Chapter 2

Polynomial Functions

2.2 Polynomial Functions and Their Graphs

Chapter 2

2.2 Polynomial Functions

2.2 p148 9, 13, 15, 23, 29, 33, 37, 41, 43, 53, 67, 87

Objectives

- Identify polynomial functions.
- Recognize characteristics of graphs of polynomial functions.
- Determine end behavior.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Use the Intermediate Value Theorem.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.

Definition of a Polynomial Function

- Let n be a nonnegative integer and let a_n, a_{n-1}, a_{n-2}, ..., a_2, a_1, a_0 be real numbers, with $a_n \neq 0$. The function defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

is called a polynomial function of **degree n**.

- The number a_n, the coefficient of the variable to highest power, is called the **leading coefficient**.

Graphs of Polynomial Functions – Smooth

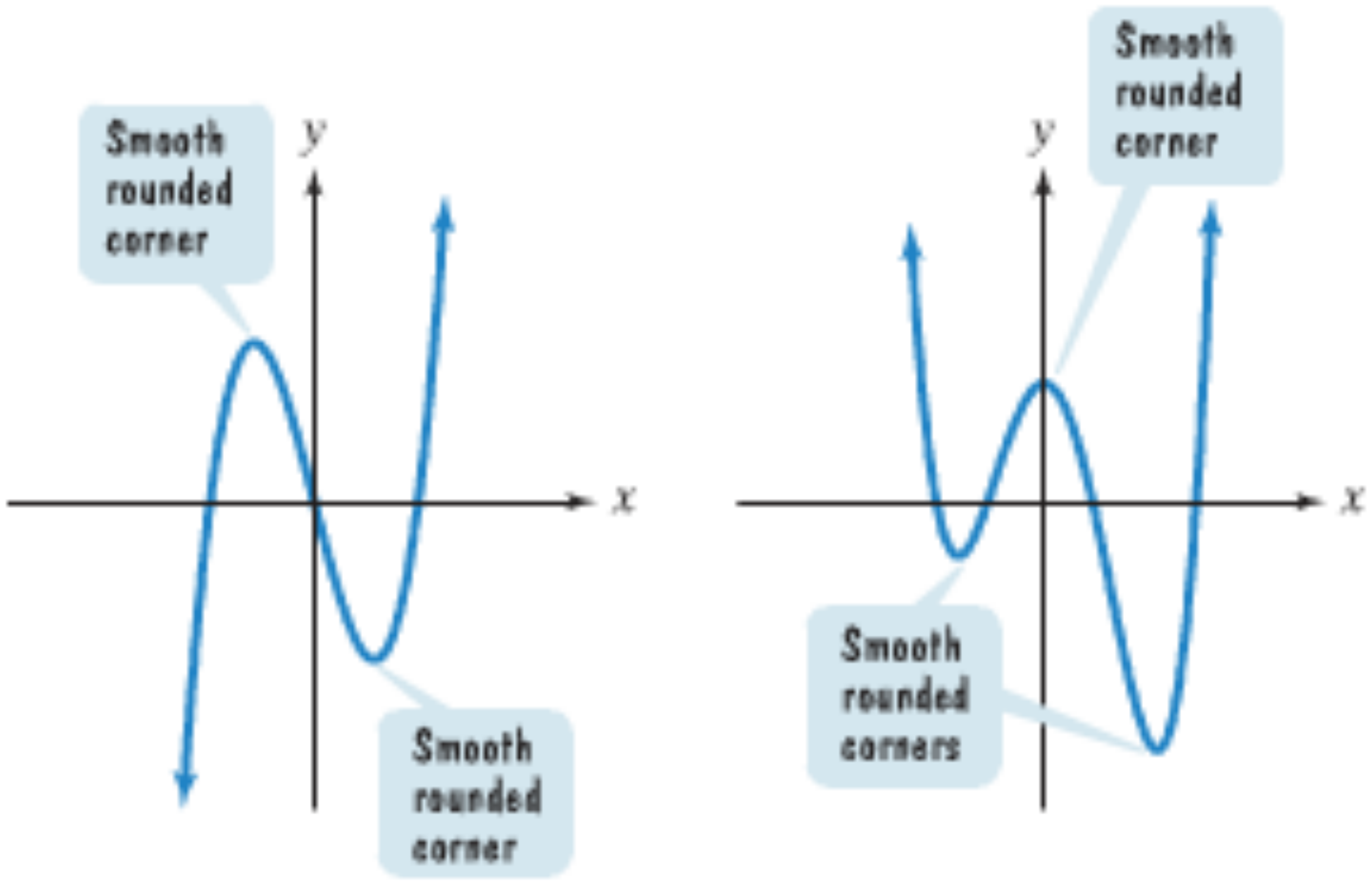
Polynomial functions of degree 2 or higher have graphs that are **smooth** and **continuous**.

By **smooth**, we mean that the graphs contain only rounded curves with no sharp corners.

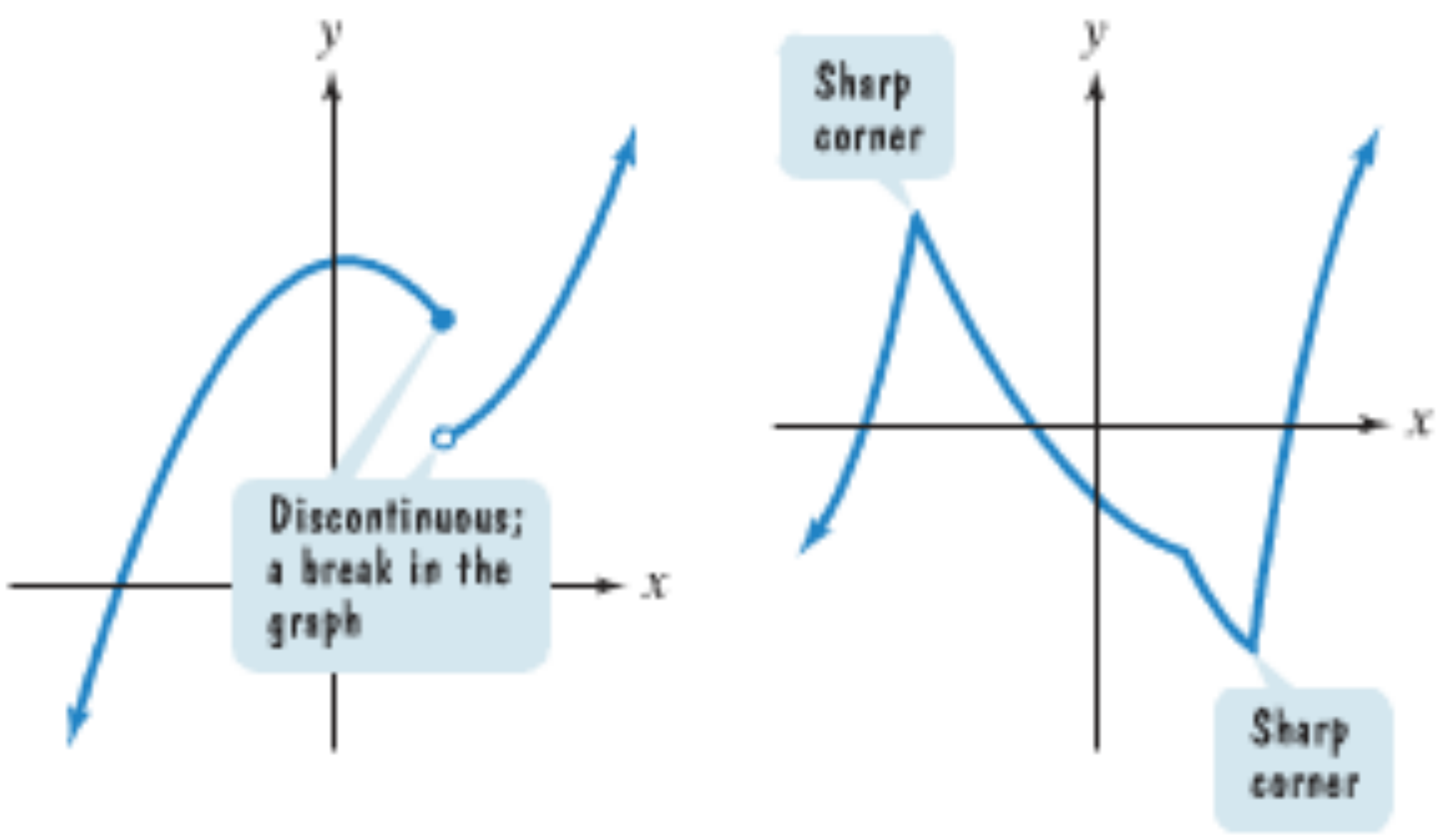
By **continuous**, we mean that the graphs have no breaks and can be drawn without lifting your pencil from the rectangular coordinate system.

Examples

Graphs of Polynomial Functions



Not Graphs of Polynomial Functions



Notice the breaks and lack of smooth curves.

End Behavior of Polynomial Functions

The tails of the graph of a function to the far left or the far right is called its end behavior.

Although the graph of a polynomial function may have intervals where it increases or decreases, the graph will eventually rise or fall without bound as it moves far to the left or far to the right.

The sign of the leading coefficient, a_n, and the degree, n, of the polynomial function reveal its end behavior.

