48x = -111$$

$$49y^2 - 294y + \quad - 4x^2 - 48x + \quad = -111 + \quad +$$

$$49(y^2 - 6y + \square) - 4(x^2 + 12x + \square) = -111 + \square + \square$$

$$49(y-3)^2 - 4(x+6)^2 = 196$$

$$\frac{(y-3)^2}{4} - \frac{(x+6)^2}{49} = 1$$

# Example

**Find the vertices, co-vertices, foci, and asymptotes of the hyperbola, then graph** $49y^2 - 4x^2 - 294y - 48x + 111 = 0$

$$\frac{(y-3)^2}{4} - \frac{(x+6)^2}{49} = 1$$

The transverse axis is vertical with **C(-6,3)**.

The center is (-6, 3)

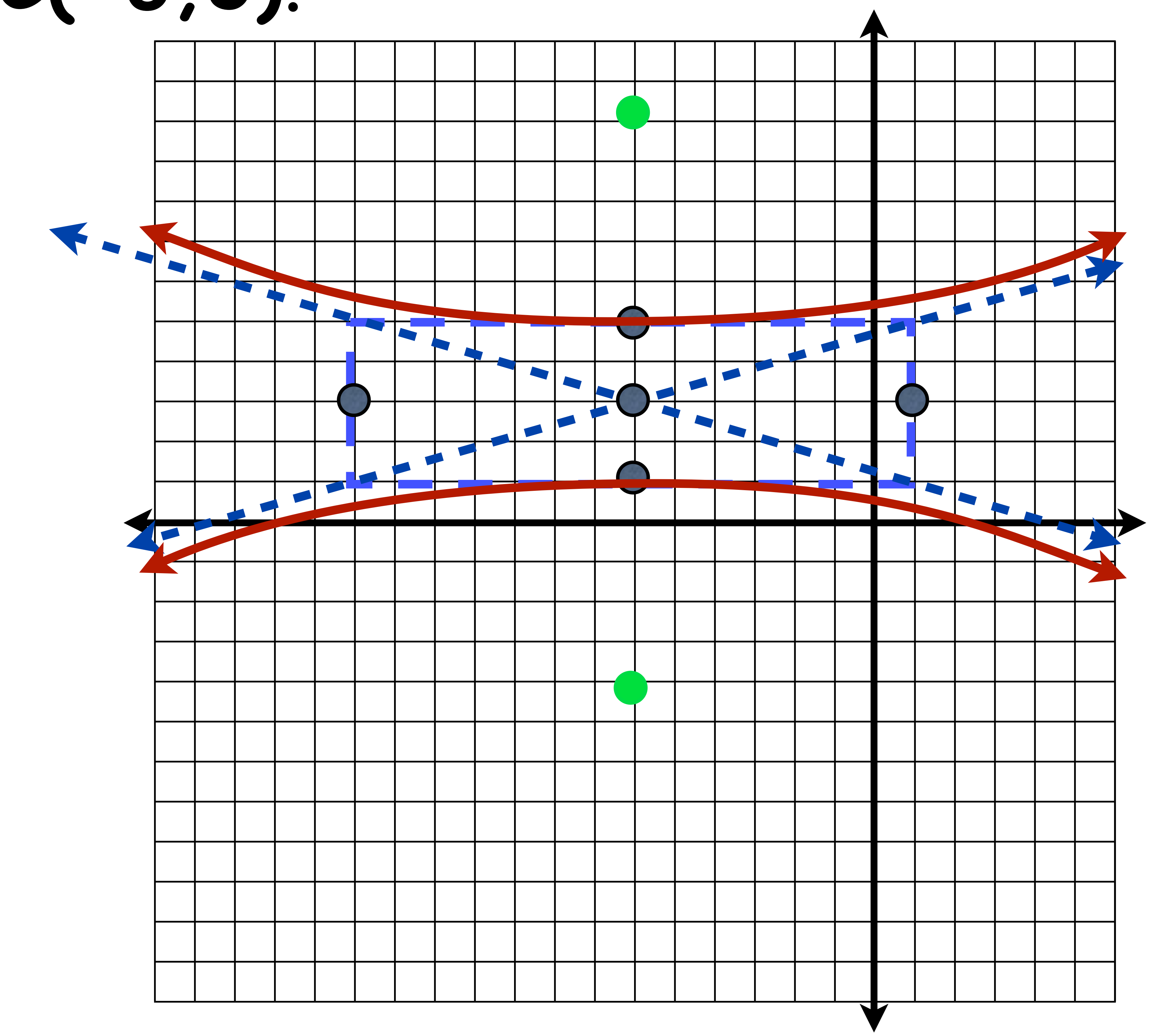
The vertices are (-6, 3±2)

