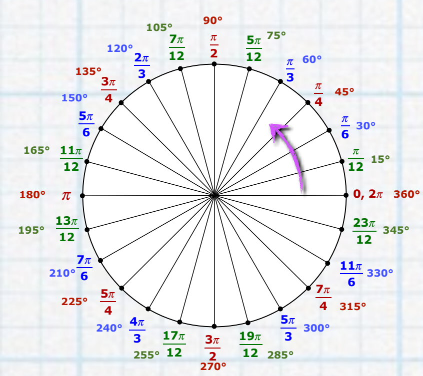
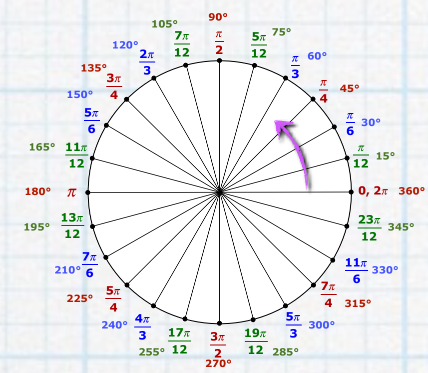


Chapter 10

Topics in Analytic Geometry

10.8 Graphs of Polar Equations

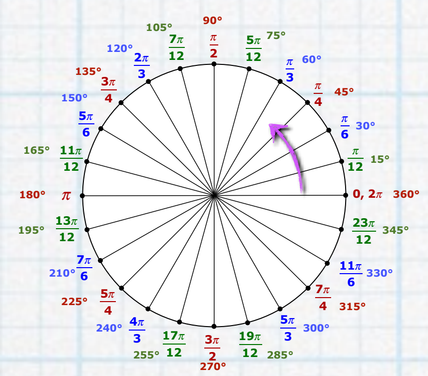


Homework

Read Sec 10.8

Complete Reading Notes

Do p791 7, 9, 11, 13, 21, 25, 33, 37, 39, 43, 47



Objectives

- Use point plotting to graph polar equations.
- Use symmetry to graph polar equations.

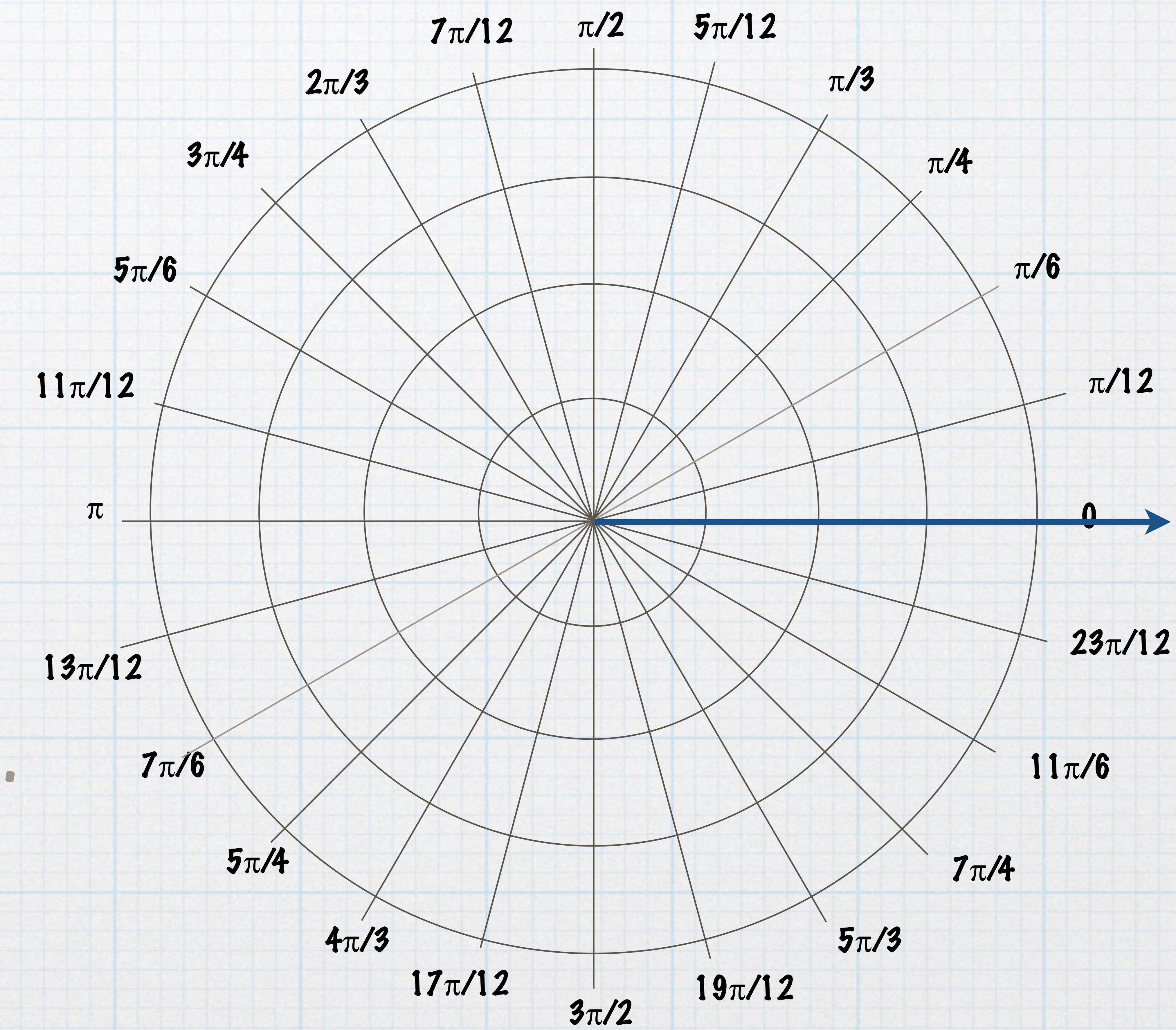
Using Polar Grids to Graph

- Use point plotting and symmetry to graph polar equations.

- A **polar equation** is an equation whose variables are **r** and **θ**. The **graph** of a polar equation is the set of all points whose polar coordinates satisfy the equation.

- We graph **polar equations** on a **polar grid.**

- As I have told you many times, the one certain way to graph any equation is by using a table of values. Polar equations are no exception.



Graphing an Equation by Plotting Points

- Use point plotting and symmetry to graph polar equations.

- Graph the equation $r = 4 \sin \theta$ with θ in radians.

- Use multiples of $\frac{\pi}{6}$ from 0 to π to generate coordinates for points.

- We construct a partial table of values for $r = 4 \sin \theta$ using multiples of $\frac{\pi}{6}$. Then we plot the points and join them in a smooth curve.

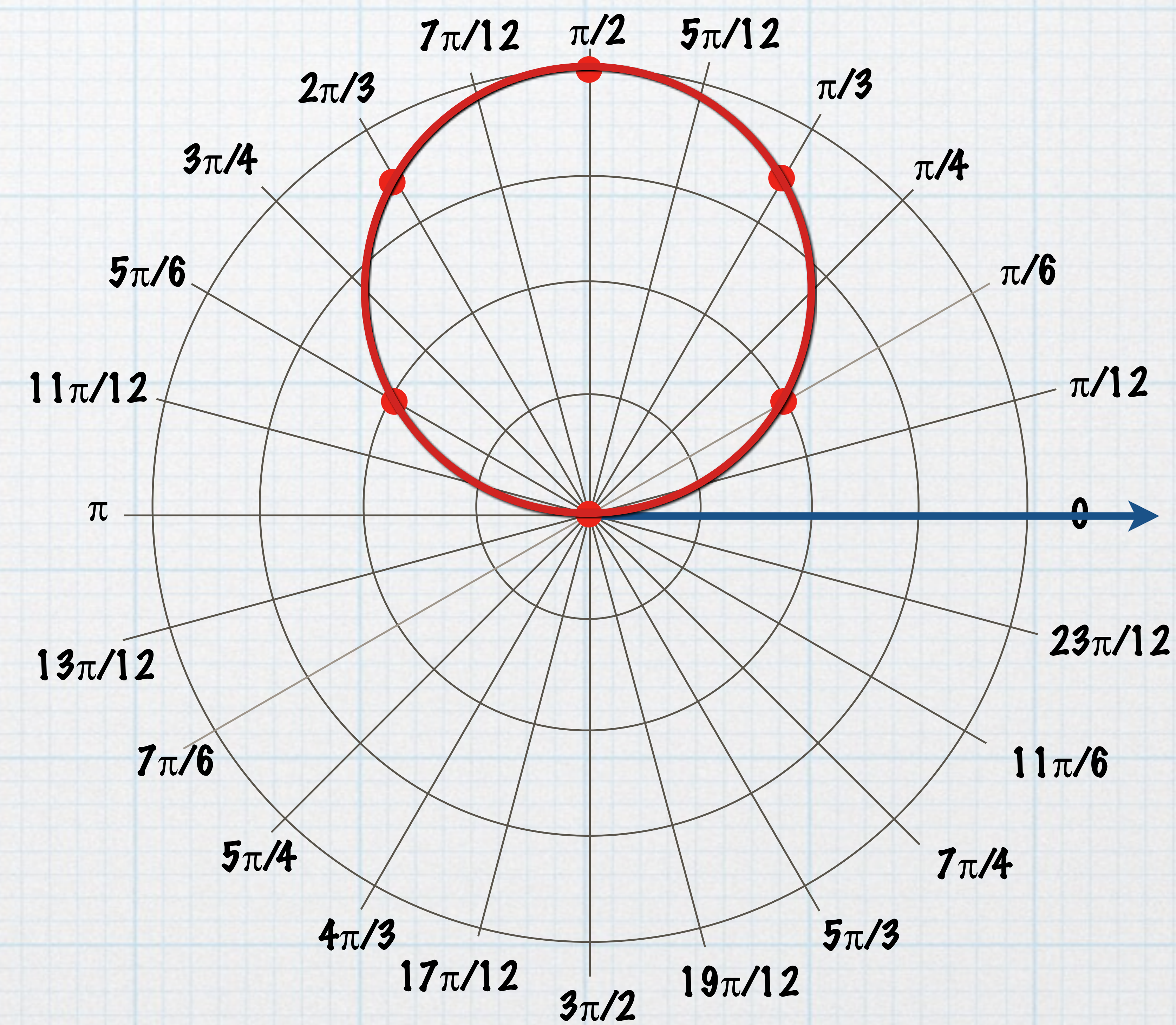
θ	$r = 4 \sin \theta$	(r, θ)
0		
$\frac{\pi}{6}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		
$\frac{2\pi}{3}$		
$\frac{5\pi}{6}$		
π		

Graphing an Equation by Plotting Points

- Use point plotting and symmetry to graph polar equations.

θ	$r = 4 \sin \theta$	(r, θ)
0	$4 \sin 0 = 4(0)$	$(0, 0)$
$\frac{\pi}{6}$	$4 \sin \frac{\pi}{6} = 4\left(\frac{1}{2}\right)$	$\left(2, \frac{\pi}{6}\right)$
$\frac{\pi}{3}$	$4 \sin \frac{\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right)$	$\left(2\sqrt{3}, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$4 \sin \frac{\pi}{2} = 4(1)$	$\left(4, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$4 \sin \frac{2\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right)$	$\left(2\sqrt{3}, \frac{2\pi}{3}\right)$
$\frac{5\pi}{6}$	$4 \sin \frac{5\pi}{6} = 4\left(\frac{1}{2}\right)$	$\left(2, \frac{5\pi}{6}\right)$
π	$4 \sin \pi = 4(0)$	$(0, \pi)$

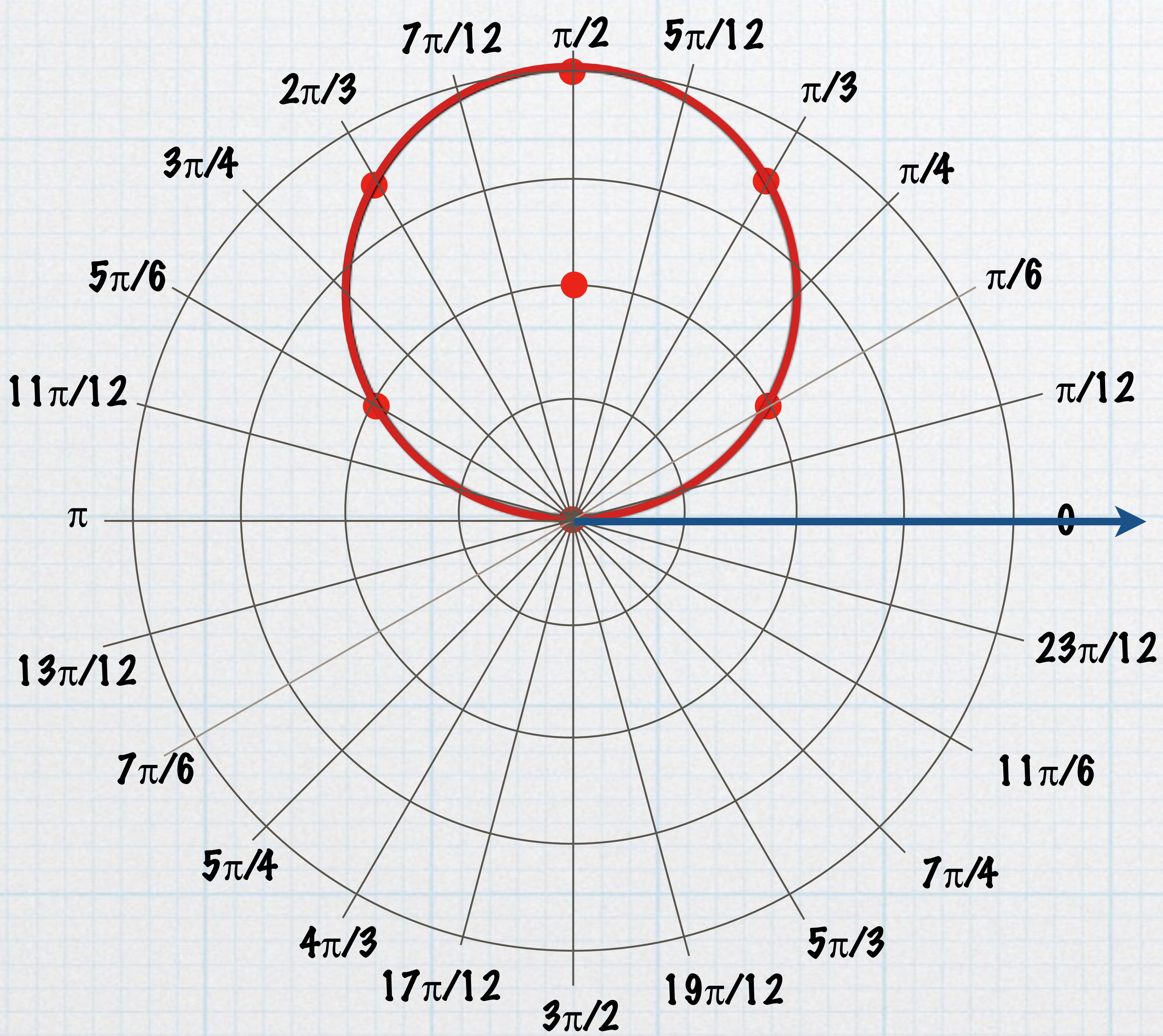
- Graph the equation $r = 4 \sin \theta$ with θ in radians.