Power Functions

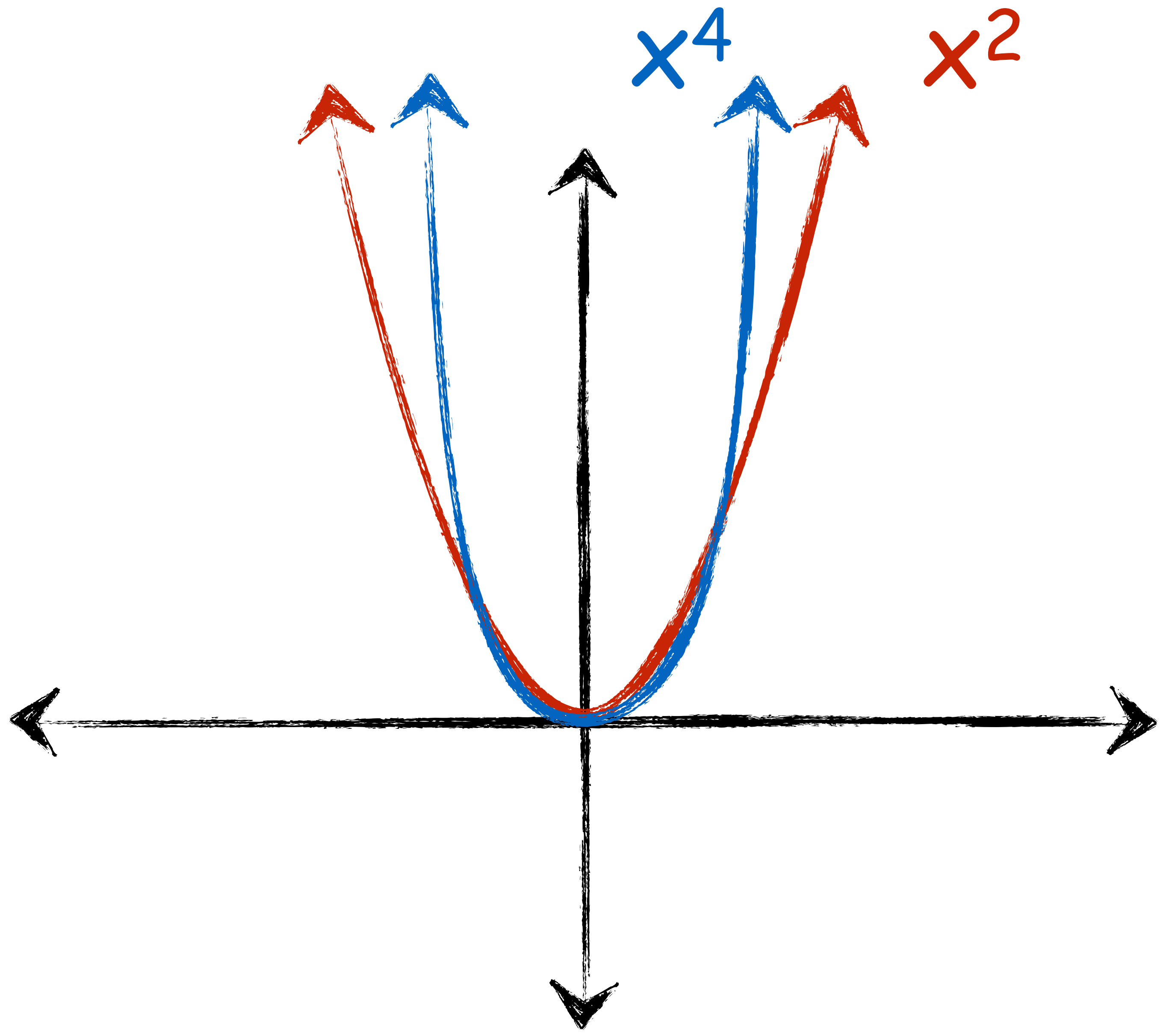
- Power functions are of the form $f(x) = x^n$.

On your calculator graph the functions, x^2, x^3, x^4, and x^5.

