Chapter 2

Polynomial and Rational Functions

2.5 Zeros of Polynomial Functions

Chapter 2

Homework

2.5 p179 1-19 odd, 23, 27, 29, 35, 37, 43, 49, 53, 59, 67, 69, 75, 77, 85, 87, 91, 93, 95, 107, 111

Objectives:

- Use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions.
- Use the Rational Zero Theorem to find possible rational zeros.
- Find conjugate pairs of complex zeros.
- Find zeros of a polynomial function.
- Solve polynomial equations.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes's Rule of Signs.

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n \geq 1$, then the equation $f(x) = 0$ has at least one complex root.

Remember: The set of complex numbers includes all the real numbers and imaginary numbers.

$3 = 3 + 0i$, and $4i = 0 + 4i$

The Linear Factorization Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ where $n \geq 1$ and $a_n \neq 0$, then

$$f(x) = a_n(x - c_1)(x - c_2)\ldots(x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers (possibly real and **not necessarily distinct**).

In words: An nth-degree polynomial can be expressed as the product of a nonzero constant and n linear factors, where each linear factor has a leading coefficient of 1.

Rational Roots

How would you go about solving $x^3 - 7x - 6 = 0$?

Suppose we could find a place to start looking, would that be helpful?

Some of you may have noticed that -1 is a solution, but what would you do next?

Recall that the degree of a polynomial indicates the maximum number of times a straight line can intersect the graph of the function. So the x-axis will intersect the graph of $f(x) = x^3 - 7x - 6$ **in at most 3 places.** That means there are potentially **3 real zeros**. How do we find (if they exist) the rest?

Rational Roots

How would you go about solving $x^3 - 7x - 6 = 0$?

We observed that –1 is a zero, but how can we verify that?

Obviously we can plug it in, plug it in. But that will simply tell us something we already know. It will not give us any indication of what other roots there may be.

One possibility is to re-write the equation using variables for the coefficients of a second factor with degree n-1.

We know -1 is a zero, so $(x + 1)$ is a factor, so write the factors:

$$x^3 - 7x - 6 = (x + 1)(ax^2 + bx + c)$$

Rational Roots

$x^3 - 7x - 6 = (x + 1)(ax^2 + bx + c)$

Multiply using the **distributive property**

$x^3 - 7x - 6 = ax^3 + ax^2 + bx^2 + bx + cx + c)$

Combine like terms using the **distributive property**

$x^3 - 7x - 6 = ax^3 + (a+b)x^2 + (b+c)x + c)$

Solve for a, b, and c $a=1$ $(a+b)=0$ $b+c=-7$ $c=-6$

$b=-1$ $b=-1$

Write the function in factored form $(x + 1)(1x^2 - 1x - 6)$

$(x+1)(x^2-x-6) = (x + 1)(x+2)(x - 3)$

The roots are -2, -1, 3

Rational Roots

Another, probably better, choice is to use synthetic substitution. The advantage being the resulting factorization.

We could divide using synthetic division to find what that other factor will be.

-1	1	0	-7	-6
		1	1	6
		-1	-6	0

We can factor: $x^3 - 7x - 6 = (x + 1)(x^2 - x - 6)$

Then $(x + 1)(x^2 - x - 6) = 0$, and we know what to do now.

$$(x + 1)(x - 3)(x + 2) = 0$$

The zeros of $f(x) = x^3 - 7x - 6$ are $-2, -1, 3$.

So we guessed what a root might be and we factored to find other roots. But what is a good starting point for our first guess?

I am glad you asked.

The Rational Zero Theorem

If $f(x) = a_nx^n + a_{n-1}x^{n-1} + . a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ has integer coefficients and $\frac{p}{q}$ (where $\frac{p}{q}$ is reduced to lowest terms) is a rational zero of f, then p is a factor of the constant term, a_0, and q is a factor of the leading coefficient, a_n.

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}}$$

Keep in mind the relationship among zeros, roots, and x-intercepts.

The zeros of function f are the roots, or solutions, of the equation $f(x) = 0$.

Also, the real zeros or real roots, are the x-intercepts of the graph of f.

Example: Using the Rational Zero Theorem

List all possible rational zeros of $f(x) = 4x^5 + 12x^4 - x - 3$.

The constant term, a_0, is -3 and the leading coefficient (a_n) is 4.