Graphing an Equation by Plotting Points

Use point plotting and symmetry to graph polar equations.

Graph the equation $r = 4 \sin \theta$ with θ in radians.



We can verify that the graph is a circle by changing from polar form to rectangular form.

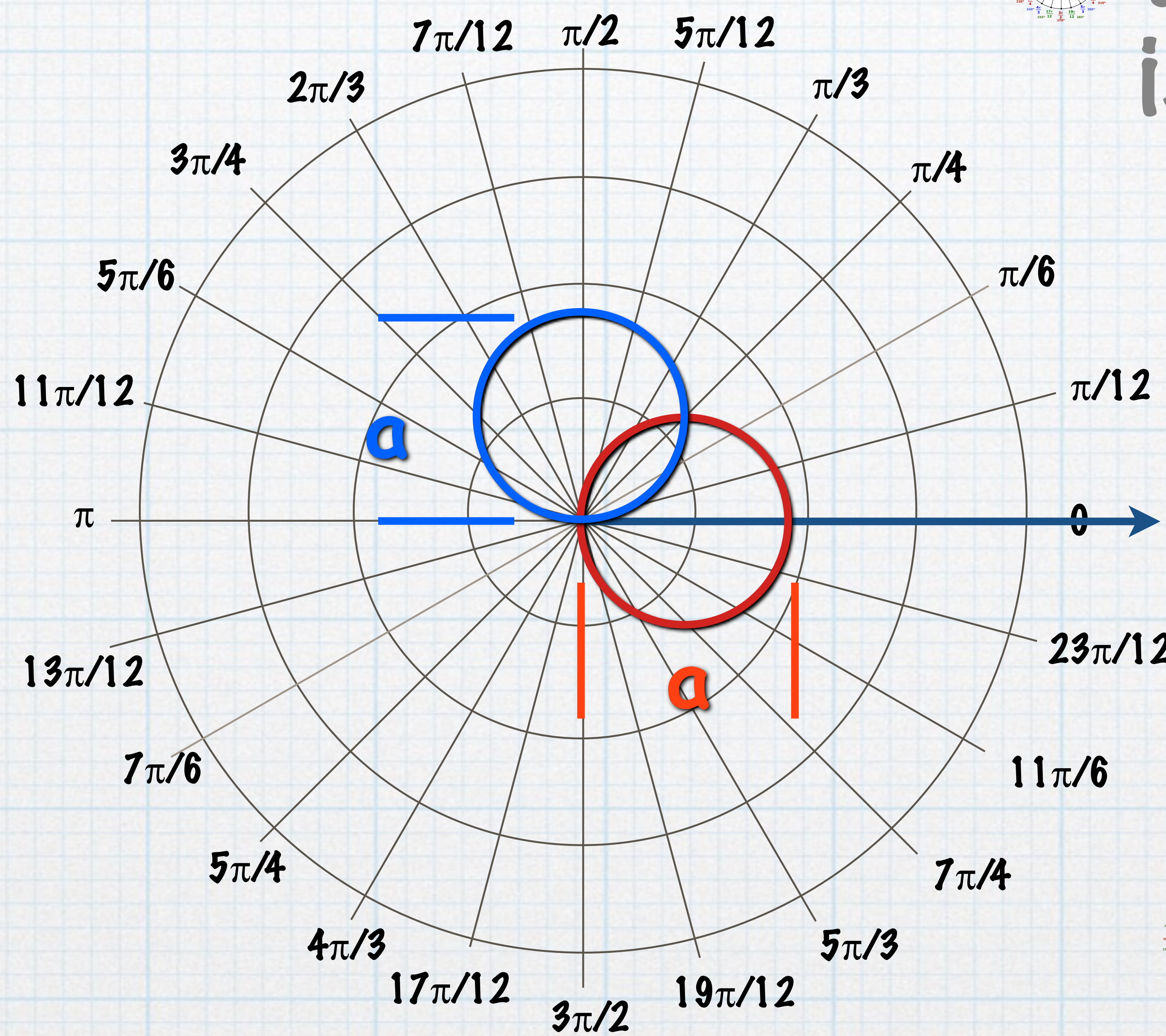
$$r = 4 \sin \theta$$
$$r^2 = 4r \sin \theta$$
$$x^2 + y^2 = 4y$$
$$x^2 + y^2 - 4y + 4 = 4$$
$$x^2 + (y - 2)^2 = 4$$
$$C(0, 2);\ r = 2$$

Using Polar Grids to Graph

- Use point plotting and symmetry to graph polar equations.

- The graph of **r = a sin θ** is a circle.

- The graph of **r = a cos θ** is a circle.



- The diameter of **r = a sin θ** is **a**.

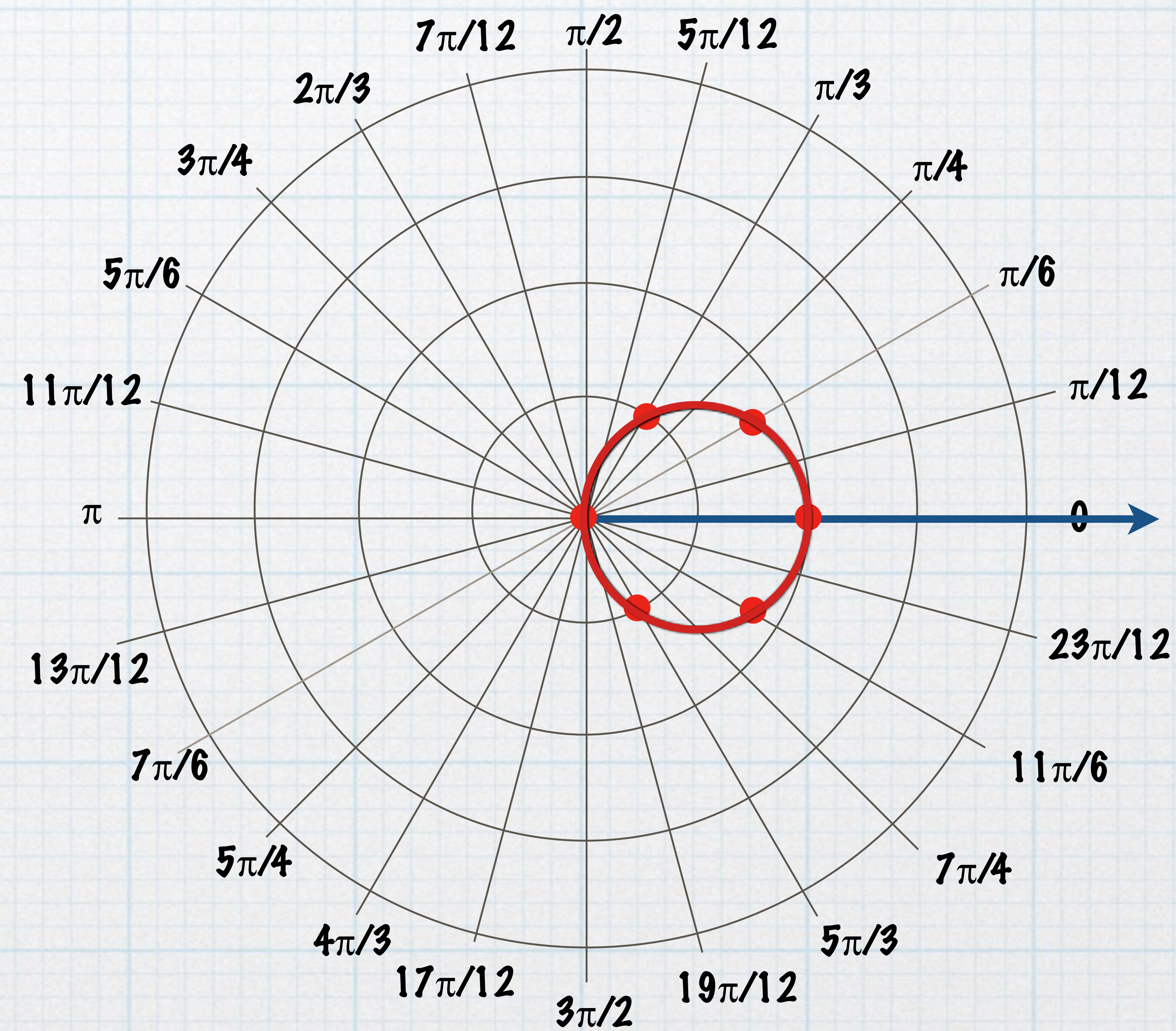
- The diameter of **r = a cos θ** is **a**.

Graphing an Equation by Plotting Points

Use point plotting and symmetry to graph polar equations.

- Graph the equation $r = 2$.
- $r = 2$ can be written $r = 2 \cos 0$

θ	$r = 2\cos\theta$	(r, θ)
0	$2\cos 0 = 2(1)$	$(2, 0)$
$\frac{\pi}{6}$	$2\cos\frac{\pi}{6} = 2\left(\frac{\sqrt{3}}{2}\right)$	$\left(\sqrt{3}, \frac{\pi}{6}\right)$
$\frac{\pi}{3}$	$2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right)$	$\left(1, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$2\cos\frac{\pi}{2} = 2(0)$	$\left(0, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right)$	$\left(-1, \frac{2\pi}{3}\right)$
$\frac{5\pi}{6}$	$2\cos\frac{5\pi}{6} = 2\left(-\frac{\sqrt{3}}{2}\right)$	$\left(-\sqrt{3}, \frac{5\pi}{6}\right)$
π	$2\cos\pi = 2(-1)$	$(-2, \pi)$



Tests for Symmetry in Polar Coordinates

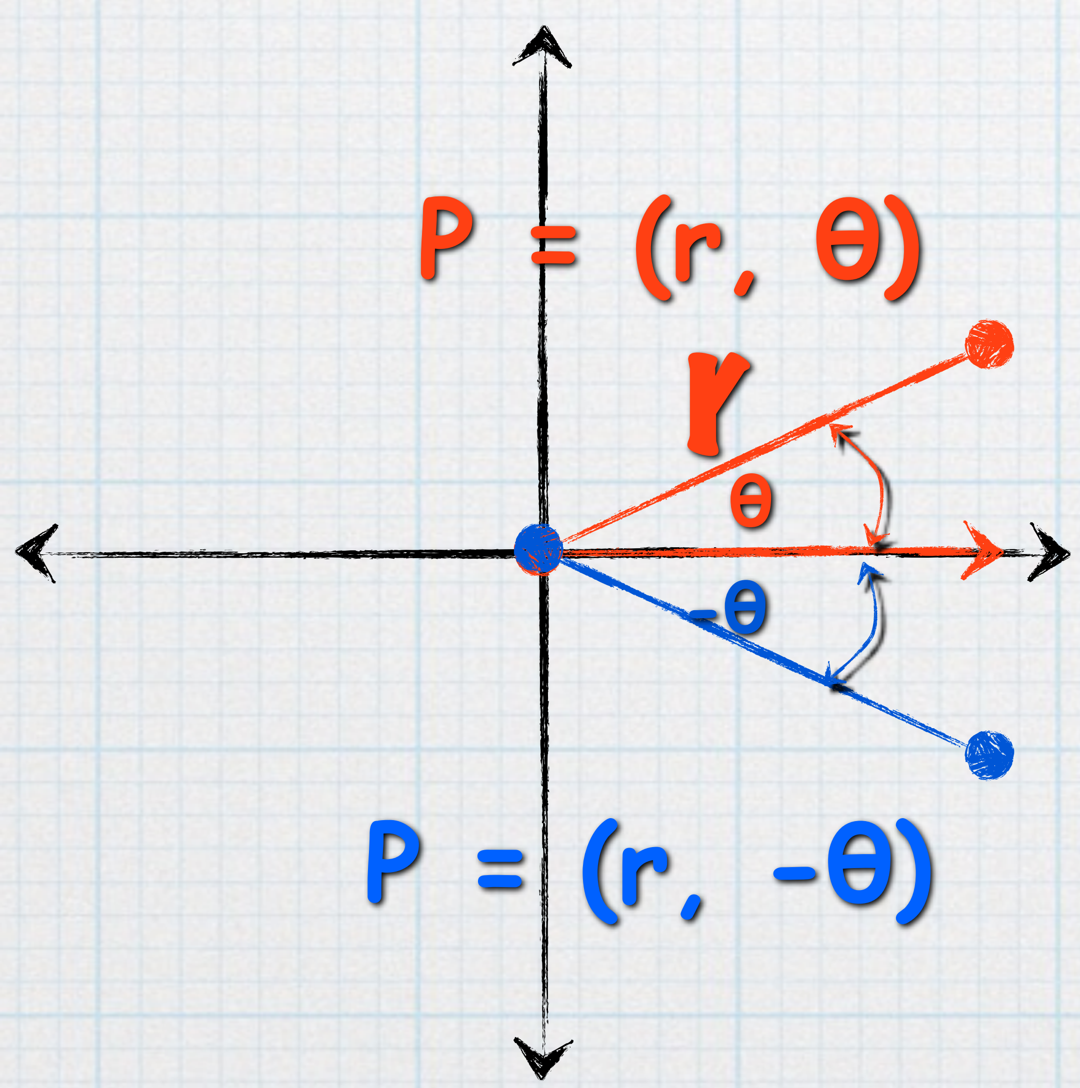
Symmetry

- The following three tests will indicate three types of symmetry in Polar Equations. The tests VERIFY the symmetry, or guarantee the symmetry, but failure of the tests does NOT guarantee lack of symmetry.