Y= X,T,θ,n ^ 2

Y= X,T,θ,n ^ 4

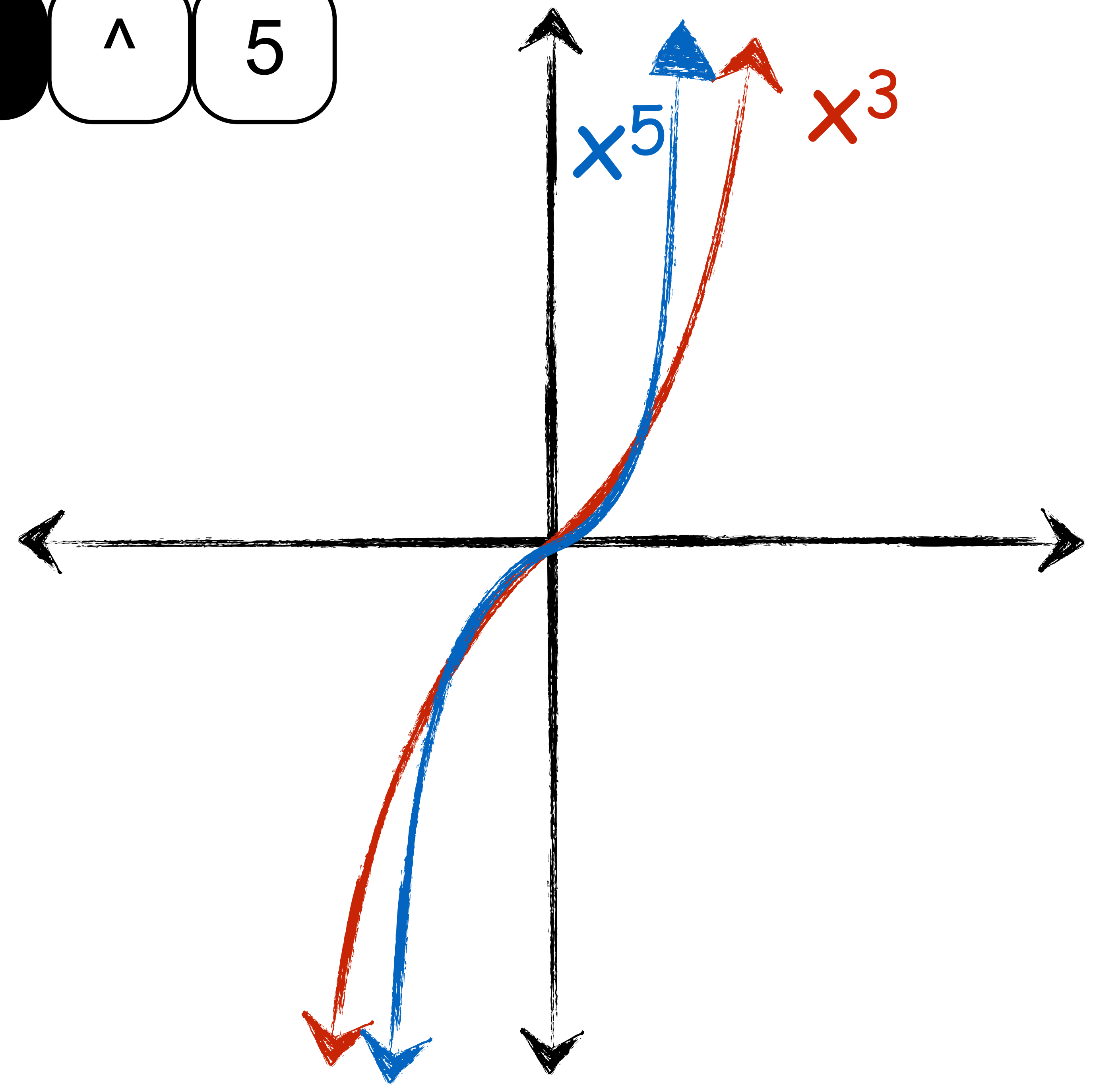
ZOOM 6



Y= X,T,θ,n ^ 3

Y= X,T,θ,n ^ 5

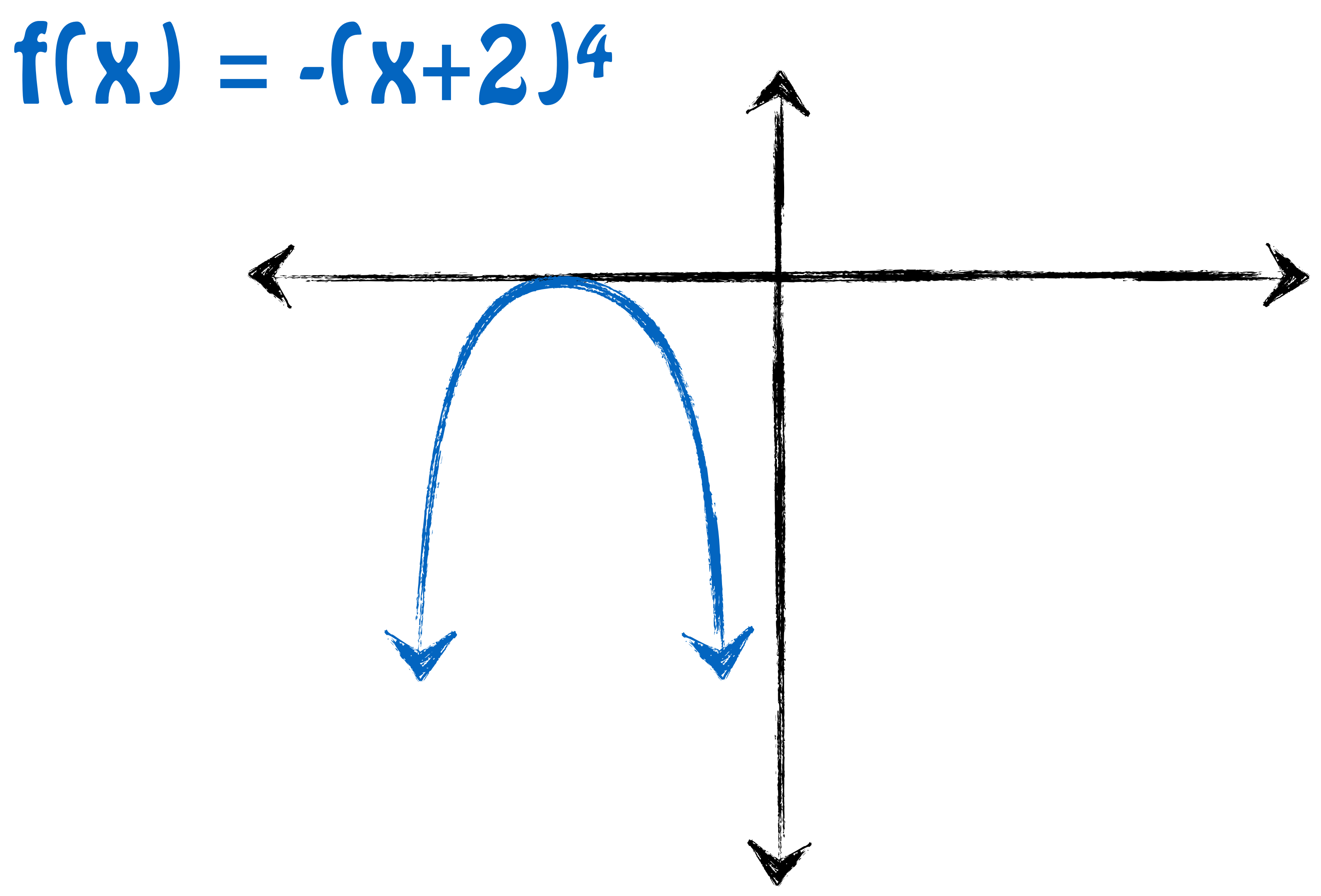
ZOOM 6



Power Functions

✦ Now graph $f(x) = -(x+2)^4$

Y= (–) (X,T,θ,n + 2) ^ 4 ZOOM 6



The Leading Coefficient Test

As x increases or decreases without bound, the graph of the polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$ eventually rises or falls.

Once again, the sign of the leading coefficient, a_n and the degree, n, reveal the end behavior of the polynomial function

Arrow Notation

As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to -\infty$

STUDY TIP

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

End Behavior

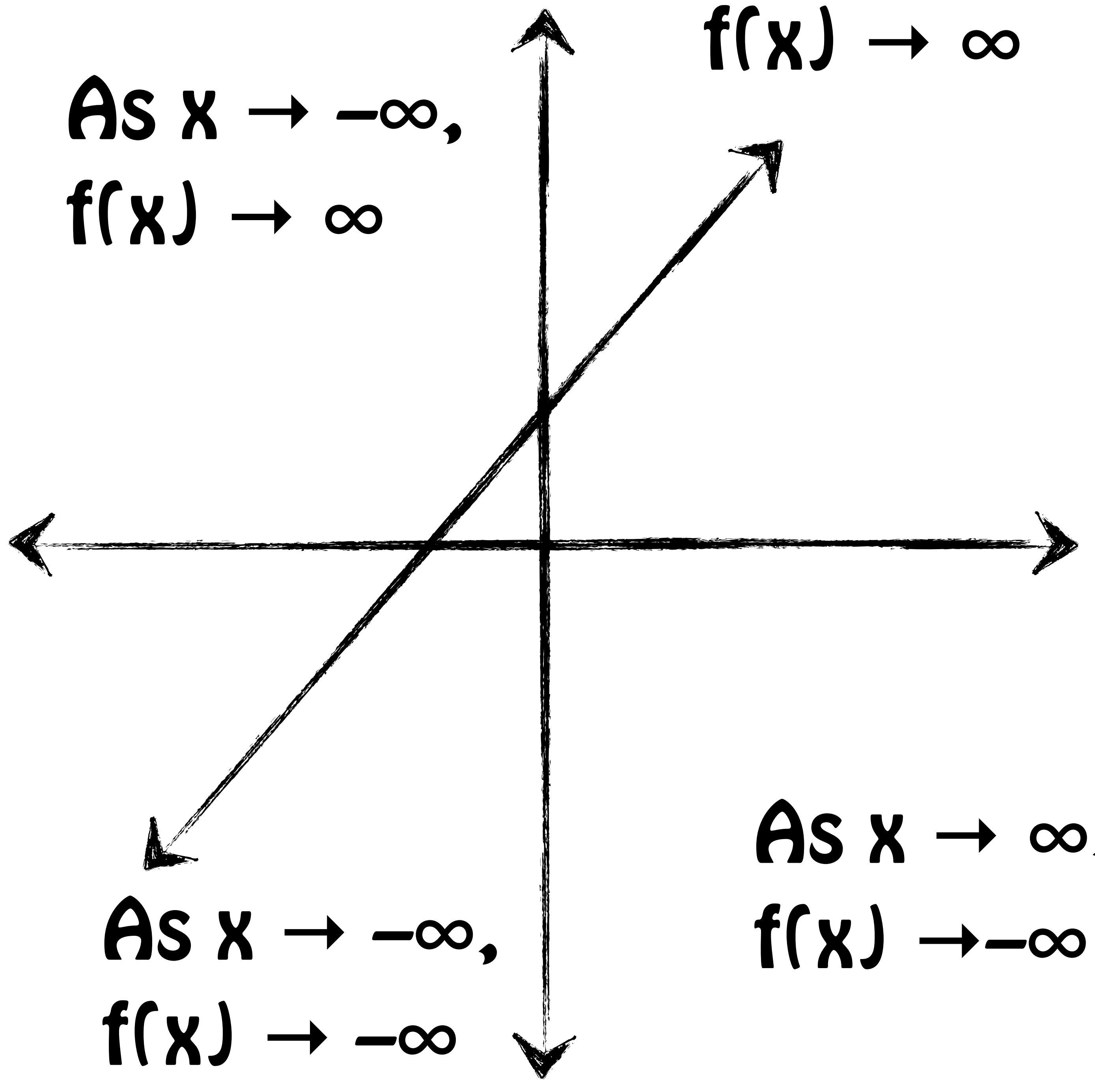
Let us start with a very familiar polynomial $y = mx + b$ and note the **end behavior**

The degree is 1, an odd number, and the ends of the graph go in opposite directions.

If the leading coefficient, m, is positive the graph rises to the right and falls to the left.

If the leading coefficient, m, is negative the graph falls to the right and rises to the left.

Consider the behavior of the graph for very large values of x in both positive ($\to\infty$) and negative direction ($\to-\infty$).



The Leading Coefficient Test

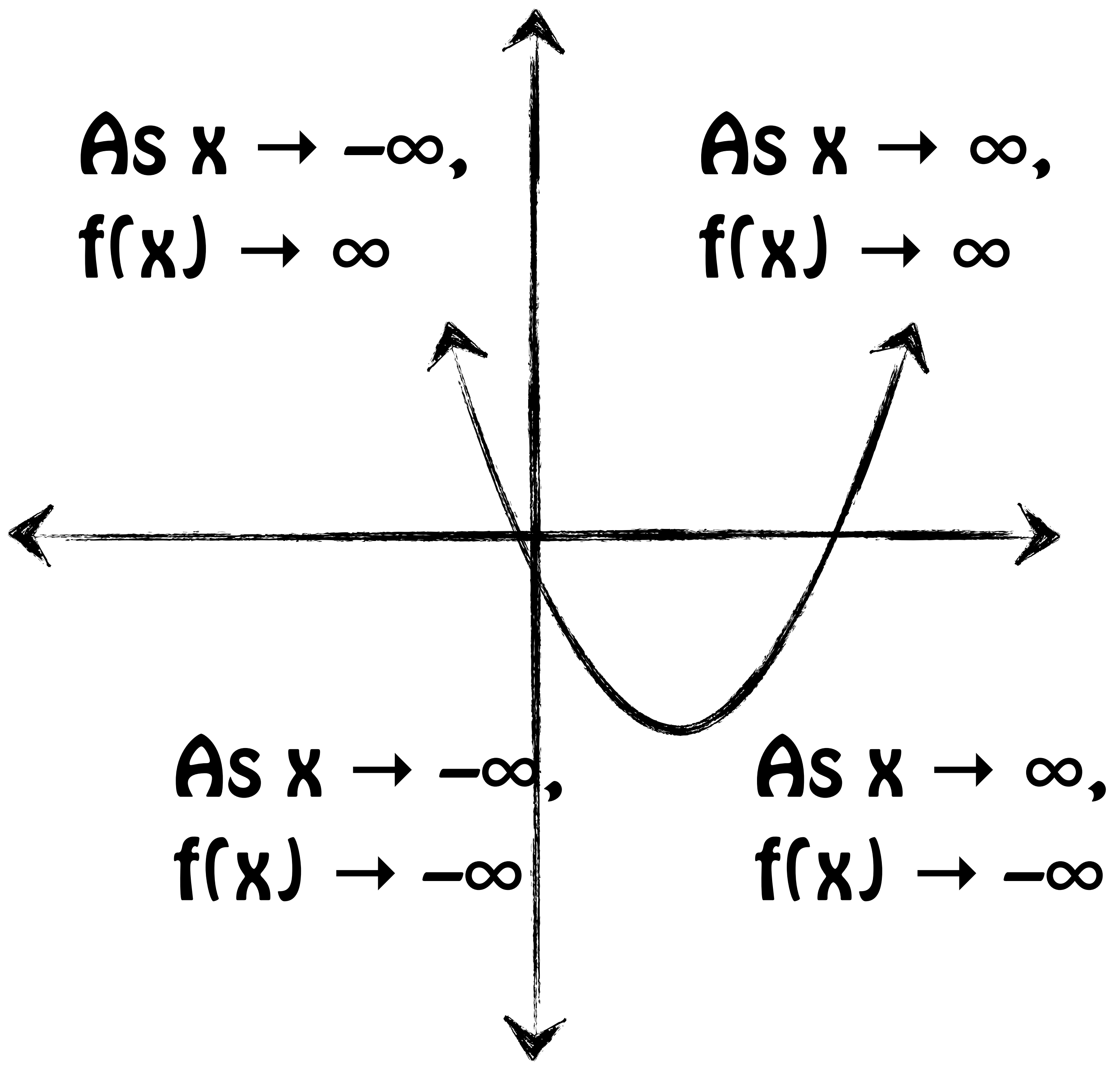
Next is another very familiar polynomial $y = ax^2 + bx + c$.