The co-vertices are (-6±7, 3)

$c^2 = a^2 + b^2 = 49 + 4 = 53 \quad c = \sqrt{53}$

The foci are $\left(-6, 3 \pm \sqrt{53}\right)$

The asymptotes are $y - 3 = \pm\frac{2}{7}(x+6)$

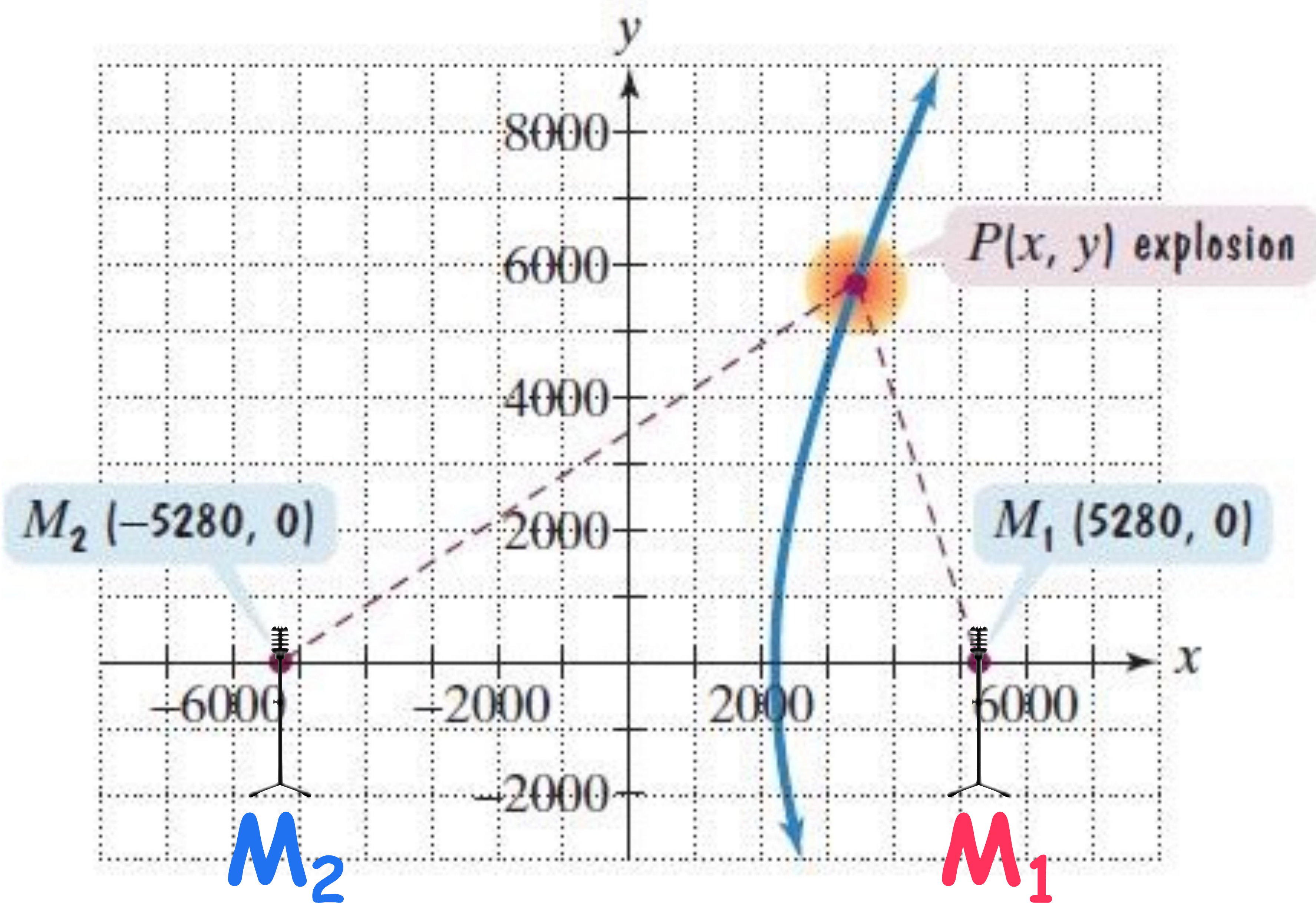


# Application

An explosion is recorded by two microphones that are 2 miles apart. Microphone $M_1$ receives the sound 3 seconds before microphone $M_2$.

Assuming sound travels at 1100 feet per second, determine the possible locations of the explosion relative to the location of the microphones.

