Chapter 2

Polynomial and Rational Functions

2.4 Complex Numbers

Chapter 2

Homework

2.4 p167 3, 13, 15, 21, 25, 33, 35, 47, 53, 55, 59, 61, 63, 71, 81, 86

Objectives:

- Use the imaginary unit I to write complex numbers.
- Add and subtract complex numbers.
- Multiply complex numbers.
- Divide complex numbers.
- Perform operations with square roots of negative numbers.
- Solve quadratic equations with complex imaginary solutions.

Complex and Imaginary Numbers

- The imaginary unit, *i*, is defined as

$$i = \sqrt{-1} \qquad \text{where} \qquad i^2 = -1$$

The set of all numbers in the form $a + bi$ eal numbers a, b, and *i*, the imaginary unit, is called the set of **complex numbers**.

The **standard form** of a complex number is $a + bi$

- Two complex numbers $a + bi$ and $c + di$ are equal **if and only if (iff)** the real and imaginary parts are equal.

$$a + bi = c + di \text{ iff } a = c \text{ and } b = d$$

Real and Imaginary Numbers

- The set of all complex numbers is in the form $a + bi$

  The set of all **real** numbers have the form $a + 0i$

  Thus a is called the real part of $a + bi$.

  The set of **pure imaginary** numbers have the form $0 + bi$

  Thus b is called the imaginary part of $a + bi$.

- Note that the additive identity of the complex numbers is zero, $0 + 0i = 0$.

- Additionally, the additive inverse of $a + bi$ is $-(a + bi) = -a - bi$.

Operations on Complex Numbers

- The form of a complex number $a + bi$ is like the binomial a + bx. To add, subtract, and multiply complex numbers, we use the same methods that we use for binomials.

- To add two complex numbers, we add the two real parts and then add the two imaginary parts. That is, $(a + bi) + (c + di) = (a + c) + (b + d)i$.

$$a.\ (5 - 2i) + (3 + 3i)$$

$$= 5 + (-2i + 3) + 3i \quad \text{Associative Property of Addition}$$

$$= 5 + (3 + -2i) + 3i \quad \text{Commutative Property of Addition}$$

$$= (5 + 3) + (-2i + 3i) \quad \text{Associative Property of Addition}$$

$$= 8 + (-2 + 3)i \quad \text{Distributive Property}$$

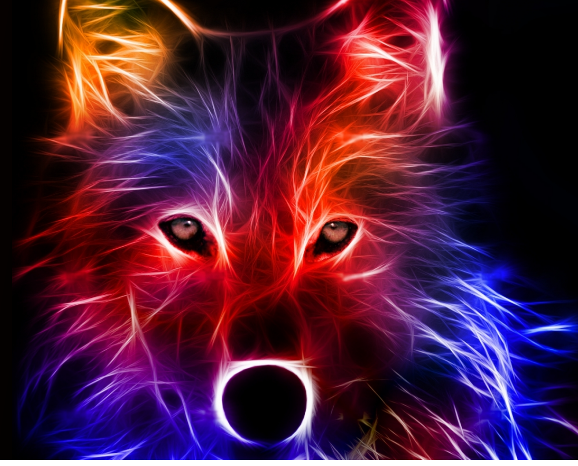
$$= 8 + i$$

Example: Subtracting Complex Numbers

- Perform the indicated operations, writing the result in standard form:

$$
\begin{aligned}
a.\ (2-6i)-(12-i) \\
&=(2-6i)+(-12+i) \\
&=(2+-12)+(-6i+i) \\
&=-10-5i
\end{aligned}
$$

Example: Multiplying Complex Numbers



- Perform the indicated operations, writing the result in standard form:

$$a.\ 7i(2-9i)$$

$$=14i-63i^2$$

$$=14i-63(-1)$$

$$=63+14i$$

$$b.\ (5+4i)(6-7i)$$

$$=5(6-7i)+4i(6-7i)$$

$$=30-35i+24i-28i^2$$

$$=30-11i-28(-1)$$

$$=30-11i+28$$

$$=58-11i$$

The F**L Word

- Do NOT let me hear anyone use the "F" Word in my classroom.
- When we multiply two binomials, or any two polynomials, …

WE USE THE DISTRIBUTIVE PROPERTY

- We, most assuredly, most emphatically, **DO NOT** use the F **L Method, which is not a "method", but simply a mnemonic device for the mathematically challenged.

Powers of Complex Numbers

- Perform the indicated operation and write the result in standard form.

$$(2-3i)^2$$

$$=(2-3i)(2-3i)$$

$$=4-6i-6i+(3i)^2$$

$$=4-12i+9i^2$$

$$=4-12i+9(-1)$$

$$=-5-12i$$

$$(a+b)^2=a^2+2ab+b^2$$

$$(2-3i)^2 \quad =4-12i+(3i)^2$$

Equity of Complex Numbers

- Two complex numbers $a + bi$ and $c + di$ are equal if and only if (iff) the real and imaginary parts are equal.

$$a + bi = c + di \quad \text{iff} \quad a = c \text{ and } b = d$$

Special Products

- Perform the indicated operation and write the result in standard form.

$$(4+5i)(4-5i) \qquad (a+b)(a-b)=a^2-b^2$$

$$=4^2-(5i)^2$$

$$=16-25(-1)$$

$$=41$$

- A real number

Conjugate of a Complex Number

- For the complex number **a + bi**, its **complex conjugate** is defined to be **a - bi**.