The degree is 2, an even number, and the ends of the graph go in the same direction.

If the leading coefficient, a, is positive the graph rises to the right and to the left.

If the leading coefficient, m, is negative the graph falls to the right and to the left.

Consider the behavior of the graph for very large values of x in both positive and negative direction.



The Leading Coefficient Test

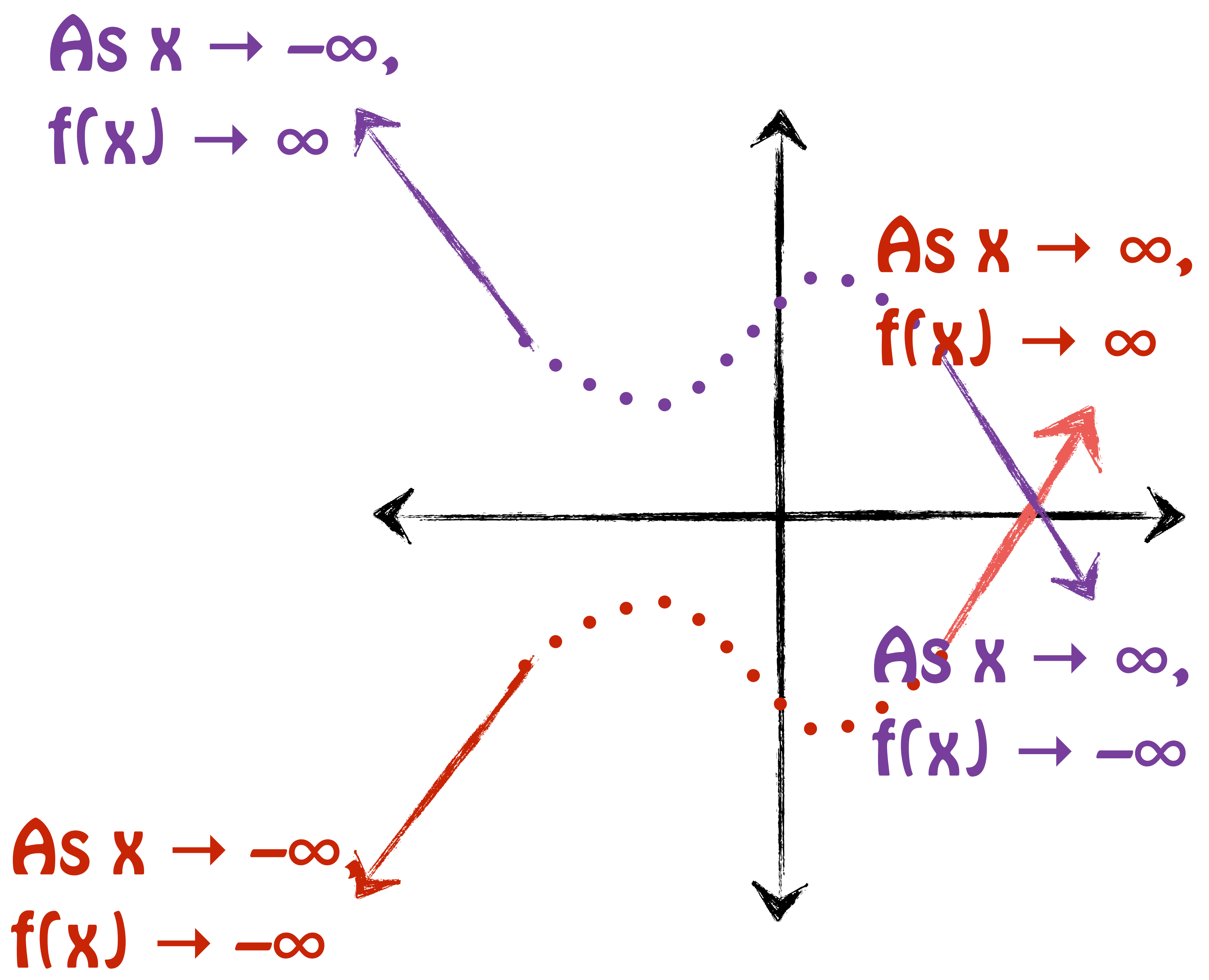
We can generalize these rules to all polynomials

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

If degree (n) an odd number, the ends of the graph go in opposite directions.

If the leading coefficient, a_n, is positive the graph rises to the right and falls to the left.

If the leading coefficient, a_n, is negative the graph falls to the right and rises to the left.



The Leading Coefficient Test

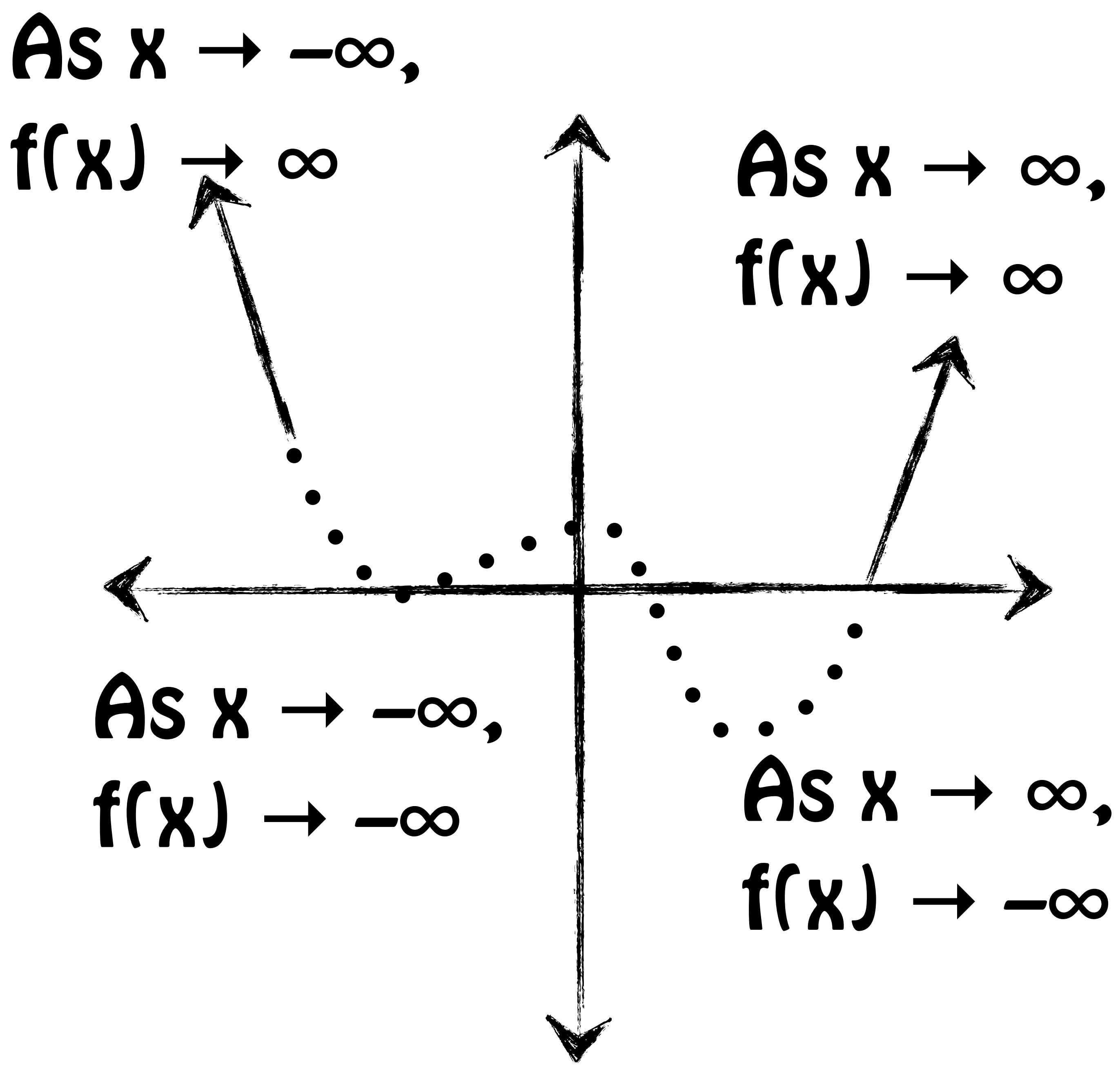
We can generalize these rules to all polynomials

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

If degree (n) an even number, the ends of the graph go in the same direction.