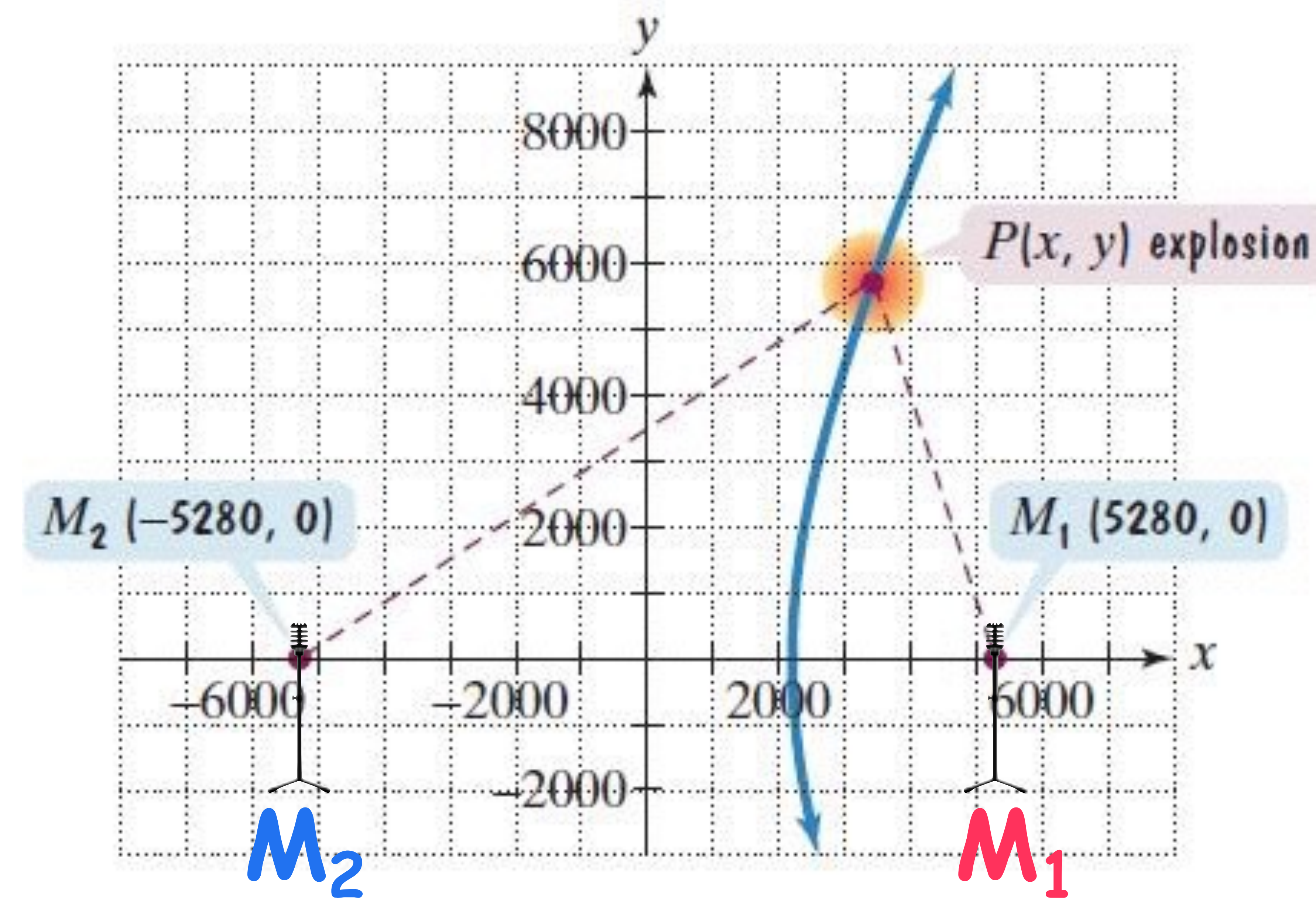
We begin by putting the microphones in a coordinate system.



# Application

We know that $M_1$ received the sound 3 seconds before $M_2$. Since sound travels at 1100 feet per second, the difference between the distance from P to $M_1$ and the distance from P to $M_2$ is 3300 feet.

The set of all points P (or locations of the explosion) satisfying these conditions fits the definition of a hyperbola, with microphones $M_1$ and $M_2$ at the foci.