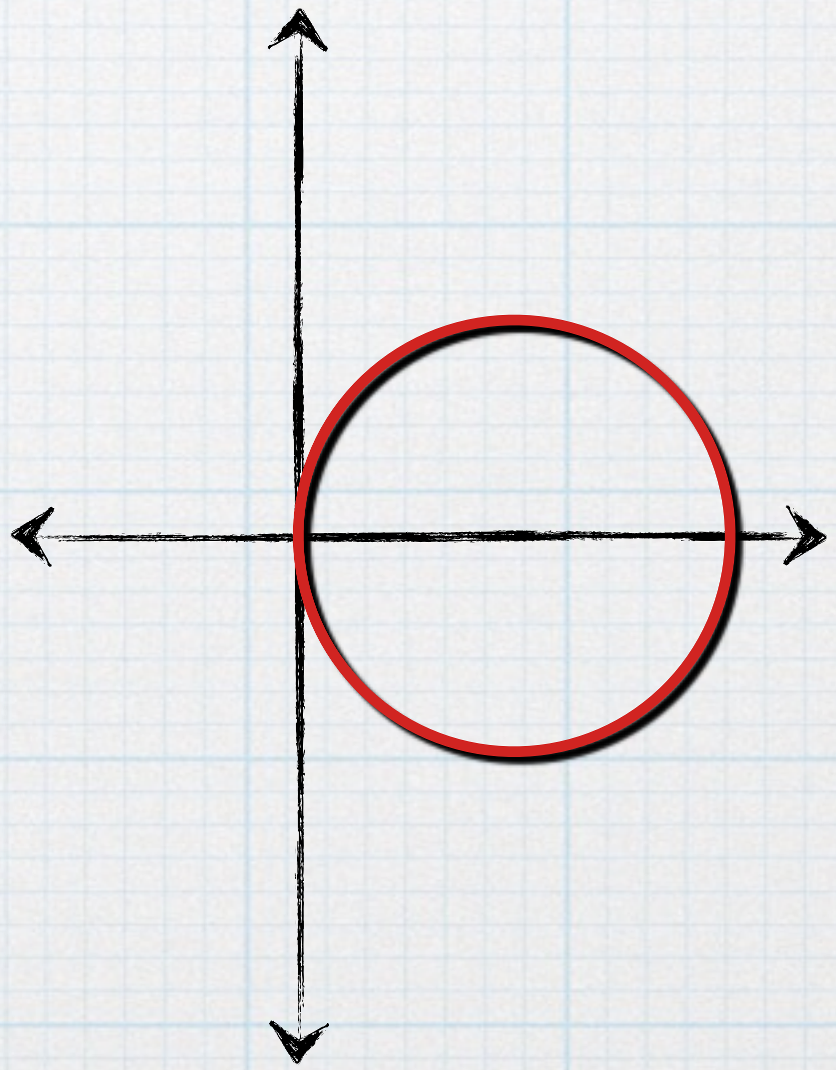
Tests for Symmetry in Polar Coordinates

- Use point plotting and symmetry to graph polar equations.

- Symmetry with respect to the Polar Axis.
- If we replace θ with $-\theta$, and we get the same graph, the graph is symmetric with respect to the polar axis.
  - By replacing θ with $-\theta$ we get an equivalent equation.

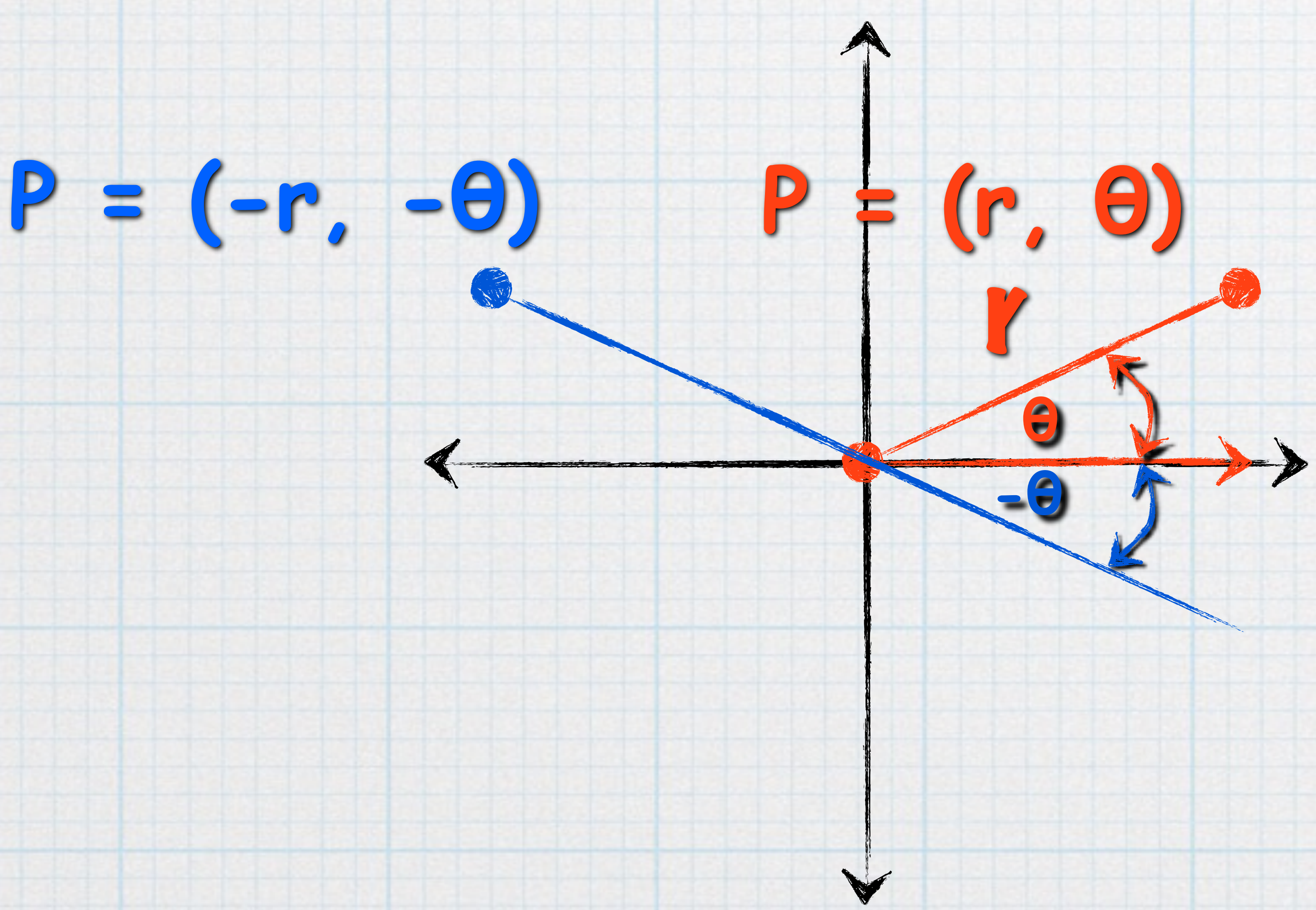


$$r = \cos\theta$$
$$r = \cos(-\theta)$$
$$\cos(-\theta) = \cos\theta$$
$$r = \cos\theta$$

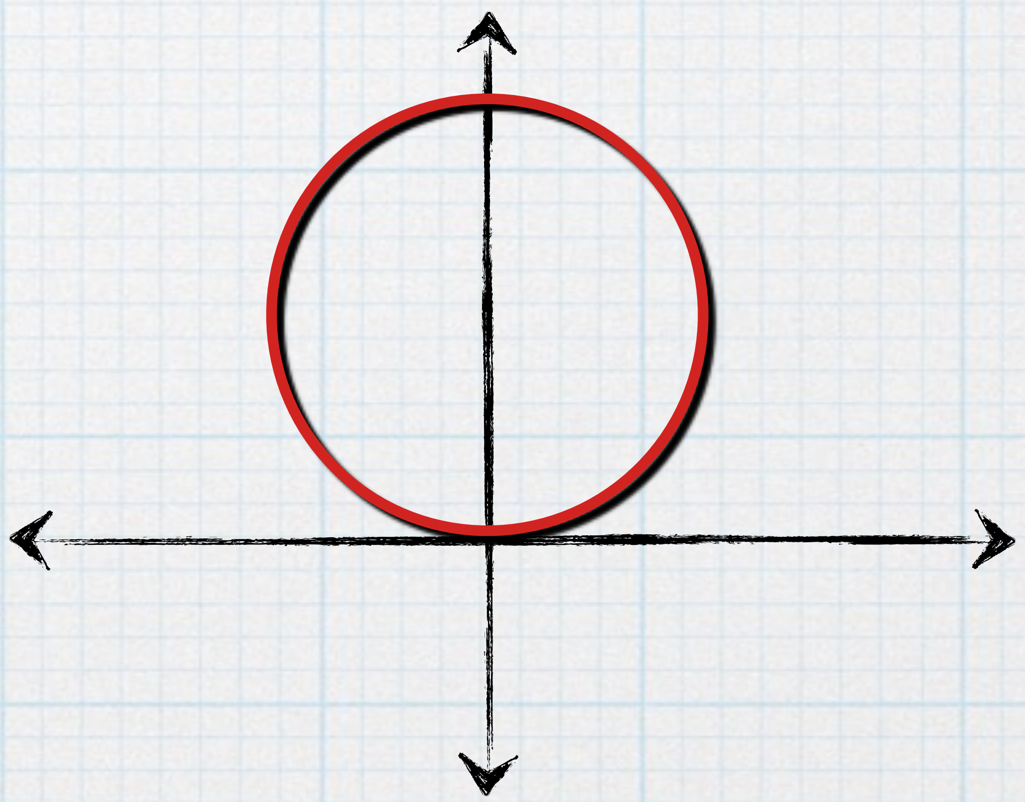


Tests for Symmetry in Polar Coordinates

- Symmetry with respect to the Y-Axis.
- If we replace θ with $-\theta$, and r with $-r$, and we get the same graph, the graph is symmetric with respect to the y-axis. By replacing (r, θ) with $(-r, -\theta)$ we get an equivalent equation.



$$r = \sin\theta$$
$$-r = \sin(-\theta)$$
$$\sin(-\theta) = -\sin\theta$$
$$-r = -\sin\theta$$
$$r = \sin\theta$$