If the leading coefficient, a_n, is positive the graph rises to the right and left.

If the leading coefficient, a_n, is negative the graph falls to the right and left.



The Leading Coefficient Test

To summarize these rules for all polynomials

degree	coefficient	ends
n even	$a_n > 0$	↖ ↗
	$a_n < 0$	↙ ↘
n odd	$a_n > 0$	↙ ↗
	$a_n < 0$	↑ ↓

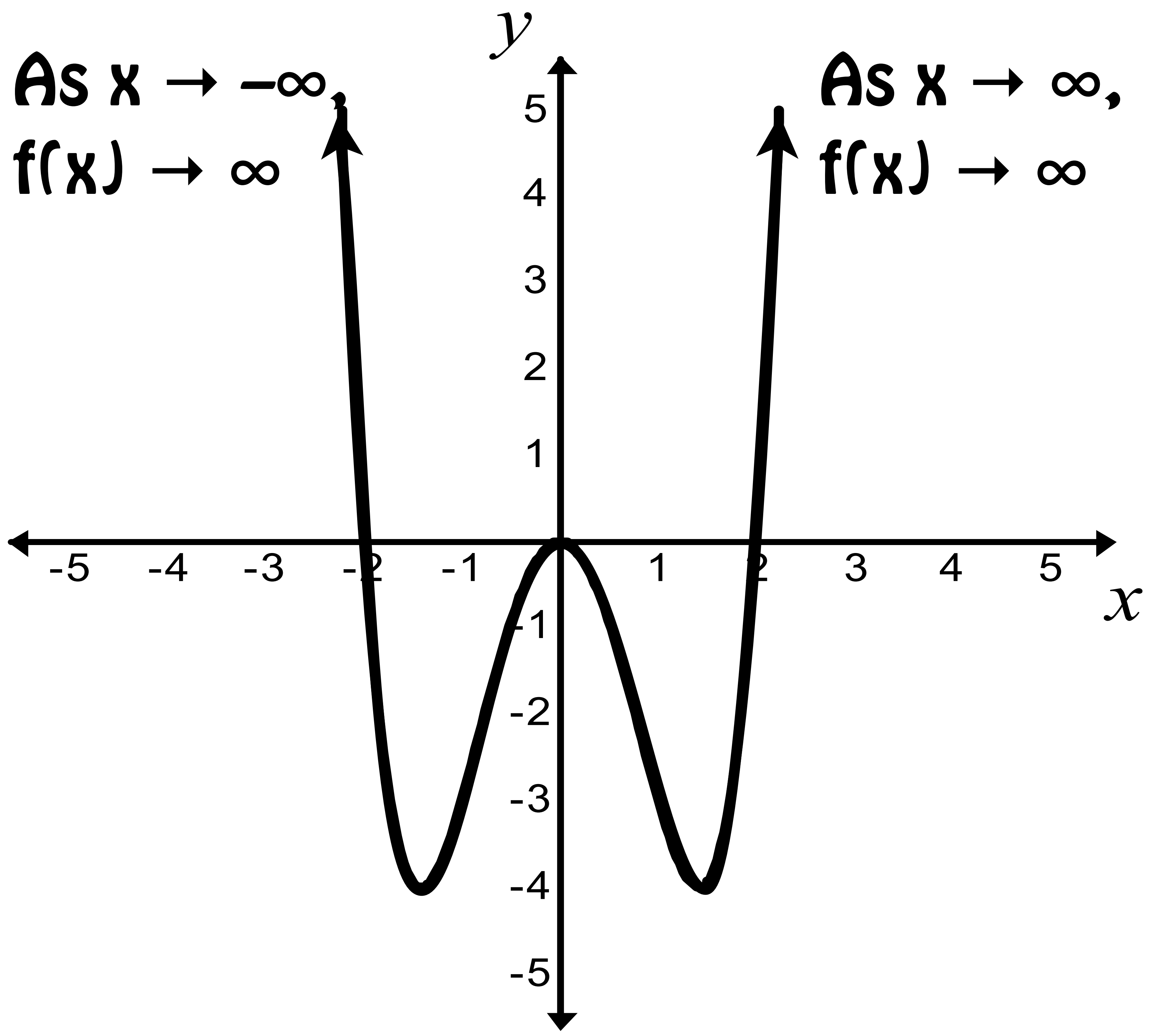
coefficient	degree	ends
$a_n > 0$	n even	↖ ↗
	n odd	↙ ↗
$a_n < 0$	n even	↙ ↘
	n odd	↑ ↓

Using the Leading Coefficient Test

Use the Leading Coefficient Test to determine the end behavior of the graph of $f(x) = x^4 - 4x^2$.

The degree of the function is 4, even. Even-degree functions have graphs with the same end behavior direction at each end.

The leading coefficient, 1, is positive. The graph rises to the left and right.



Zeros of Polynomial Functions

If f is a polynomial function, the **values of x** for which f(x) is equal to 0 (**f(x) = 0**) are called the **zeros** of f.

These **values of x** are the roots, or solutions, of the polynomial equation **f(x) = 0**.

Each real root of the polynomial equation appears as an **x-intercept** of the graph of the polynomial function.

Zeros of Polynomial Functions

Repeat

The zeros of a function are the roots of the polynomial equation. Real roots appear as an x-intercepts of the graph of the polynomial function.

Zeros, roots, and x-intercepts (when real) all refer to the same values.

Finding Zeros of a Polynomial Function

✦ Find all zeros of $f(x) = x^3 + 2x^2 - 4x - 8$.

We find the zeros of f by setting f(x) = 0 and solving the resulting equation.

$$f(x) = x^3 + 2x^2 - 4x - 8$$

$$0 = x^3 + 2x^2 - 4x - 8$$

$$0 = (x^3 + 2x^2) + (-4x - 8)$$

$$0 = x^2(x+2) + -4(x+2)$$

$$0 = (x^2 - 4)(x+2)$$

$$0 = (x-2)(x+2)(x+2)$$

$$0 = x - 2 \text{ or } 0 = x + 2$$

$$x = 2 \text{ or } x = -2$$

the zeros of f(x) are 2 and –2.

Zero -2 has a multiplicity of 2.