Factors of the constant term, -3: $\pm 1, \pm 3$

Factors of the leading coefficient, 4: $\pm 1, \pm 2, \pm 4$

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}}$$

$$\text{Possible rational zeros} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2, \pm 4} = \pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{3}{4}$$

Finding Zeros of a Polynomial Function

Find all zeros of $f(x) = x^3 + x^2 - 5x - 2$.

We begin by listing all possible rational zeros.

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}}$$

$$\text{Possible rational zeros} = \frac{\pm 1, \pm 2}{\pm 1} = \pm\frac{1}{1}, \pm\frac{2}{1} = \pm 1, \pm 2$$

We now use synthetic division to see if we can find a rational zero among the four possible rational zeros.

Finding Zeros of a Polynomial Function

Find all zeros of $f(x) = x^3 + x^2 - 5x - 2$.

Possible rational zeros $= \pm 1, \pm 2$

We now use synthetic division to see if we can find a rational zero among the four possible rational zeros.

Neither -2 nor -1 is a zero. We continue testing possible rational zeros.

-2	1	1	-5	-2
		-2	2	6
	1	-1	-3	4

-2 is not a zero

-1	1	1	-5	-2
		-1	0	5
	1	0	-5	3

-1 is not a zero

1	1	1	-5	-2
		1	2	-3
	1	2	-3	-5

1 is not a zero

2	1	1	-5	-2
		2	6	2
	1	3	1	0

2 is a zero

$f(x) = x^3 + x^2 - 5x - 2 = (x - 2)(x^2 + 3x + 1)$.

Finding Zeros of a Polynomial Function

Find all zeros of $f(x) = x^3 + x^2 - 5x - 2$.

$f(x) = x^3 + x^2 - 5x - 2 = (x - 2)(x^2 + 3x + 1)$. 2 is a zero

We now solve $x^2 + 3x + 1 = 0$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)} = \frac{-3 \pm \sqrt{5}}{2}$$

The solution set is $\left\{2, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}\right\}$

Finding Zeros of a Polynomial Function

Find all zeros of $f(x) = 3x^3 - 20x^2 + 23x + 10$.

Possible **rational** roots

$$= \frac{\pm 1, \pm 2, \pm 5, \pm 10}{\pm 1, \pm 3} = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{5}{1}, \pm\frac{10}{1}, \pm\frac{1}{3}, \pm\frac{2}{3} \pm\frac{5}{3}, \pm\frac{10}{3}$$

It is pretty obvious 1 and −1 are not zeros. 10 is even so we can try 2.

$$\begin{array}{r|rrrr} 2 & 3 & -20 & 23 & 10 \\ & & 6 & -28 & -10 \\ \hline & 3 & -14 & -5 & 0 \end{array}$$

$3x^3 - 20x^2 + 23x + 10 = (x - 2)(3x^2 - 14x - 5) = 0.$

$(x - 2)(3x + 1)(x - 5) = 0.$

2 is a zero

The solution set is $\left\{\frac{-1}{3}, 2, 5\right\}$

Properties of Roots of Polynomial Equations

1. If a polynomial equation is of degree n, then counting multiple roots separately, the equation has n roots.

2. If $a + bi$ is a root of a polynomial equation with real coefficients $(b \neq 0)$ then the imaginary number $a - bi$ is also a root

Imaginary roots, if they exist, occur in **conjugate pairs**.

Example: Solving a Polynomial Equation

Solve: $x^4 - 6x^3 + 22x^2 - 30x + 13 = 0$.

1. List all possible rational roots: $\dfrac{\pm 1, \pm 13}{\pm 1}$

   Possible rational roots are 1, –1, 13, and –13.

2. Use synthetic division to test the possible rational zeros.

-1	1	-6	22	-30	13
		-1	7	-29	59
		-7	29	-59	72

-1 is not a root

1	1	-6	22	-30	13
		1	-5	17	-13
		-5	17	-13	0

1 is a root

$x^4 - 6x^3 + 22x^2 - 30x + 13 = (x - 1)(x^3 + -5x^2 + 17x + -13)$.

Example: Solving a Polynomial Equation

Solve: $x^4 - 6x^3 + 22x^2 - 30x + 13 = 0$.

$(x - 1)(x^3 + -5x^2 + 17x + -13)$. 1 is a root

We now solve $x^3 + -5x^2 + 17x + -13 = 0$

1. list all possible rational roots: $\dfrac{\pm 1, \pm 13}{\pm 1}$

Possible rational roots are 1, ~~-1~~, 13, and -13.