The product of a complex number and its **conjugate** is a **real number**.

$$
\begin{aligned}
&(a+bi)(a-bi) \\
&\quad = a^2-(bi)^2 \\
&\quad = a^2-b^2(-1) \\
&\quad = a^2+b^2
\end{aligned}
$$

Complex Number Division

- The goal of complex number division is to obtain a real number in the denominator (rationalize the denominator).

- We multiply the numerator and denominator of a complex number quotient by a value (usually the conjugate of the denominator) to obtain a real number in the denominator.

Example: Dividing Complex Numbers

- Divide and express the result in standard form:

$$\frac{5+4i}{4-i} = \frac{5+4i}{4-i} \bullet \frac{4+i}{4+i}$$

$$= \frac{20+5i+16i+4i^2}{4^2-i^2}$$

$$= \frac{16+21i}{17}$$ In standard form $$\frac{16}{17}+\frac{21}{17}i$$

STUDY TIP

Note that when you multiply the numerator and denominator of a quotient of complex numbers by

$$\frac{c-di}{c-di}$$

you are actually multiplying the quotient by a form of 1. You are not changing the original expression, you are only creating an expression that is equivalent to the original expression.

Principal Square Root of a Negative

- For any positive real number b, the principal square root of the negative number −b is defined by

$$\sqrt{-b} = i\sqrt{b}$$

- Remember the order of operations, square root is an exponent and must be done first, thus

take care of the negative square root before any other operation.

$$i\sqrt{25}\cdot i\sqrt{4} = i^2\sqrt{(25)(4)}$$
$$= -\sqrt{100}$$
$$= -10$$

AHHHHH, much better

STUDY TIP

The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid if *both* a and b are negative. For example,

$$\begin{aligned}\sqrt{-5}\sqrt{-5} &= \sqrt{5(-1)}\sqrt{5(-1)}\\ &= \sqrt{5}i\sqrt{5}i\\ &= \sqrt{25}i^2\\ &= 5i^2 = -5\end{aligned}$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

To avoid problems with square roots of negative numbers, be sure to convert complex numbers to standard form *before* multiplying.

Example

- Rewrite in standard form $\left(1-\sqrt{-14}\right)^2$
  - Remember, take care of the negative first!

$$=\left(1-i\sqrt{14}\right)^2$$

$$=1^2-2i\sqrt{14}+\left(i\sqrt{14}\right)^2$$

$$=1-2i\sqrt{14}+14i^2$$

$$=1-2i\sqrt{14}+14(-1)$$

$$=-13-2i\sqrt{14}$$

Example: Square Roots of Negatives

- Perform the indicated operations and write the result in standard form.

a. $\sqrt{-27}+\sqrt{-48}$

$=i\sqrt{27}+i\sqrt{48}$

$=3i\sqrt{3}+4i\sqrt{3}$

$=7i\sqrt{3}$

b. $\left(-2+\sqrt{-3}\right)^2$

$=\left(-2+i\sqrt{3}\right)^2$

$=(-2)^2+2(-2)(i\sqrt{3})+(i\sqrt{3})^2$

$=4+(-4i\sqrt{3})+3i^2$

$=1-4i\sqrt{3}$

Solve $x^2 + 9 = 0$

$$x^2 = -9$$

$$x = \pm\sqrt{-9}$$

$$x = \pm i\sqrt{9}$$

$$x = \pm 3i$$

- Do not forget that a negative number squared is positive.

Quadratic Equations with complex roots

- Solve $x^2 + 6x + 15 = 0$

$$x^2 + 6x + \_\_ = -15 + \_\_$$

- Complete the square

$$x^2 + 6x + 9 = -15 + 9$$

$$(x+3)^2 = -6$$

$$x + 3 = \pm\sqrt{-6}$$

$$x + 3 = \pm i\sqrt{6}$$

$$x = -3 \pm i\sqrt{6}$$

- Complex Conjugates

Quadratic Equations with Complex Imaginary Solutions

- A quadratic equation may be expressed in the general form

$$ax^2 + bx + c = 0$$

- The quadratic can be solved using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- **$b^2 - 4ac$** is called the **discriminant**. If the discriminant is negative, a quadratic equation has no real solutions. Quadratic equations with negative discriminants have two solutions that are complex conjugates.

Example: A Quadratic Equation with Imaginary Solutions

- Solve using the quadratic formula: $x^2 - 2x + 2 = 0$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$= \frac{--2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2} = 1 \pm i$$

The solutions are complex conjugates.

The solution set is {1 + i, 1 - i}.