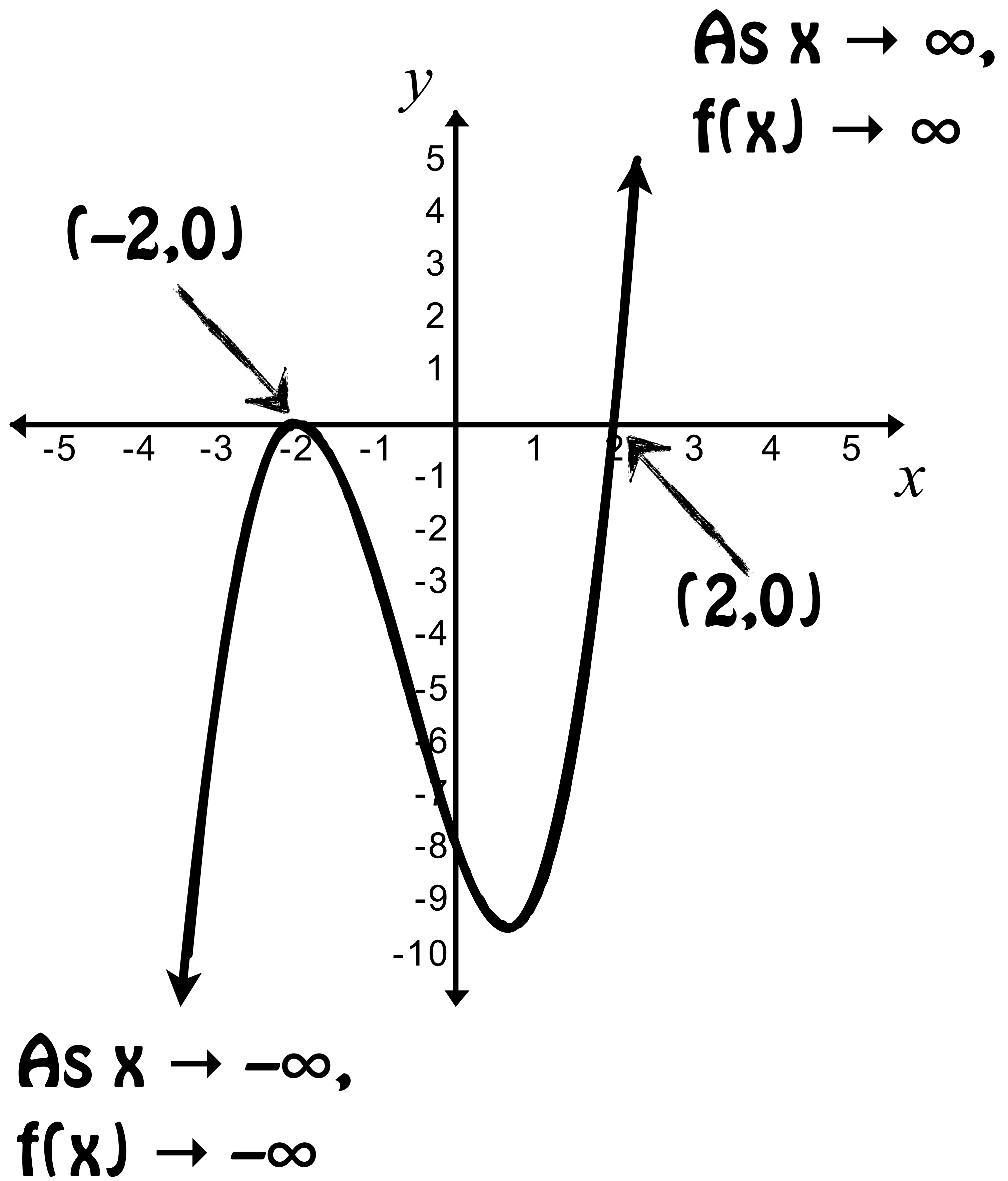
Finding Zeros of a Polynomial Function

✦ Find all zeros of $f(x) = x^3 + 2x^2 - 4x - 8$.

the zeros of f(x) are 2 and –2.

The zeros are the x-intercepts of the graph of f(x). The graph passes through (–2,0) and (2,0).

The value -2 has a multiplicity of 2, meaning it occurs twice in the solution.



Multiplicity

- For $f(x) = -x^2(x-2)^2$, notice that each factor occurs twice. When factoring this equation for f, if the same factor (x-r) occurs k times, but not k+1 times, we call r a zero with multiplicity k.

In the example, $f(x) = -x^2(x-2)^2$ both 0 and 2 are zeros with multiplicity 2.

In $f(x) = 4(x+3)^2(x-1)^3$, the zero -3 has multiplicity 2, the zero 1 has multiplicity 3.

Multiplicity and x-Intercepts

Multiplicity is the number of occurrences of a root, or zero.

If r is a zero of even multiplicity, then the graph touches the x-axis and turns around (bounces off) at r.

If r is a zero of odd multiplicity, then the graph crosses the x-axis at r.

Regardless of whether the multiplicity of a zero is even or odd, graphs tend to flatten out near zeros with multiplicity greater than one.

Finding Zeros and their Multiplicities

Find the zeros of $f(x) = -4\left(x + \frac{1}{2}\right)^2 (x-5)^3$

Give the multiplicities of each zero. State whether the graph crosses the x-axis or touches the x-axis and turns around at each zero.

To find the zeros, f(x) = 0.

$$0 = -4\left(x + \frac{1}{2}\right)^2 (x-5)^3$$

$0 = x + \frac{1}{2}$ $x = -\frac{1}{2}$ multiplicity 2

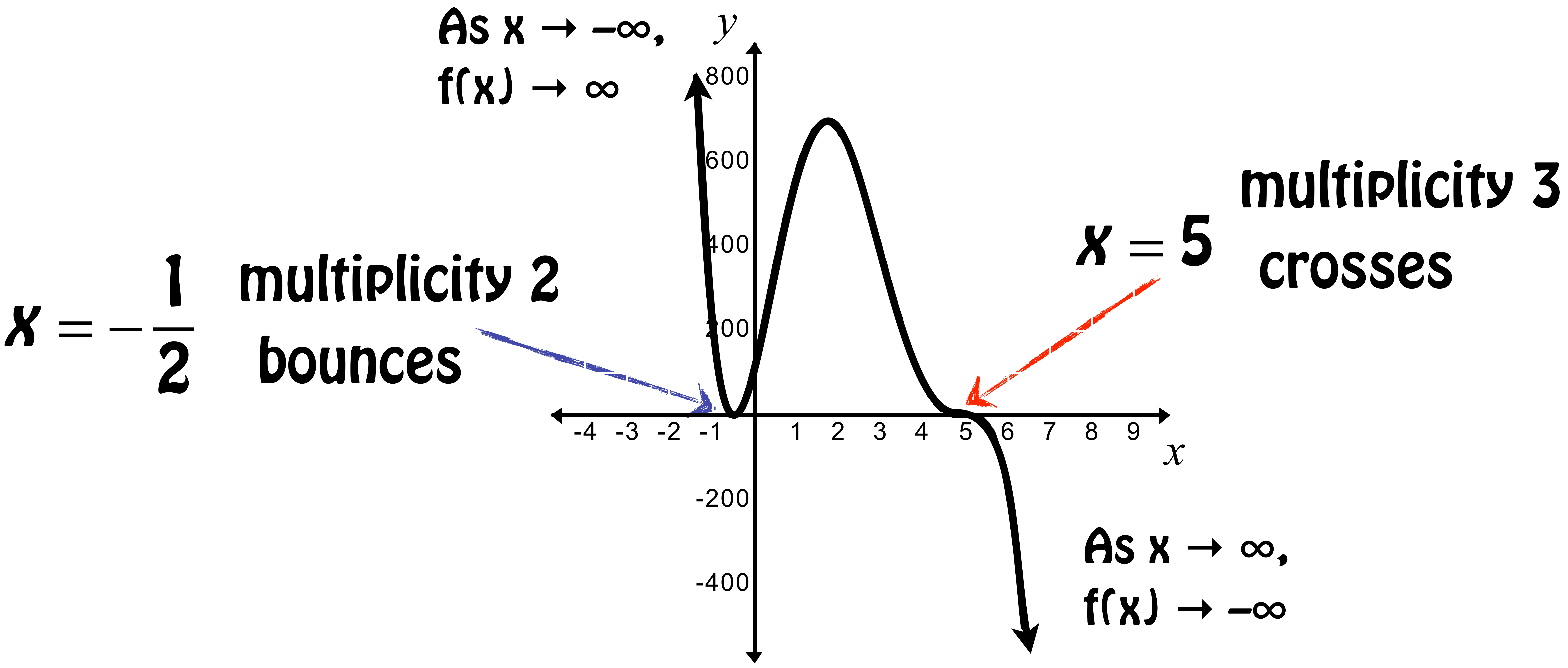
bounces

$0 = x - 5$ $x = 5$ multiplicity 3

crosses

Finding Zeros and their Multiplicities

Find the zeros of $f(x) = -4\left(x + \frac{1}{2}\right)^2 (x-5)^3$



Above or Below

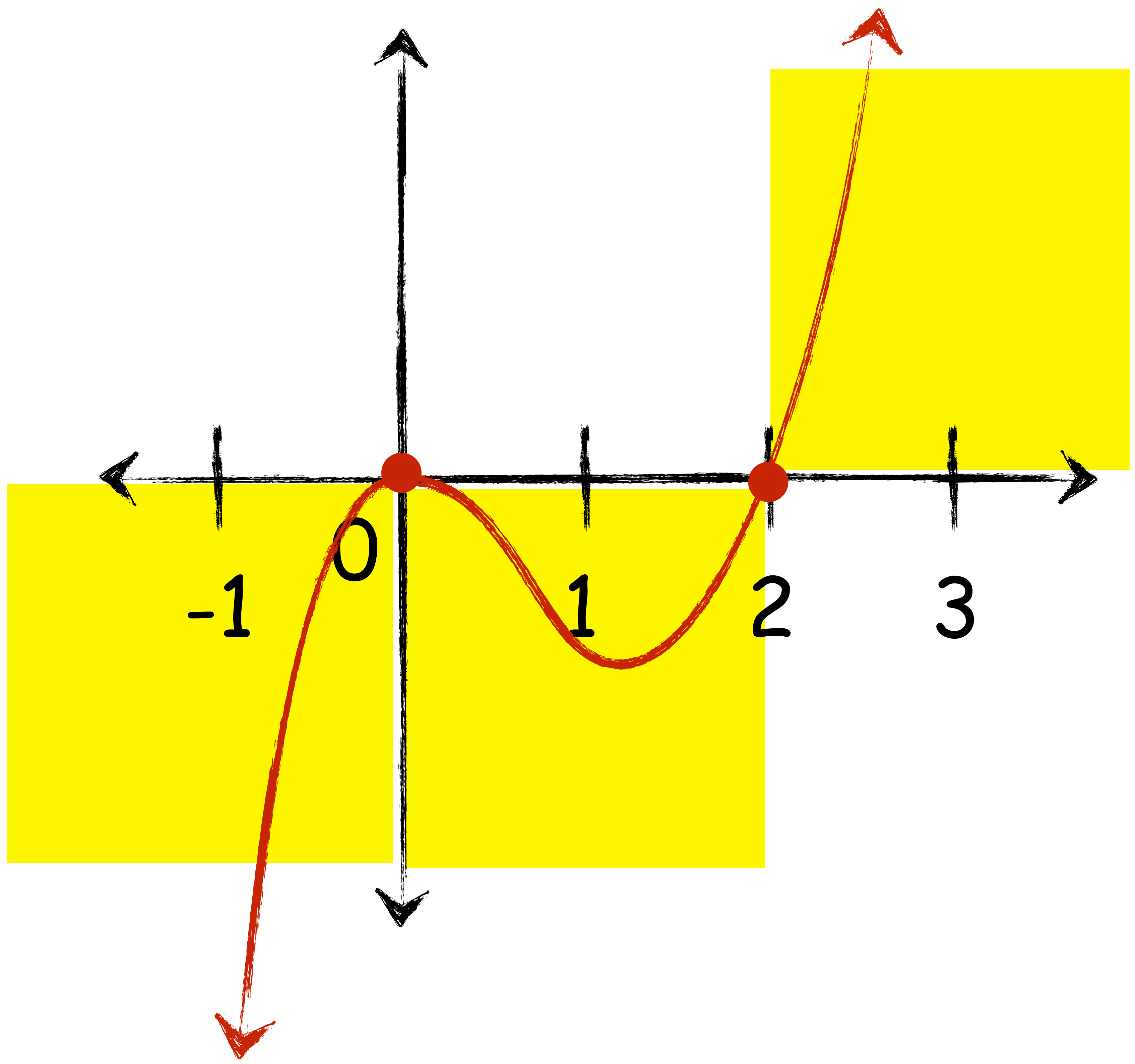
- To graph a polynomial function, you can use the fact that the function can change signs only at its zeros. Between two consecutive zeros, the polynomial must be either entirely positive or entirely negative.
  - If the real zeros are put in order, they divide the number line (x-axis) into test intervals on which the function has no sign changes.
  - By picking a representative x-value in each test interval, you can determine whether that portion of the graph lies above the x-axis (positive value of f) or below the x-axis (negative value of f).