2. Use synthetic division to test the possible rational zeros.

1	1	-5	17	-13
		1	-4	13
	1	-4	13	0

1 is a root again

$x^4 - 6x^3 + 22x^2 - 30x + 13 =$

$(x - 1)(x - 1)(x^2 + -4x + 13)$.

Example: Solving a Polynomial Equation

Solve: $x^4 - 6x^3 + 22x^2 - 30x + 13 = 0.$

$$x^4 - 6x^3 + 22x^2 - 30x + 13 = (x - 1)(x - 1)(x^2 + -4x + 13).$$

1 is a root 1 is a root again

Using the quadratic formula for $x^2 + -4x + 13 = 0$

$$x = \frac{--4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

The solution set is {1, 2-3i, 2+3i}. With 1 having a multiplicity of 2, and the imaginary roots are a conjugate pair.

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n \geq 1$, then the equation $f(x) = 0$ has at least one complex root.

Remember: The set of complex numbers includes all the real numbers and imaginary numbers.

$3 = 3 + 0i$, and $4i = 0 + 4i$

The Linear Factorization Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ where $n \geq 1$ and $a_n \neq 0$, then

$$f(x) = a_n(x - c_1)(x - c_2)\ldots(x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers (possibly real and not necessarily distinct).

In words: An nth-degree polynomial can be expressed as the product of a nonzero constant and n linear factors, where each linear factor has a leading coefficient of 1.

Finding a Polynomial Function Given Zeros

Find a third-degree polynomial function $f(x)$ with **real coefficients** that has -3 and i as zeros and such that $f(1) = 8$.

Because i is a zero and the polynomial has real coefficients, the conjugate, $-i$, must also be a zero.

We can now use the Linear Factorization Theorem.

$$f(x) = a_n(x - c_1)(x - c_2)\dots(x - c_n)$$

$$f(x) = a_n(x - -3)(x - i)(x - -i)$$

$$f(x) = a_n(x + 3)(x - i)(x + i)$$

$$f(x) = a_n(x + 3)(x^2 - i^2)$$

$$f(x) = a_n(x + 3)(x^2 + 1) \quad = a_n(x^3 + 3x^2 + x + 3)$$

Finding a Polynomial Function Given Zeros

Find a third-degree polynomial function $f(x)$ with **real coefficients** that has -3 and i as zeros and such that **$f(1) = 8$**.

$$f(x) = a_n(x^3 + 3x^2 + x + 3)$$

$$f(1) = a_n(1^3 + 3(1)^2 + 1 + 3) = 8$$

$$f(1) = a_n(1 + 3 + 1 + 3) = 8$$

$$f(1) = a_n(8) = 8 \qquad a_n = 1$$

$$f(x) = 1(x^3 + 3x^2 + x + 3)$$

$$\mathbf{f(x) = x^3 + 3x^2 + x + 3}$$

Finding a Polynomial Function with Given Zeros

Find all the zeros of $f(x) = x^4 - 4x^3 + 12x^2 + 4x - 13$ given that $2 + 3i$ is a zero.

If $2 + 3i$ is a zero, then $2 - 3i$ is a zero as well.

$x - (2 + 3i)$ and $x - (2 - 3i)$ are factors.

$[x - (2 + 3i)][x - (2 - 3i)] = \quad x^2 - 4x + 13$

$$
\begin{array}{r}
[x - 2 - 3i] \\
[x - 2 + 3i] \\
\hline
3ix - 6i - 9i^2 \\
-2x + 4 \qquad + 6i \\
x^2 - 2x \qquad - 3ix \\
\hline
x^2 - 4x \; + 4 \qquad\qquad + 9
\end{array}
$$

$$
\begin{array}{r|l}
 & x^2 \qquad\qquad - 1 \\
\hline
x^2 - 4x + 13 & x^4 - 4x^3 + 12x^2 + 4x - 13 \\
 & x^4 - 4x^3 + 13x^2 \\
\hline
 & -1x^2 + 4x - 13 \\
 & -1x^2 + 4x - 13 \\
\hline
 & 0
\end{array}
$$

$(x^2 - 1)$ is a factor

Finding a Polynomial Function with Given Zeros

Find all the zeros of $f(x) = x^4 - 4x^3 + 12x^2 + 4x - 13$ given that $2 + 3i$ is a zero.