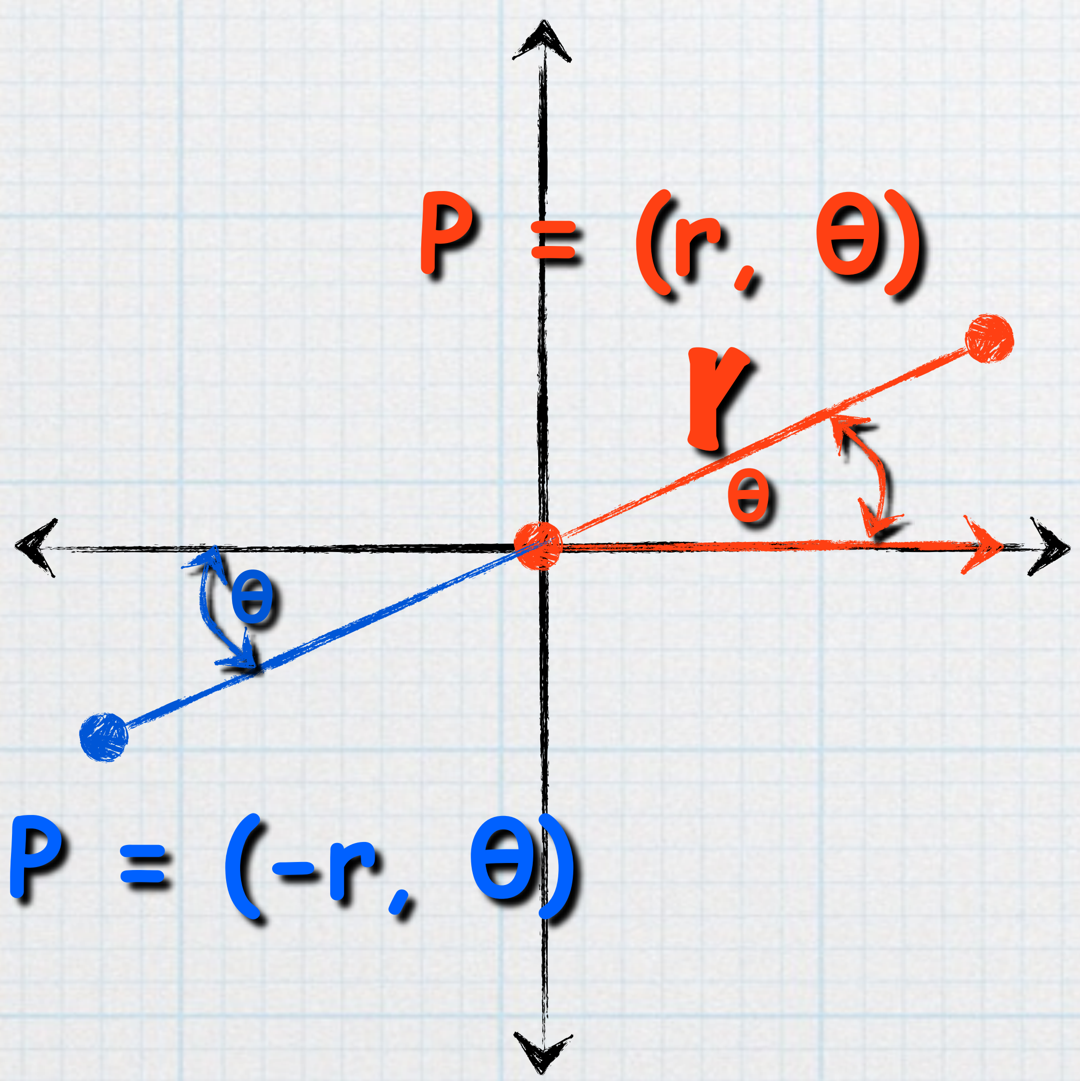


Tests for Symmetry in Polar Coordinates

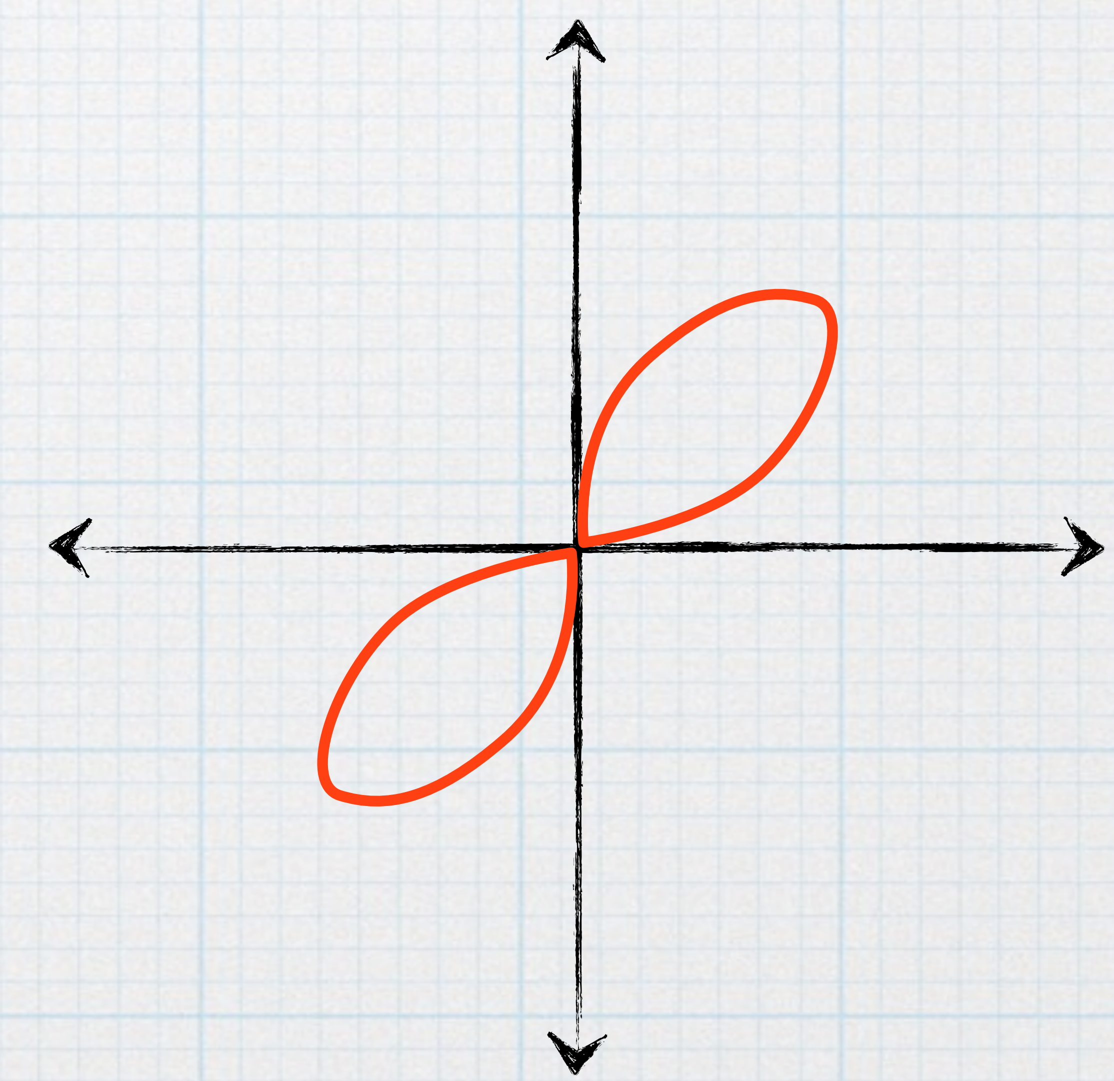
- Use point plotting and symmetry to graph polar equations.

- Symmetry with respect to the Origin (Pole).

- If we replace r with $-r$, and we get the same graph, the graph is symmetric with respect to the origin. By replacing (r, θ) with $(-r, \theta)$ we get an equivalent equation.



$$r^2 = \sin\theta$$
$$(-r)^2 = \sin\theta$$
$$(-r)^2 = r^2$$
$$r^2 = \sin\theta$$



Symmetry

Use point plotting and symmetry to graph polar equations.

- Symmetry with respect to the Polar Axis
  - Replace (r, θ) with $(r, -\theta)$ or $(-r, \pi-\theta)$ results in an equivalent equation.
- Symmetry with respect to the Y-Axis ($\theta = \pi/2$).
  - replacing (r, θ) with $(-r, -\theta)$ or $(r, \pi-\theta)$ we get an equivalent equation
- Symmetry with respect to the Origin (Pole).
  - replacing (r, θ) with $(-r, \theta)$ or $(r, \pi+\theta)$ we get an equivalent equation
- It is important to understand that there will always be symmetry of some sort, EVEN IF ALL THREE TESTS FAIL. Graphing the equation will verify symmetry.

0 and Maximum Value

- Use point plotting and symmetry to graph polar equations.

- There is another aid in graphing polar equations that is the result of what we know about trigonometric functions $y = \cos\theta$ and $y = \sin\theta$. We know the range of $\cos\theta$ and $\sin\theta$ are both $[-1, 1]$.
  - We can find points at which $|r|$ is maximum and points at which $r = 0$.
  - For the polar equation, $r = 2\sin\theta$, we find
    - Maximum value of r: $r = 2\sin\frac{\pi}{2} = 2$ $\quad \left(2, \frac{\pi}{2}\right)$
    - $r = 0$ $\quad r = 2\sin\theta = 0$

      $\theta = 0, \pi$ $\quad (0,0);(0,\pi)$

Example: Graphing a Polar Equation Using Symmetry

Use point plotting and symmetry to graph polar equations.

Check for symmetry and then graph the polar equation $r = 1 + \cos\theta$:

First replace θ with $-\theta$.

$r = 1 + \cos(-\theta)$ $\quad \cos(-\theta) = \cos\theta$ $\quad r = 1 + \cos\theta$

Symmetric with respect to the x (polar) axis.

Replace θ with $-\theta$, and r with $-r$

$-r = 1 + \cos(-\theta)$ $\quad \cos(-\theta) = \cos\theta$ $\quad r = -1 - \cos\theta$

Not symmetric with respect to the y-axis. $\left(\theta = \frac{\pi}{2}\right)$

Replace r with $-r$

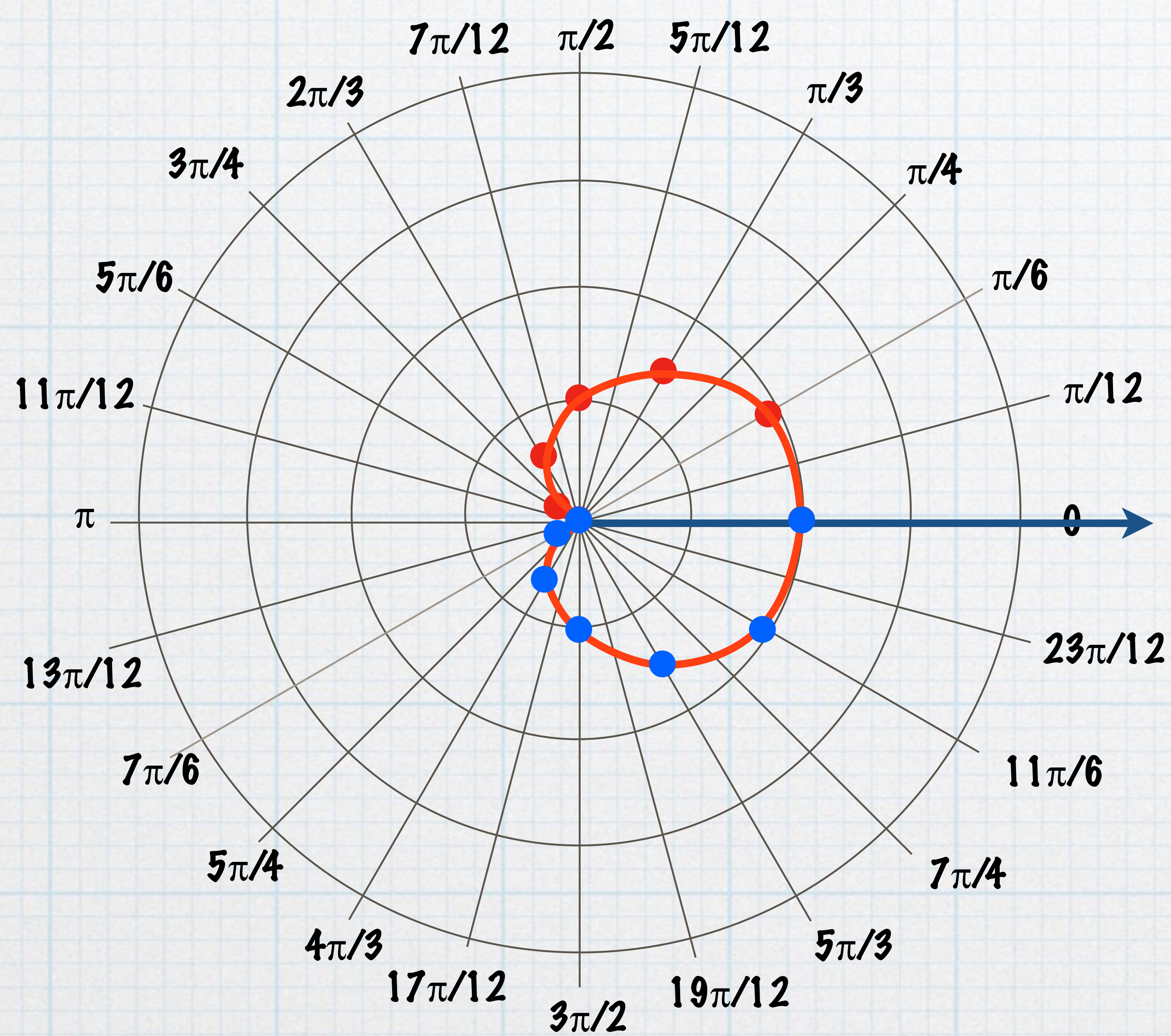
$-r = 1 + \cos\theta$ $\quad r = -1 - \cos\theta$

Not symmetric with respect to the origin (pole).

Example: Graphing a Polar Equation Using Symmetry

- Use point plotting and symmetry to graph polar equations.

- Graph the equation $r = 1 + \cos\theta$ with θ in radians.



θ	$r = 1+\cos\theta$	(r, θ)
0	$1+\cos 0 = 1+1$	$(2, 0)$
$\frac{\pi}{6}$	$1+\cos\frac{\pi}{6} = 1+\frac{\sqrt{3}}{2}$	$\left(1.87, \frac{\pi}{6}\right)$
$\frac{\pi}{3}$	$1+\cos\frac{\pi}{3} = 1+\frac{1}{2}$	$\left(1\frac{1}{2}, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$1+\cos\frac{\pi}{2} = 1+0$	$\left(1, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$1+\cos\frac{2\pi}{3} = 1+\left(-\frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{2\pi}{3}\right)$
$\frac{5\pi}{6}$	$1+\cos\frac{5\pi}{6} = 1+\left(-\frac{\sqrt{3}}{2}\right)$	$\left(.13, \frac{5\pi}{6}\right)$
π	$1+\cos\pi = 1+(-1)$	$(0, \pi)$

- Fill in the rest with symmetry about the polar axis.

Example: Graphing a Polar Equation Using Symmetry

Use point plotting and symmetry to graph polar equations.