Sketch the graph of $f(x) = x^3 - 2x^2$

$f(x) = x^3 - 2x^2 = x^2(x-2)$

zero at 0 multiplicity 2

zero at 2 multiplicity 1



$f(-1) = (-1)^3 - 2(-1)^2 = -3$ negative

$f(1) = (1)^3 - 2(1)^2 = -1$ negative

$f(3) = (3)^3 - 2(3)^2 = 9$ positive

The Intermediate Value Theorem

Let f be a polynomial function with real coefficients.

If f(a) and f(b) have opposite signs, then there is at least one value of c between a and b for which f(c) = 0.

In other words, the graph of f(x) touches the x-axis between a and b.

Equivalently, the equation f(x) = 0 has at least one real root between a and b.

Intermediate Value Theorem

- Show that the polynomial function $f(x) = 3x^3 - 10x + 9$ has a real zero between -3 and -2.

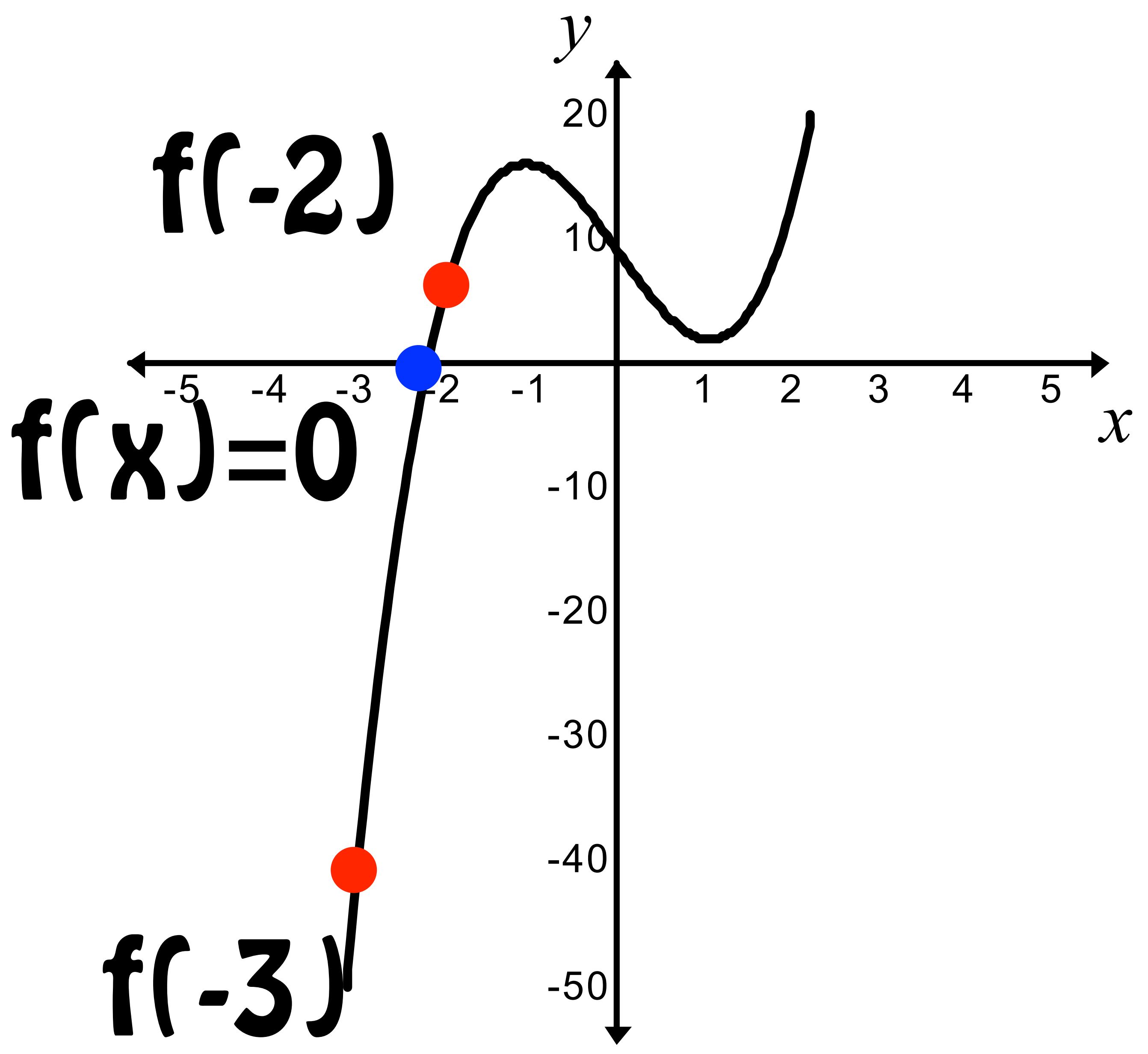
We evaluate f(-3) and f(-2). If f(−3) and f(−2) have opposite signs, then there is at least one real zero between −3 and −2.

$$f(x) = 3x^3 - 10x + 9$$

$$f(-3) = 3(-3)^3 - 10(-3) + 9 = -42$$

$$f(-2) = 3(-2)^3 - 10(-2) + 9 = 5$$

f(-3) and f(-2) have opposite signs

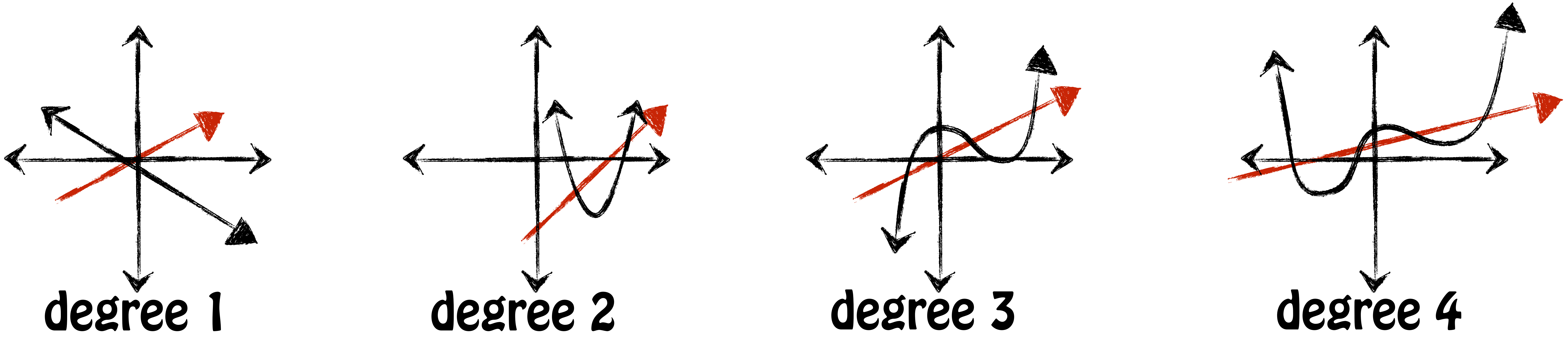


Turning Points of Polynomial Functions

In general, if f is a polynomial function of degree n, then the graph of f has at most n – 1 turning points.

In other words, the graph of f(x) changes direction one fewer times than the degree of f(x).

Another way to think of this is that a straight line will intersect the graph of the function in **at most n places**.