$$x^4 - 4x^3 + 12x^2 + 4x - 13 = [x - 2 - 3i]\ [x - 2 + 3i]\ (x^2 - 1)$$

$$= [x - 2 - 3i][x - 2 + 3i](x + 1)(x - 1)$$

The solution set is $\{2 + 3i, 2 - 3i, -1, 1\}$

We have seen that there can be many potential rational roots. There are a couple of ways of dealing with a very large list generated by the Rational Root Theorem. The first of these is **Descartes's Rule of Signs.**

Two things to keep in mind about Descartes's Rule of Signs:

1. A variation in sign means that two consecutive coefficients have opposite signs.
2. When we count the zeros, we must count their multiplicities.

Descartes' Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients.

1. The number of positive real zeros of f is either

   a. the same as the number of sign changes of $f(x)$

   or

   b. less than the number of sign changes of $f(x)$ by a positive even integer.

If $f(x)$ has only one variation in sign, then f has exactly one positive real zero.

Descartes' Rule of Signs

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients.

2. The number of negative real zeros of f is either

   a. the same as the number of sign changes of $f(-x)$

   or

   b. less than the number of sign changes of $f(-x)$ by a positive even integer.

$$f(-x) = a_n(-x)^n + a_{n-1}(-x)^{n-1} + \ldots + a_2(-x)^2 + a_1(-x) + a_0$$

If $f(-x)$ has only one variation in sign, then f has exactly one negative real zero

Example: Using Descartes' Rule of Signs

Determine the possible number of positive and negative real zeros of
$f(x) = x^4 - 14x^3 + 71x^2 - 154x + 120$.

1. To find possibilities for positive real zeros, count the number of sign changes in the equation for f(x).

$f(x) = x^4 - 14x^3 + 71x^2 - 154x + 120$ There are 4 variations in sign.

The number of positive real zeros of f is either 4, 2, or 0.

Example: Using Descartes' Rule of Signs

Determine the possible number of positive and negative real zeros of
$f(x) = x^4 - 14x^3 + 71x^2 - 154x + 120$.

2. To find possibilities for negative real zeros, count the number of sign changes in the equation for $f(-x)$.

$$f(-x) = (-x)^4 - 14(-x)^3 + 71(-x)^2 - 154(-x) + 120$$

$$f(-x) = x^4 + 14x^3 + 71x^2 + 154x + 120$$

There are no variations in sign.

There are no negative real roots for f.

We ain't gonna find the roots for this miserable function. You can try on your own.

Example

Find all the zeros of $f(x) = x^4 - 3x^3 + x - 3$

There are 3 changes in sign so we have 1 or 3 positive real roots.

$f(-x) = x^4 + 3x^3 - x - 3$

There is 1 change in sign so we have exactly 1 negative real root.

list all possible rational roots: $\frac{\pm 1, \pm 3}{\pm 1}$

Possible rational roots are 1, –1, 3, –3.

I can see -1 is a zero.

-1	1	-3	0	1	-3
		-1	4	-4	3
	1	-4	4	-3	0

-1 is a root

Example

Find all the zeros of $f(x) = x^4 - 3x^3 + x - 3$

$$x^4 - 3x^3 + x - 3 = (x-1)(x^3-4x^2+4x-3)$$

-1	1	-3	0	1	-3
		-1	4	-4	3
	1	-4	4	-3	0

-1 is a root

We have found our negative root, so we need only check for positive values, 1 or 3. An examination of the function shows 1 is not a choice.

3	1	-4	4	-3
		3	-3	3
	1	-1	1	0

3 is a root

$$x^4 - 3x^3 + x - 3 = (x+1)(x-3)(x^2-x+1)$$

$$x = \frac{1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

The solution set is $\left\{-1, 3, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right\}$

Too Much Monkey Business

The second way of dealing with a very large list generated by the Rational Root Theorem is the **Upper and Lower Bound Rules**.

Bounds are boundaries beyond which (above or below) we will find no values.

A real number b is an **upper bound** for the real zeros of f if there are no zeros of f greater than b. A real number b is a **lower bound** for the real zeros of f if there are no zeros of f less than b.

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial function with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$ using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, then c is an upper bound for the real zeros of f.

2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as either), then c is a lower bound for the real zeros of f.

Upper and Lower Bound Rules

These are pretty obvious if you stop to think about them.