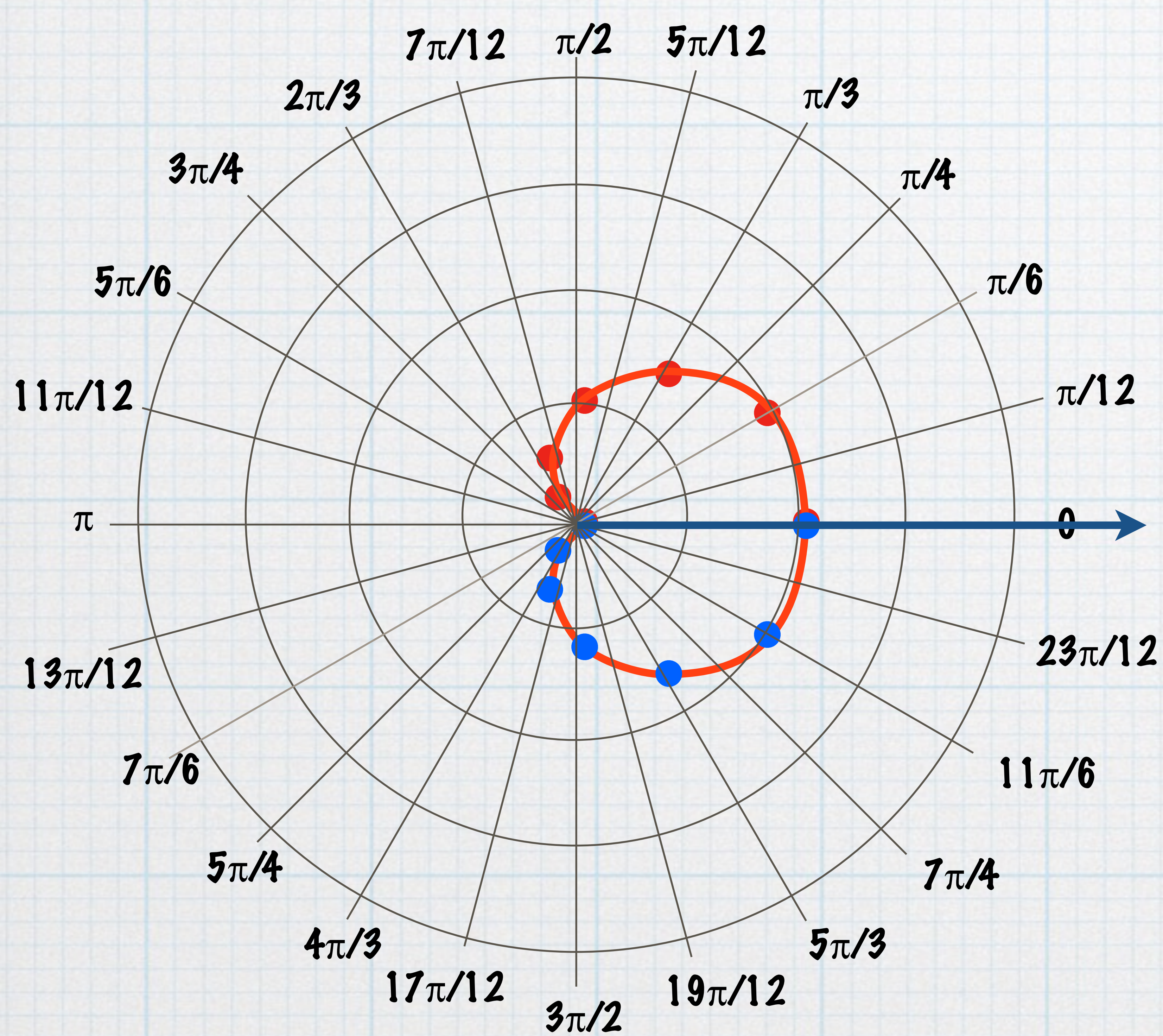
θ	r = 1+cosθ	(r, θ)
0	$1+\cos 0 = 1+1$	$(2, 0)$
$\frac{\pi}{6}$	$1+\cos\frac{\pi}{6} = 1+\frac{\sqrt{3}}{2}$	$\left(1.87, \frac{\pi}{6}\right)$
$\frac{\pi}{3}$	$1+\cos\frac{\pi}{3} = 1+\frac{1}{2}$	$\left(1\frac{1}{2}, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$1+\cos\frac{\pi}{2} = 1+0$	$\left(1, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$1+\cos\frac{2\pi}{3} = 1+\left(-\frac{1}{2}\right)$	$\left(-\frac{1}{2}, \frac{2\pi}{3}\right)$
$\frac{5\pi}{6}$	$1+\cos\frac{5\pi}{6} = 1+\left(-\frac{\sqrt{3}}{2}\right)$	$\left(.13, \frac{5\pi}{6}\right)$
π	$1+\cos\pi = 1+(-1)$	$(0, \pi)$

θ	r = 1+cosθ	(r, θ)
0	$1+\cos 0 = 1+1$	$(2, 0)$
$-\frac{\pi}{6}$	$1+\cos\left(-\frac{\pi}{6}\right) = 1+\frac{\sqrt{3}}{2}$	$\left(1.87, -\frac{\pi}{6}\right)$
$-\frac{\pi}{3}$	$1+\cos\left(-\frac{\pi}{3}\right) = 1+\frac{1}{2}$	$\left(1\frac{1}{2}, -\frac{\pi}{3}\right)$
$-\frac{\pi}{2}$	$1+\cos\left(-\frac{\pi}{2}\right) = 1+0$	$\left(0, -\frac{\pi}{2}\right)$
$-\frac{2\pi}{3}$	$1+\cos\left(-\frac{2\pi}{3}\right) = 1+\left(-\frac{1}{2}\right)$	$\left(\frac{1}{2}, -\frac{2\pi}{3}\right)$
$-\frac{5\pi}{6}$	$1+\cos\left(-\frac{5\pi}{6}\right) = 1+\left(-\frac{\sqrt{3}}{2}\right)$	$\left(.13, -\frac{5\pi}{6}\right)$
$-\pi$	$1+\cos\pi = 1+(-1)$	$(0, -\pi)$

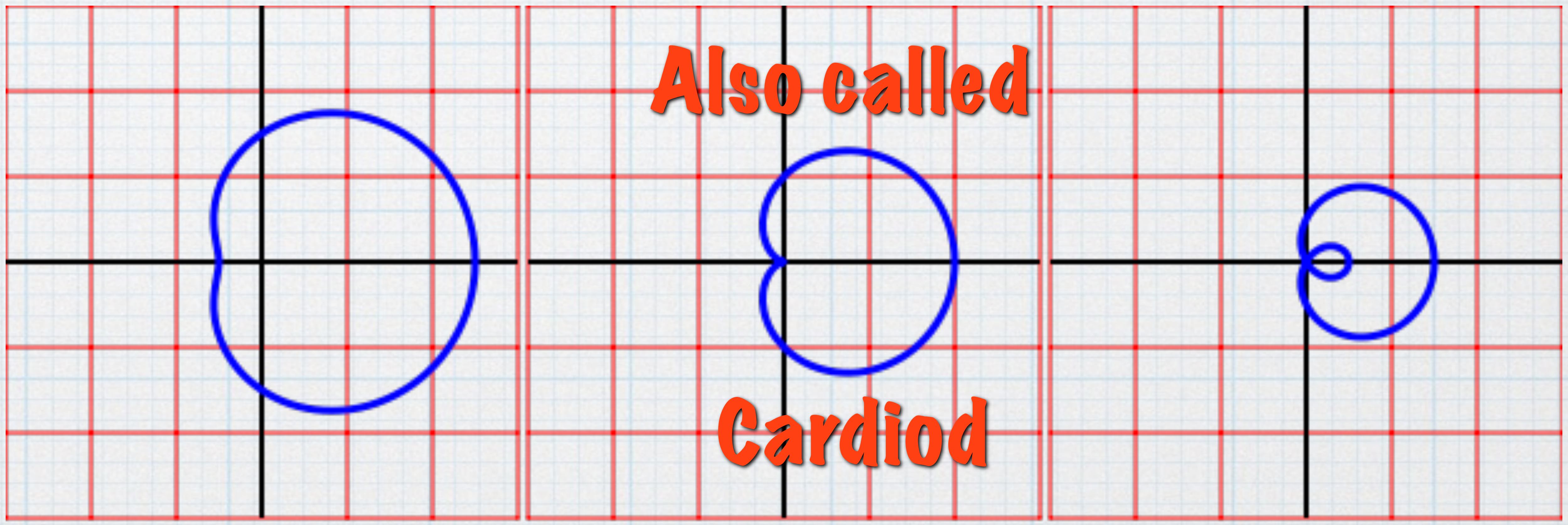
Limaçons

Use point plotting and symmetry to graph polar equations.

- The graph of $r = 1 + \cos\theta$ is an example of a **limaçon**.



The graphs of $r = a - b\sin\theta$, $r = a + b\sin\theta$, $r = a - b\cos\theta$, $r = a + b\cos\theta$ are **limaçons** ($a > 0$, $b > 0$).

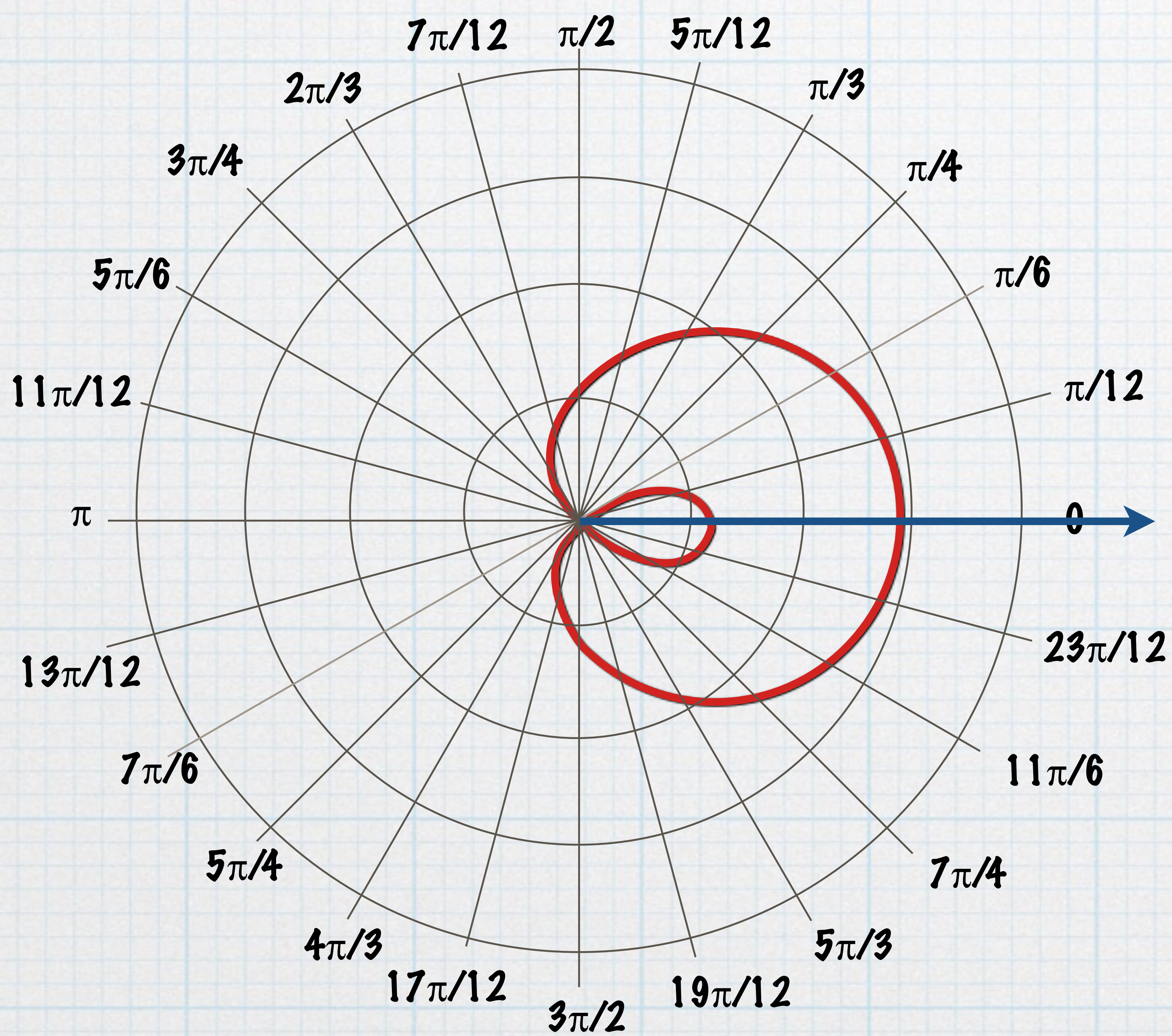


$r = 1.5 + \cos\theta$ $\quad$ $r = 1 + \cos\theta$ $\quad$ $r = 0.5 + \cos\theta$

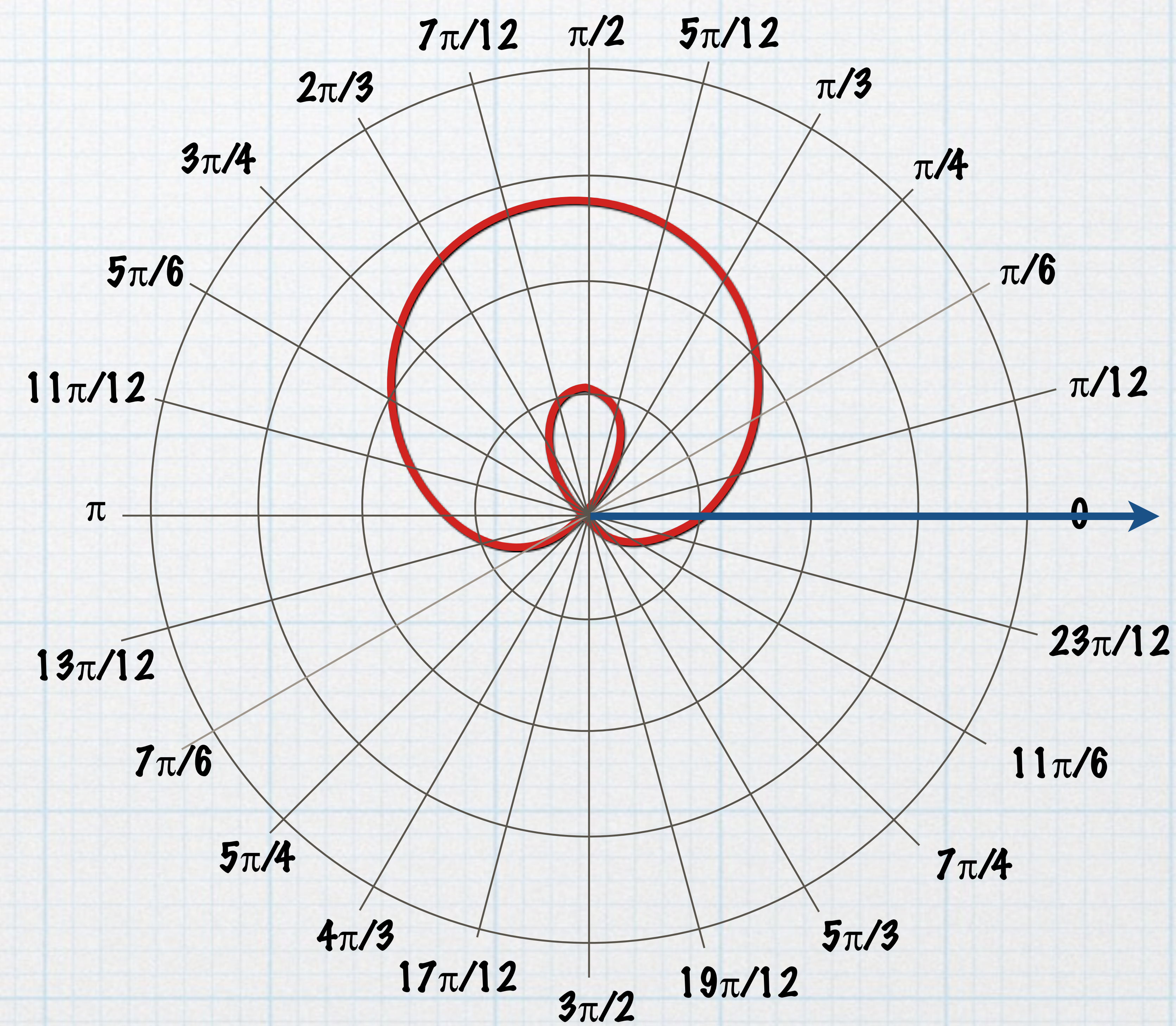
Limaçons

Use point plotting and symmetry to graph polar equations.