Strategy for Graphing Polynomials

Graphing $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$

1. Use the leading coefficient to determine the graph's end behavior.

2. Find x-intercepts by setting f(x) = 0 and solving. If there is an x-intercept at r as a result of $(x-r)^k$ being a factor of f(x), then

   2a. If k is even, the graph bounces at r

   2b. If k is odd, the graph crosses at r

   2c. If k > o, the graph flattens near (r,0).

   2d. Test the intervals between the zeros to determine if the graph is above or below the x-axis.

Strategy for Graphing Polynomials

Graphing $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$

3. Find y-intercepts by finding f(0).

4. When possible, use symmetry.

   4a. Reflection across y-axis, $f(-x) = f(x)$.

   4b. Reflection across origin, $f(-x) = -f(x)$.

5. The maximum number of turning points (changes in direction) is n-1, where n is the degree of f(x).

STUDY TIP

If you are unsure of the shape of a portion of the graph of a polynomial function, plot some additional points, such as the point $(0.5, -0.3125)$ as shown in Figure 2.21.

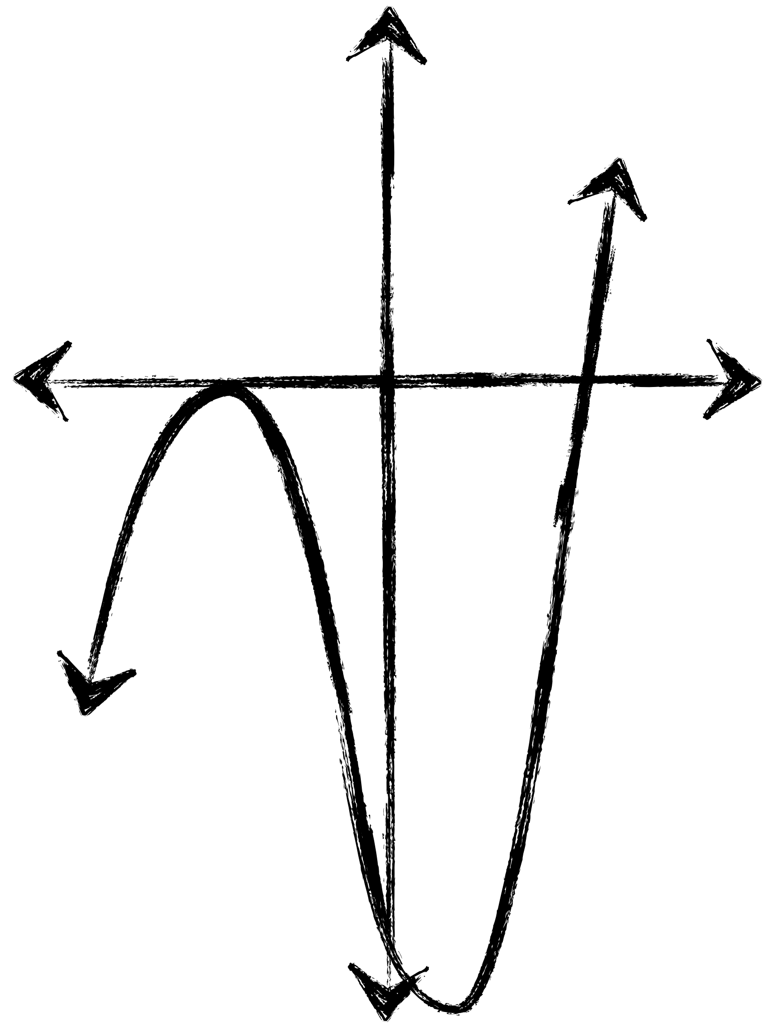
Graphing a Polynomial Function

Graphing $f(x) = 2(x+2)^2(x-3)$

1. End behavior - leading coefficient is 2, up to right.

   degree is 2 + 1 = 3 odd. ends opposite direction.

2. x-intercepts $0 = 2(x+2)^2(x-3)$. $x = -2$ multiplicity 2 even, bounces.
   $x = 3$, multiplicity 1 odd, crosses.



3. y-intercepts $f(0) = -24$.

4. No symmetry

5. changes in direction 3-1 = 2 times

   $f(1) = 2(1+2)^2(1-3) = -36$ $f(2) = 2(2+2)^2(2-3) = -32$

- Let the TI do a lot of the work for you by using the table feature of the calculator.

Enter a function into the Y= window. You can do more than one at a time but we will restrict ourselves to a single table.

Now enter the table setup

2nd | Window / TBLSET

TblStart= Enter the first value for x in the table

ΔTbl= Enter the increment between x values

Indpnt: Auto

Depend:: Auto

To see your table

2nd | Graph / TABLE

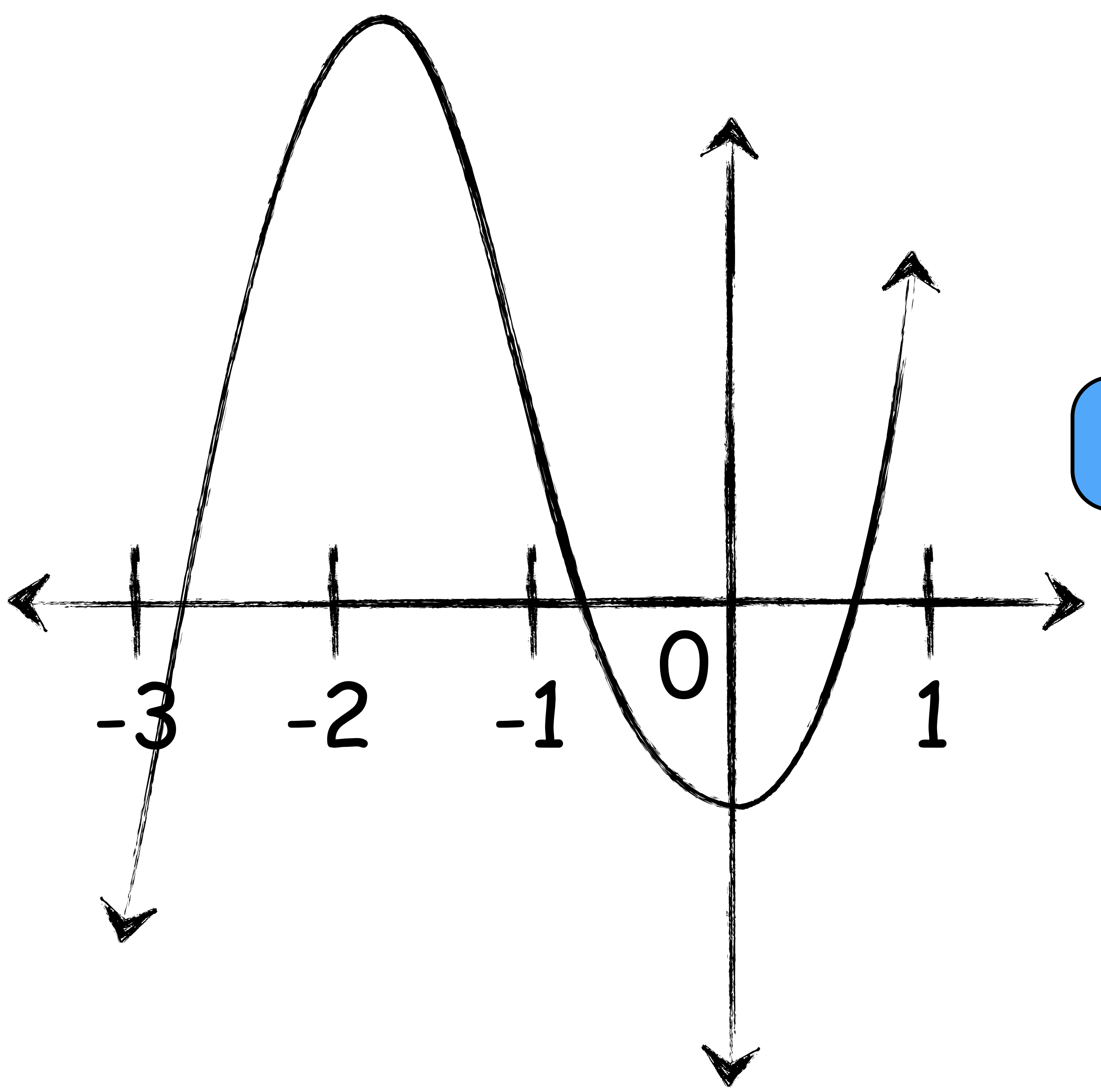
To hone in on the x intercepts Start close to the x and decrease the increment

Finding Zeros

✦ Estimate the zeros of the function $f(x) = x^3 + 3x^2 - 1$

Graph $y = x^3 + 3x^2 - 1$

You will note the intercepts are not integers.



One zero appears to be near -3.

2nd Window TBLSET

TblStart= -3

ΔTbl= .5

Indpnt: Auto

Depend:: Auto

2nd Graph TABLE

-3	-1
-2.5	2.125

Yeperdoo, tween -3 and -2.5

Finding Zeros

✦ Estimate the zeros of the function $f(x) = x^3 + 3x^2 - 1$

Let's dial it in

2nd | Window TBLSET

TblStart= -3

ΔTbl= .1

Indpnt: Auto

Depend:: Auto

2nd | Graph TABLE

-3	-1
-2.9	-.159
-2.8	.568

Aaah, between -2.9 and -2.8

I will let you estimate the other zeros