If $c > 0$ and the last row is all positive, the coefficients of the polynomial are all positive and a larger positive value for x will simply return a larger value for $f(x)$.

If $c < 0$ and the last row is all alternates, the coefficients of the polynomial are alternating, a more negative value for x will simply return a larger absolute value for $f(x)$.

Upper and Lower Bound Rules

Find all the zeros of $f(x) = x^4 - 7x^3 + x^2 + 63x - 90$

$$\frac{\pm1, \pm2, \pm3, \pm5, \pm6, \pm9, \pm10, \pm15, \pm18, \pm30, \pm45, \pm90}{\pm1}$$

-1	1	-7	1	63	-90
		-1	8	-9	-54
		-8	9	54	-144

1	1	-7	1	63	-90
		1	-6	-5	58
		-6	-5	58	-32

-5	1	-7	1	63	-90
		-5	60	-305	1210
		-12	61	-242	1120

$-5 < 0$ and Last line alternates, no value below -5

Upper and Lower Bound Rules

Find all the zeros of $f(x) = x^4 - 7x^3 + x^2 + 63x - 90$

Possible rational roots

$$\frac{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 9, \pm 10, \pm 15, \pm 18, \pm 30, \pm 45, \pm 90}{\pm 1}$$

9	1	-7	1	63	-90
		9	18	171	2106
		2	19	234	2016

$9 > 0$ and Last line all > 0, no value above 9

-3	1	-7	1	63	-90
		-3	30	-93	90
		-10	31	-30	0

$(x + 3)(x^3 + -10x^2 + 31x + -30)$.

Upper and Lower Bound Rules

Find all the zeros of $f(x) = x^4 - 7x^3 + x^2 + 63x - 90$

Possible rational roots

$$\frac{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 9, \pm 10, \pm 15, \pm 18, \pm 30, \pm 45, \pm 90}{\pm 1}$$

-3	1	-7	1	63	-90
		-3	30	-93	90
2	1	-10	31	-30	0
		2	-16	30	
		-8	15	0	

$(x + 3)(x^3 + -10x^2 + 31x + -30).$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30}{\pm 1}$$

$(x + 3)(x - 2)(x^2 + -8x + 15).$

The rest is easy

$(x + 3)(x - 2)(x - 5)(x - 3).$

The zeros are $\{-3, 2, 3, 5\}$

Example

Find all the zeros of $f(x) = x^4 - 3x^3 + x - 3$

There are 3 changes in sign so we have 1 or 3 positive real roots.

$f(-x) = x^4 + 3x^3 - x - 3$

There is 1 change in sign so we have exactly 1 negative real root.

list all possible rational roots: $\frac{\pm 1, \pm 3}{\pm 1}$

Possible rational roots are 1, –1, 3, –3.

I can see -1 is a zero.

I can also see that -1 is a lower bound.

-1	1	-3	0	1	-3
		-1	4	-4	3
	1	-4	4	-3	0

-1 is a root

Example

Find all the zeros of $f(x) = x^4 - 3x^3 + x - 3$

$$\begin{array}{r|rrrrr} -1 & 1 & -3 & 0 & 1 & -3 \\ & & -1 & 4 & -4 & 3 \\ \hline & 1 & -4 & 4 & -3 & 0 \end{array}$$

−1 is a root

$$x^4 - 3x^3 + x - 3 = (x-1)(x^3 - 4x^2 + 4x - 3)$$

We have found our negative root, so we need only check for positive values, 1 or 3. An examination of the function shows 1 is not a choice.

$$\begin{array}{r|rrrr} 3 & 1 & -4 & 4 & -3 \\ & & 3 & -3 & 3 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

3 is a root

$$x^4 - 3x^3 + x - 3 = (x-1)(x-3)(x^2 - x + 1)$$

$$x = \frac{1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

The solution set is $\left\{ -1, 3, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \right\}$

TI-84

Find all the zeros of $f(x) = x^4 - 3x^3 + x - 3$ using the TI-84.

Enter the function and graph it. Now you can ask for zeros

2nd TRACE CALC ∨ 2:Zero Set your left boundary by moving the cursor. ENTER Set your right boundary by moving the cursor. ENTER Guess? ENTER

Or you can use the table function to list the possible values

2nd Window TBLSET

TblStart= -3

ΔTbl= 1

Indpnt: Ask

Depend: Auto

2nd Graph TABLE

Enter the possible values

-3	156
-1	0
1	-4
3	0