$r = a + b\cos\theta$



$r = a + b\sin\theta$



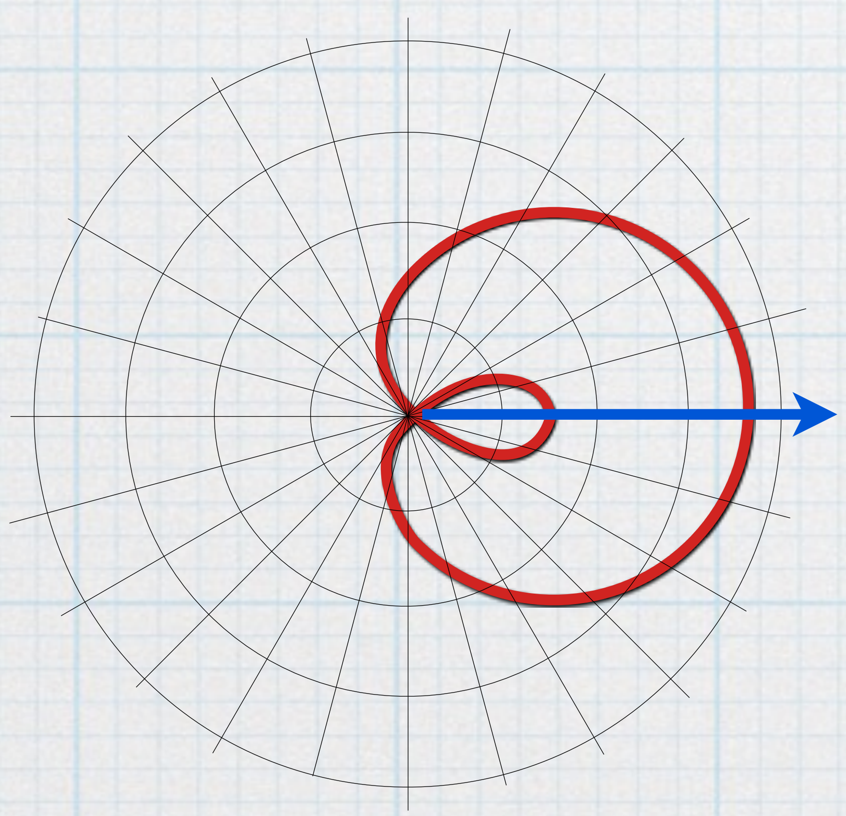
Limaçons

Use point plotting and symmetry to graph polar equations.

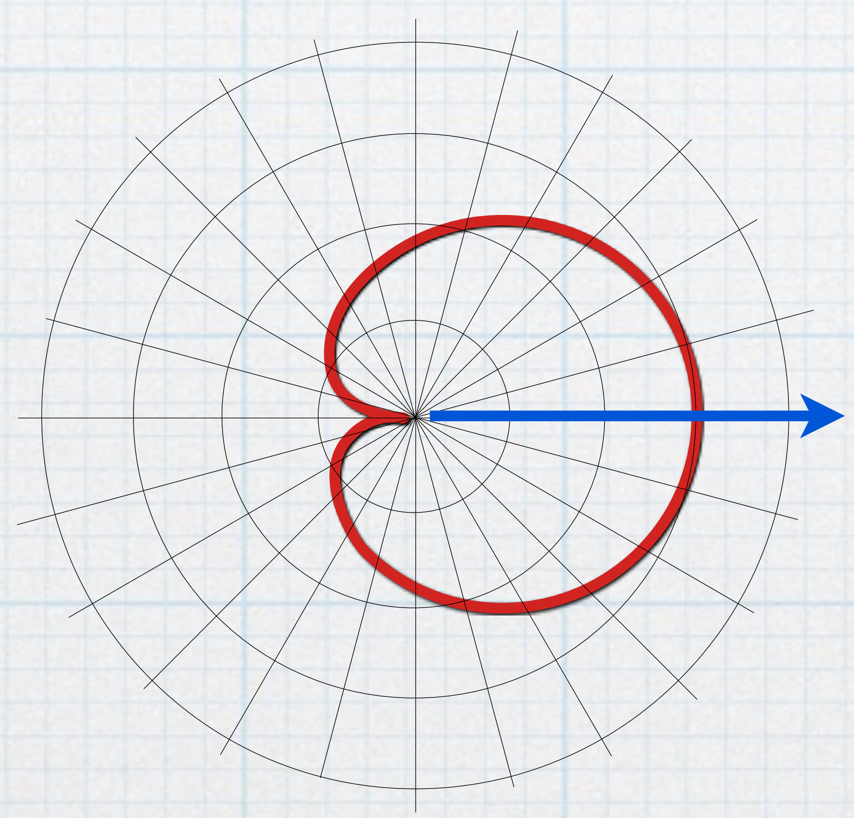
$r = a + b\cos\theta$

The shape of the limaçon is determined by the ratio $\frac{a}{b}$

If $\frac{a}{b} < 1$ you have an inner loop.

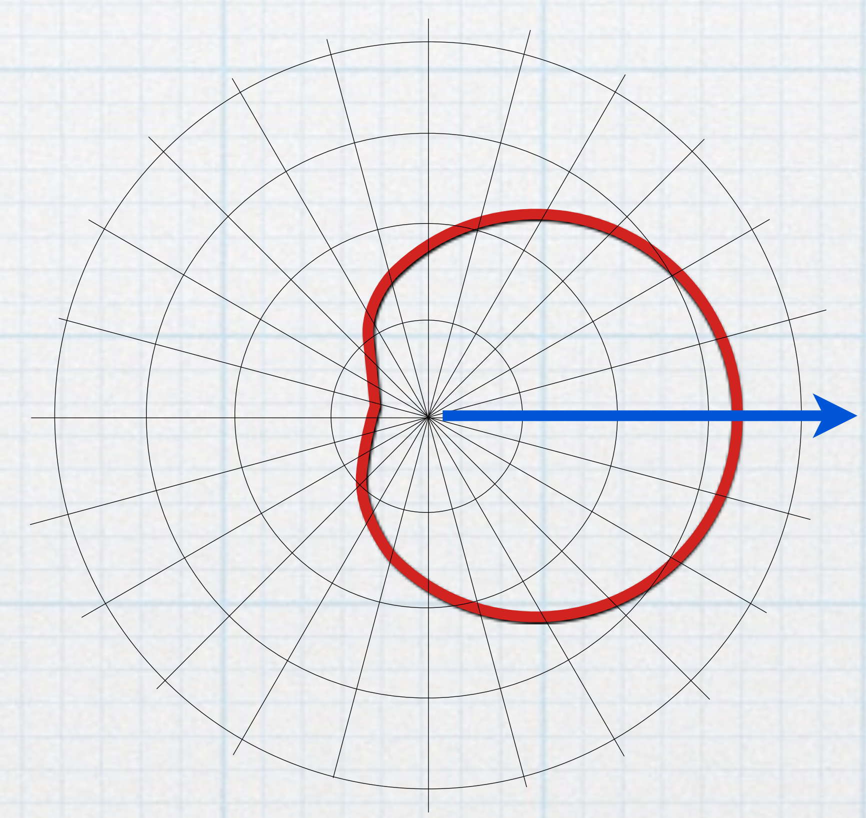


If $\frac{a}{b} = 1$ you have a heart shape.

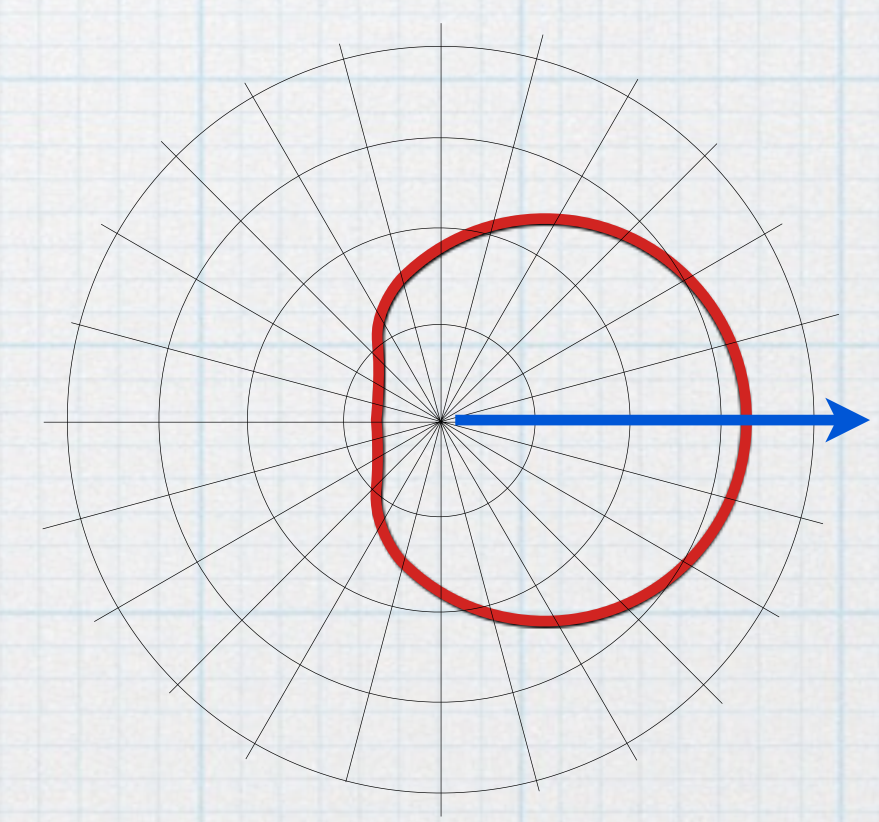


Also called a cardioid. (get it?)

If $1 < \frac{a}{b} < 2$ you have a dimple.



If $\frac{a}{b} \geq 2$ you have no dimple.



Example: Graphing a Polar Equation

Use point plotting and symmetry to graph polar equations.

Graph the polar equation: $r = 3\cos 2\theta$

First check for symmetry:

Replace θ with $-\theta$.

$r = 3\cos(-2\theta)$ $\quad \cos(-2\theta) = \cos 2\theta$ $\quad r = 3\cos 2\theta$

Symmetric with respect to the x (polar) axis.