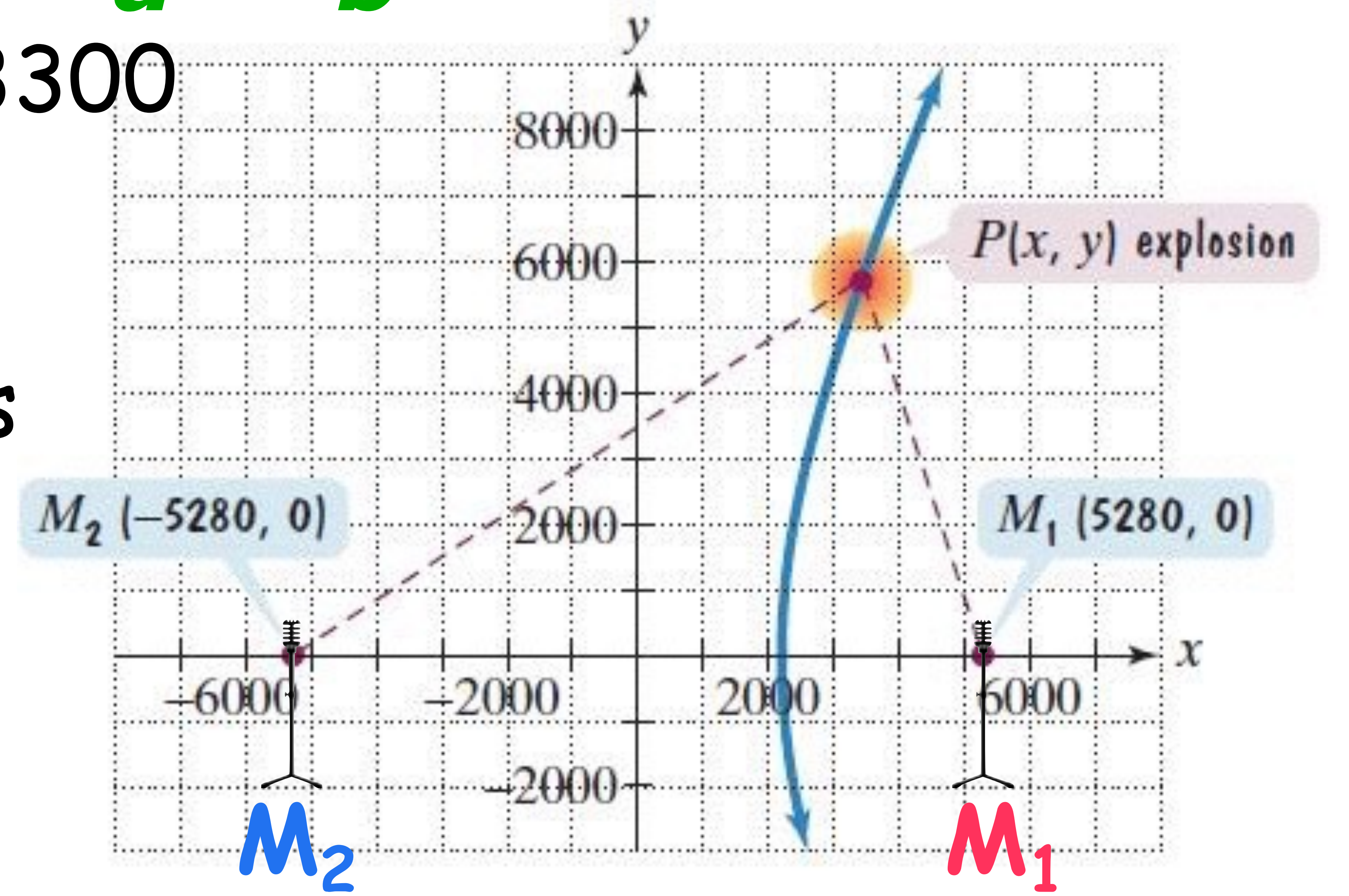


# Application

We will use the standard form of the hyperbola's equation. P(x, y), the explosion point, lies on this hyperbola. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The differences between the distances is 3300 feet (3 sec). Thus, 2a = 3300 and a = 1650.

The distance from the center to each focus is 5280 ft (1 mi). Thus c = 5280.




$b^2 = c^2 - a^2 = 5280^2 - 1650^2 = 25155900$

$$\frac{x^2}{2722500} - \frac{y^2}{25155900} = 1$$

The explosion occurred somewhere on the branch of this hyperbola closest to $M_1$.

# General Equations for Conics

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is

A circle … when $A = C$

An ellipse … when $AC > 0$

A parabola … when $AC = 0$

A hyperbola … when $AC < 0$

Identify

$4x^2 + 5y^2 - 9x + 8y = 0$ ellipse ($AC = 20$)

$2x^2 - 5x + 7y - 8 = 0$ parabola ($AC = 0$)

$7x^2 + 7y^2 - 9x + 8y - 16 = 0$ circle ($A = C$)

$4x^2 - 5y^2 - x + 8y + 1 = 0$ hyperbola ($AC < 0$)