Replace θ with $-\theta$, and r with $-r$

$-r = 3\cos(-2\theta)$ $\quad \cos(-2\theta) = \cos 2\theta$ $\quad r = -3\cos 2\theta$

Not symmetric with respect to the y-axis. $\left(\theta = \frac{\pi}{2}\right)$

Replace r with $-r$

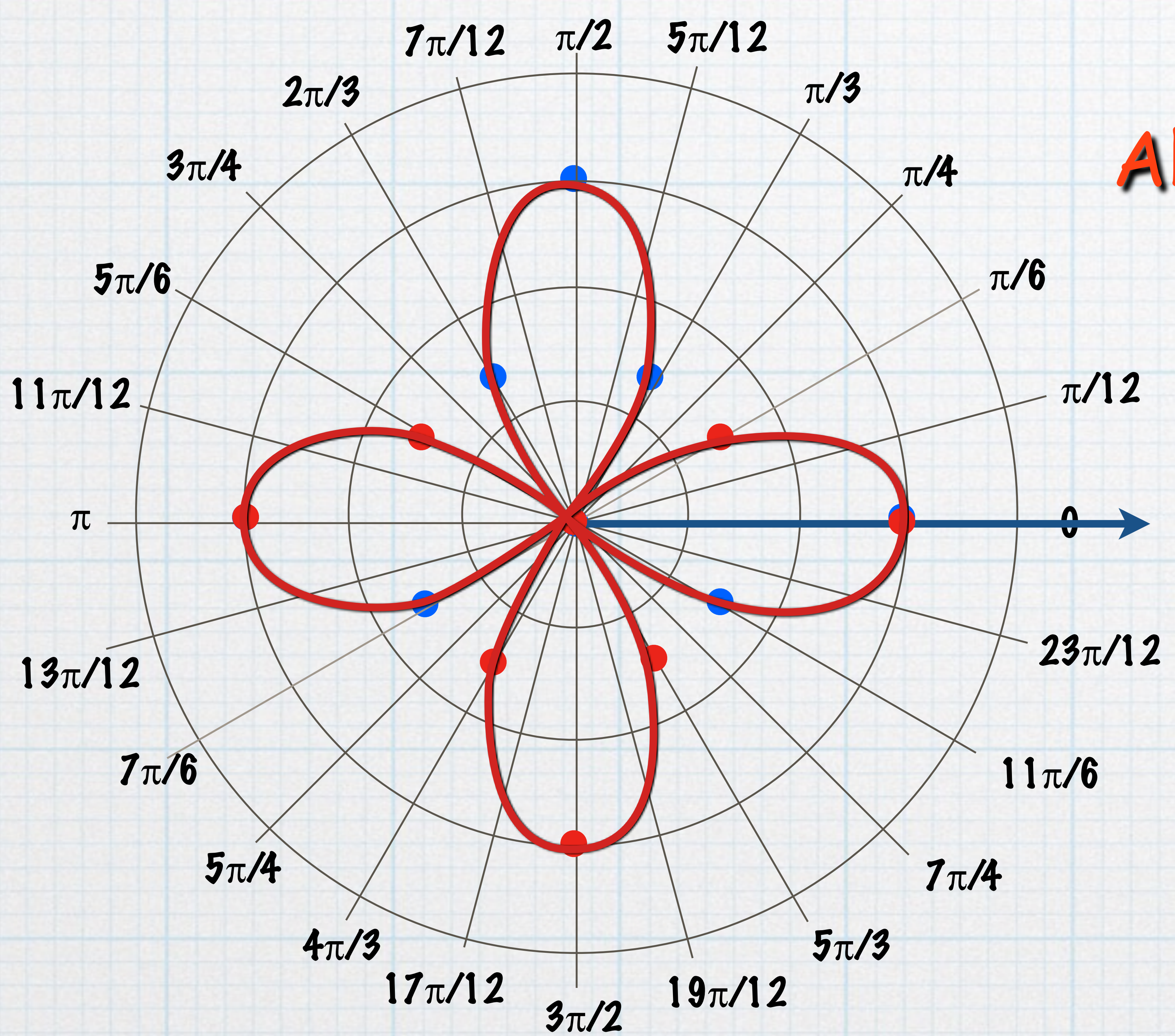
$-r = 3\cos\theta$ $\quad r = -3\cos 2\theta$

Not symmetric with respect to the origin (pole).

Example: Graphing a Polar Equation

- Use point plotting and symmetry to graph polar equations.

- Graph the polar equation: $r = 3\cos 2\theta$



Ahhh, a beautiful rose curve.

θ	$r = 3\cos 2\theta$	(r, θ)
0	$3\cos 2(0) = 3(1)$	$(3, 0)$
$\frac{\pi}{6}$	$3\cos\frac{2\pi}{6} = 3\left(\frac{1}{2}\right)$	$\left(\frac{3}{2}, \frac{\pi}{6}\right)$
$\frac{\pi}{3}$	$3\cos\frac{2\pi}{3} = 3\left(-\frac{1}{2}\right)$	$\left(-\frac{3}{2}, \frac{\pi}{3}\right)$
$\frac{\pi}{4}$	$3\cos\frac{2\pi}{4} = 3(0)$	$\left(0, \frac{\pi}{4}\right)$
$\frac{\pi}{2}$	$3\cos\frac{2\pi}{2} = 3(-1)$	$\left(-3, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$3\cos\frac{4\pi}{3} = 3\left(-\frac{1}{2}\right)$	$\left(-\frac{3}{2}, \frac{2\pi}{3}\right)$
$\frac{3\pi}{4}$	$3\cos\frac{6\pi}{4} = 3(0)$	$\left(0, \frac{3\pi}{4}\right)$
$\frac{5\pi}{6}$	$3\cos\frac{10\pi}{6} = 3\left(\frac{1}{2}\right)$	$\left(\frac{3}{2}, \frac{5\pi}{6}\right)$
π	$3\cos 2\pi = 3(1)$	$(3, \pi)$

- We could continue around the circle but we have symmetry about the polar axis.

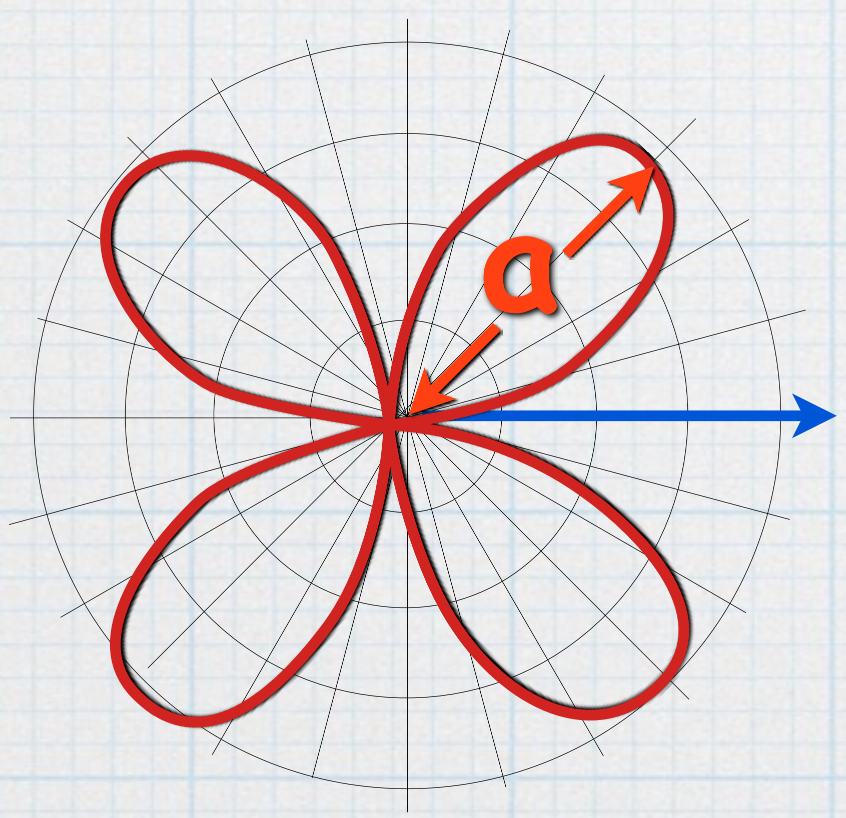
Rose Curves

Use point plotting and symmetry to graph polar equations.

- The graph of **r = a sin nθ, and r = a cos nθ** are called **rose curves**. The number of petals is determined by **n**. If **n** is **odd**, the rose has **n** petals. If **n** is **even**, the rose has **2n** petals..

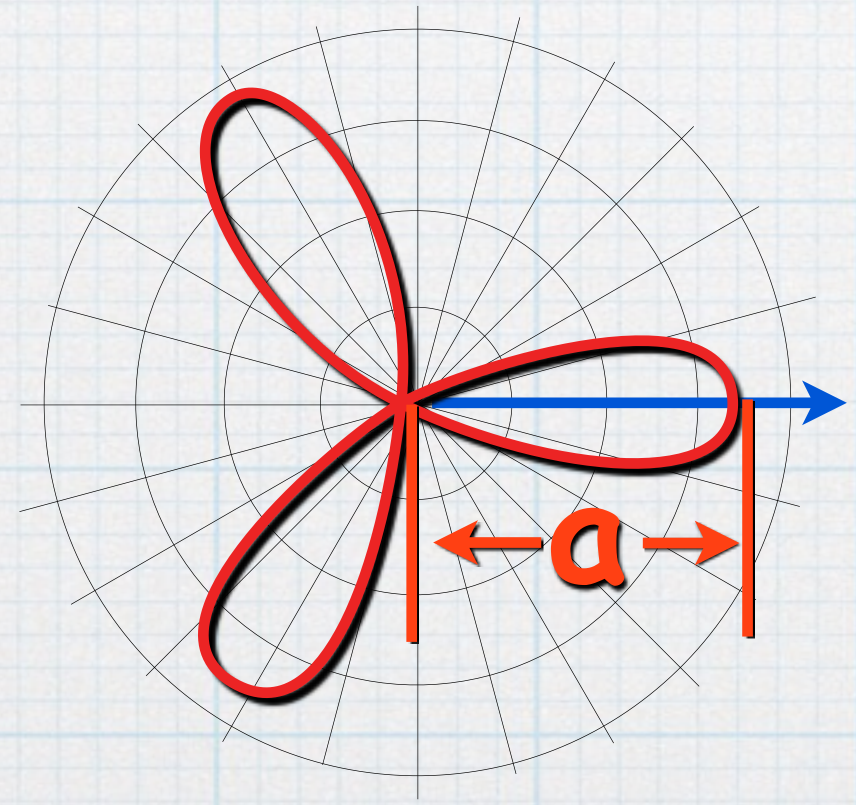
$r = a \sin 2\theta$

$n = 2$



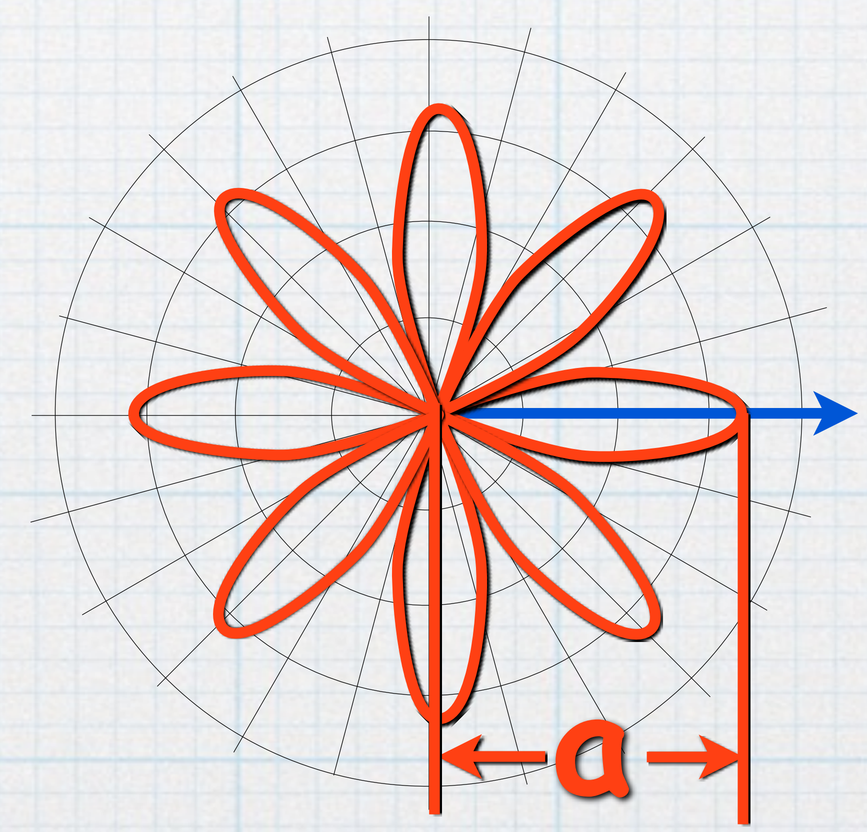
$r = a \cos 3\theta$

$n = 3$



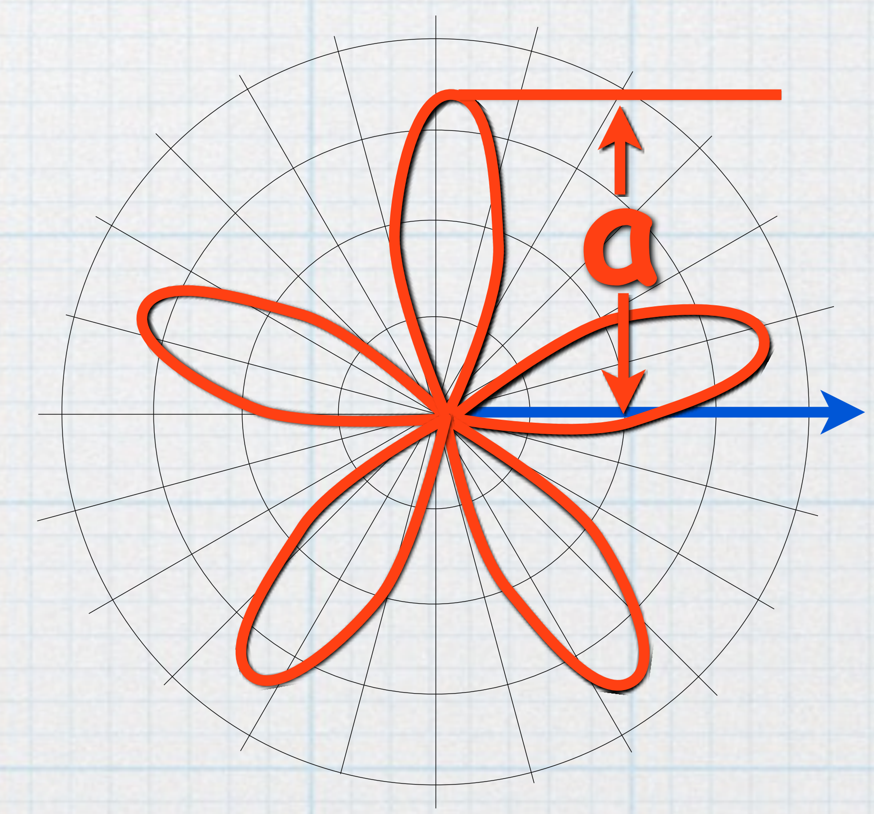
$r = a \cos 4\theta$

$n = 4$



$r = a \sin 5\theta$

$n = 5$



Rose Curves

Use point plotting and symmetry to graph polar equations.

Graph the polar equation: $r^2 = 4\cos 2\theta$

First check for symmetry:

Replace θ with $-\theta$.

$r^2 = 4\cos(-2\theta)$ $\quad \cos(-2\theta) = \cos 2\theta$ $\quad r^2 = 4\cos 2\theta$

Symmetric with respect to the x (polar) axis.

Replace θ with $-\theta$, and r with $-r$

$(-r)^2 = 4\cos(-2\theta)$ $\quad \cos(-2\theta) = \cos 2\theta$ $\quad (-r)^2 = r^2$ $\quad r^2 = 4\cos\theta$

Symmetric with respect to the y-axis. $\left(\theta = \frac{\pi}{2}\right)$

Replace r with $-r$

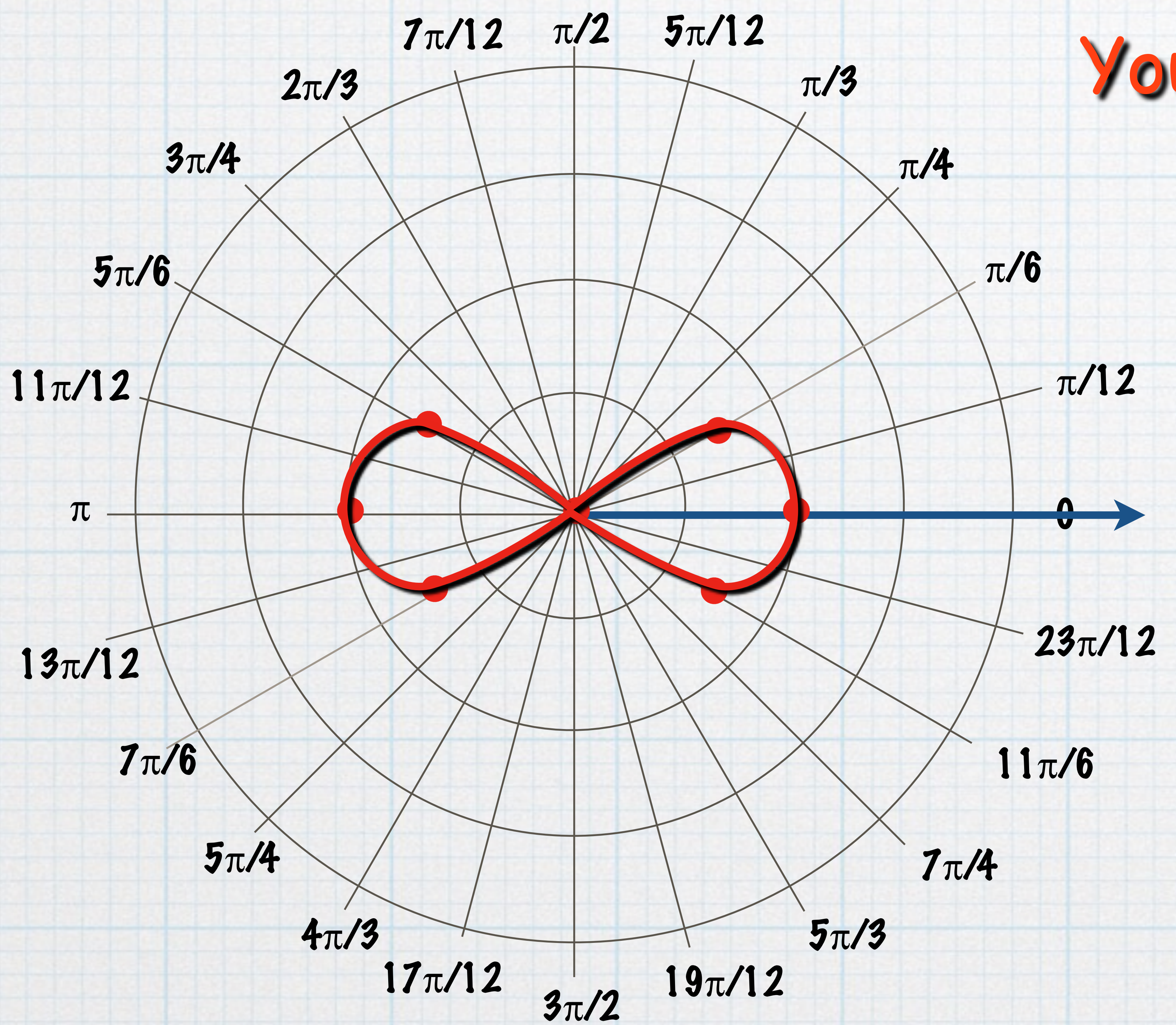
$(-r)^2 = 4\cos 2\theta$ $\quad r^2 = 4\cos 2\theta$

Symmetric with respect to the origin (pole).

Example: Graphing a Polar Equation

- Use point plotting and symmetry to graph polar equations.

- Graph the polar equation: $r^2 = 4 \cos 2\theta$



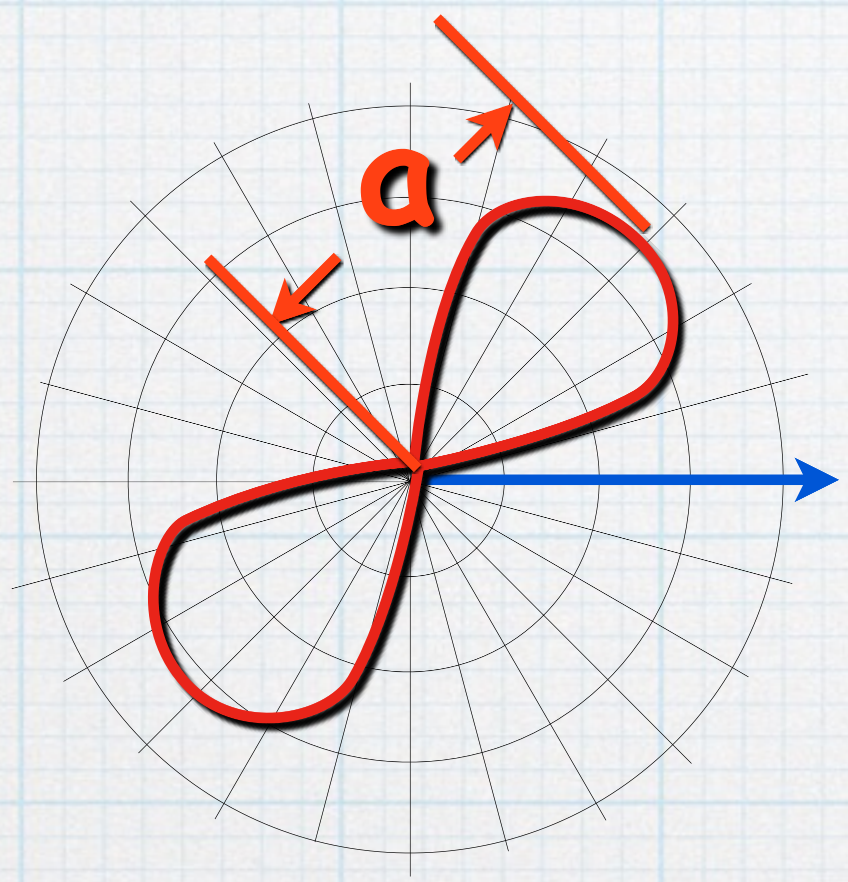
You cannot escape the lovely lemniscate.

θ	$r^2 = 4\cos 2\theta$	(r, θ)
0	$4\cos 2(0) = 4(1)$	$(\pm 2, 0)$
$\frac{\pi}{6}$	$4\cos\frac{2\pi}{6} = 4\left(\frac{1}{2}\right)$	$\left(\pm\sqrt{2}, \frac{\pi}{6}\right)$
$\frac{\pi}{3}$	$4\cos\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right)$	$\varnothing$
$\frac{\pi}{4}$	$4\cos\frac{2\pi}{4} = 4(0)$	$\left(0, \frac{\pi}{4}\right)$
$\frac{\pi}{2}$	$4\cos\frac{2\pi}{2} = 4(-1)$	$\varnothing$
$\frac{2\pi}{3}$	$4\cos\frac{4\pi}{3} = 4\left(-\frac{1}{2}\right)$	$\varnothing$
$\frac{3\pi}{4}$	$4\cos\frac{6\pi}{4} = 4(0)$	$\left(0, \frac{3\pi}{4}\right)$
$\frac{5\pi}{6}$	$4\cos\frac{10\pi}{6} = 4\left(\frac{1}{2}\right)$	$\left(\pm\sqrt{2}, \frac{5\pi}{6}\right)$
π	$4\cos 2\pi = 4(1)$	$(\pm 2, \pi)$

Lemniscates

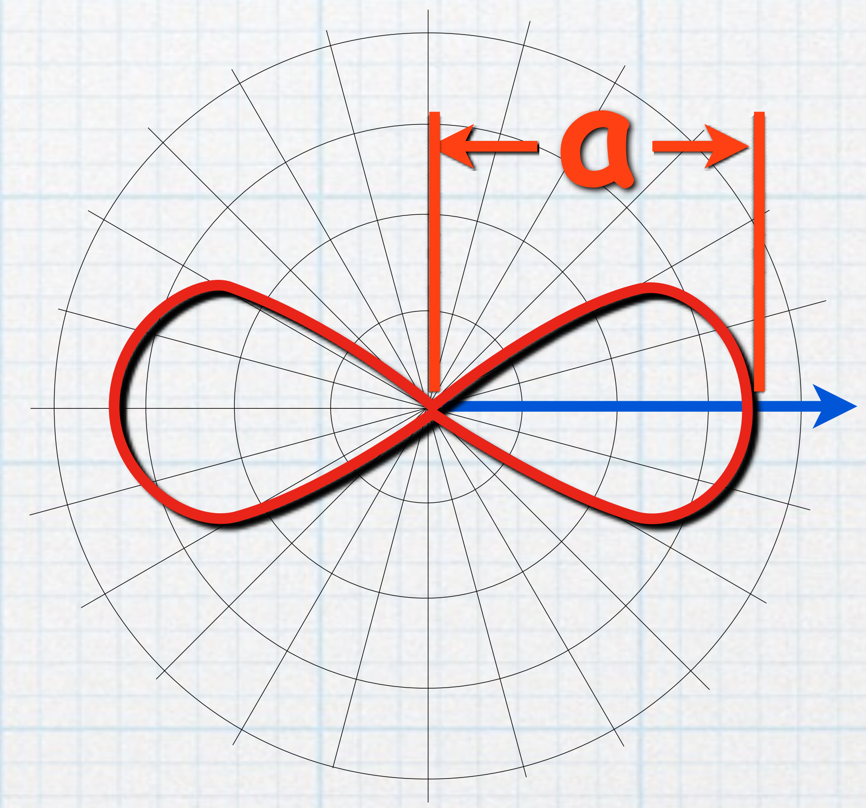
- The graphs of $r^2 = a^2 \sin 2\theta$ and $r^2 = a^2 \cos 2\theta$ are examples of **lemniscates**.

$$r^2 = a^2 \sin 2\theta$$



- Symmetric with respect to the pole.

$$r^2 = a^2 \cos 2\theta$$



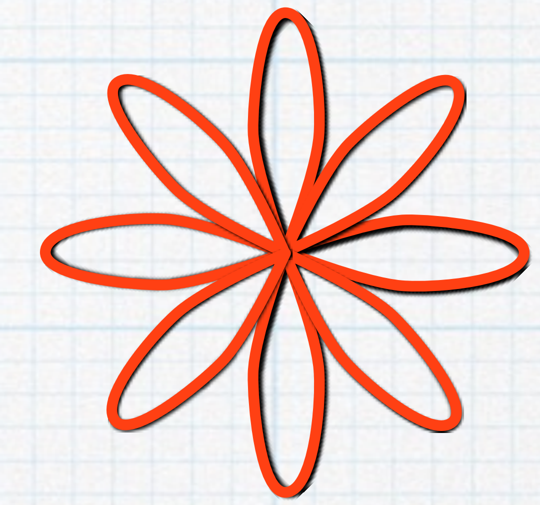
- Symmetric with respect to the polar axis, the y axis, and the pole.

Graphing Polar Equations

Use point plotting and symmetry to graph polar equations.

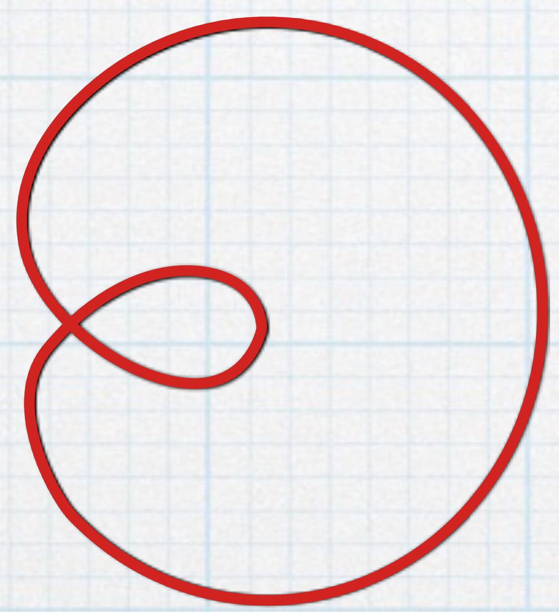
$r = a \sin \theta$
$r = a \cos \theta$ }

$r = a \sin n\theta$
$r = a \cos n\theta$



Rose

$r = a + b\cos \theta,$
$r = a - b\cos \theta,$
$r = a + b\sin \theta,$
$r = a - b\cos \theta$
$(a > 0, b > 0)$



limaçon

$r^2 = a^2 \sin 2\theta$
$r^2 = a^2 \cos 2\theta$

Lemniscate

1. Check symmetry
2. Table of Values

TI-84

As is most often the case, your very capable calculator can graph polar equations. This will require a little more instruction from you.

MODE ⋎ ➢ FUNCTION PARAMETRIC POLAR SEQ ENTER

While in MODE verify the angle measure radians or degrees.

y= \R1= Enter polar function 2 . 5 − 1 . 5 COS X,T,θ,n)

Zoom 4 or Zoom 5

Graphing circles and lemniscates requires graphing two equations ±√r.