Chapter 3

Exponential and Logarithmic Functions

3.1 Exponential Functions

Chapter 3.1

Homework

3.1 p226 7-10,11-21 odd, 37, 43, 45, 47, 63, 66

Chapter 3.1

Objectives

- Evaluate exponential functions.
- Graph exponential functions.
- Evaluate functions with base e.
- Use compound interest formulas.

Definition of the Exponential Function

- The **exponential function** **f** **with** **base b** is defined by

$$\boxed{f(x) = b^x} \quad \text{or} \quad \boxed{y = b^x}$$

- where b is a positive constant other than 1 ($b > 0$ and $b \neq 1$) and x is any real number.

- The parent <u>**exponential function**</u> is $f(x) = b^x$, where the **base b** is a constant and the **exponent x** is the independent variable.

$$\boxed{f(x) = b^x, \text{ where } b > 0,\ b \neq 1}$$

Evaluating an Exponential Function

- The exponential function $f(x) = 42.2(1.56)^x$ models the average amount spent, $f(x)$, in dollars, at a shopping mall after x hours.

  - What is the average amount spent, to the nearest dollar, after **three** hours at a shopping mall?

$$f(x) = 42.2(1.56)^x$$

$$f(3) = 42.2(1.56)^3 \approx 160.21$$

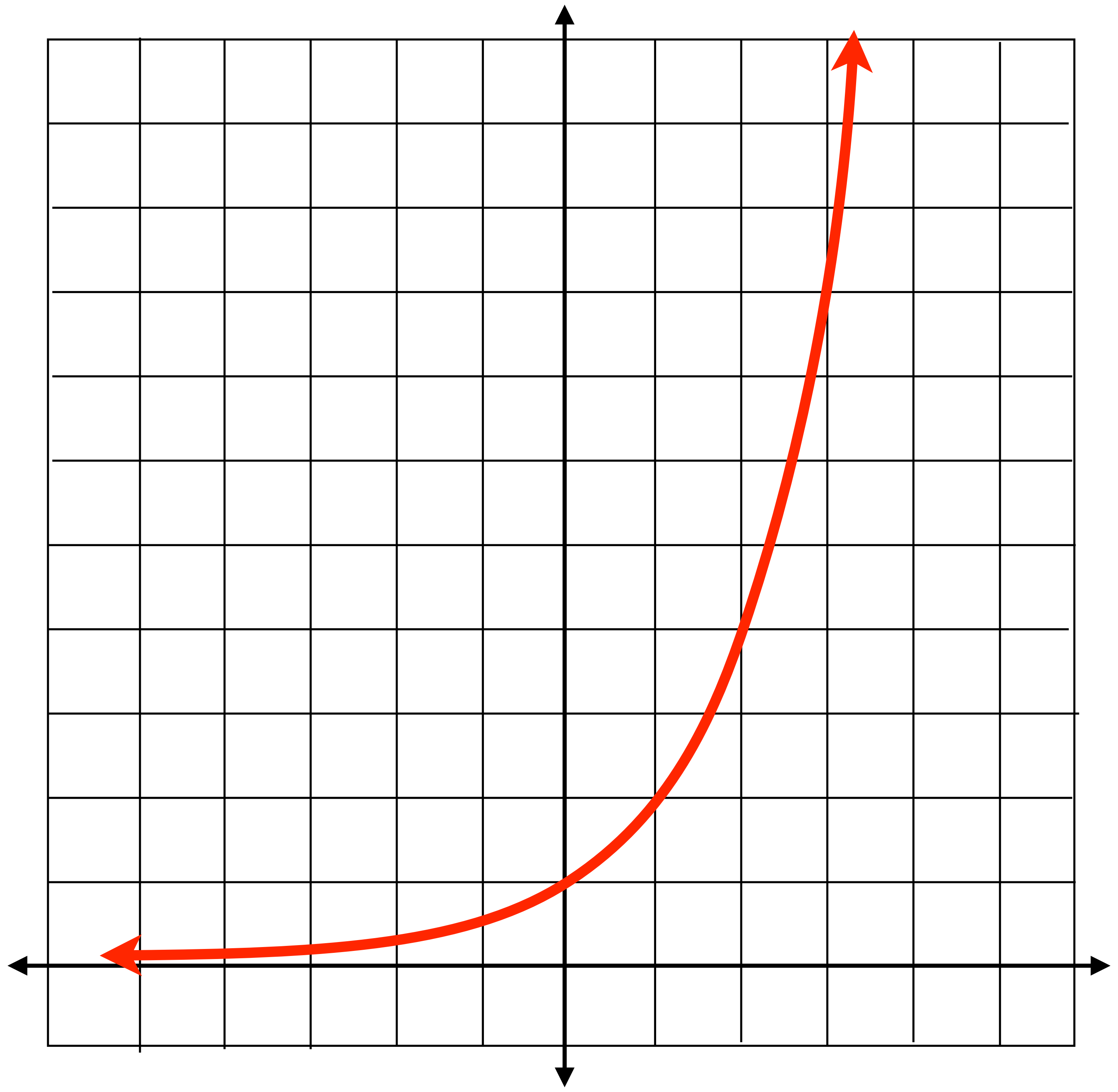
  - After 3 hours at a shopping mall, the average amount spent is $160.

Graphing an Exponential Function

- Graph: $f(x) = 2^x$
  - You should know how to graph the parent exponential function $f(x) = 2^x$.

    The domain is all real numbers $(-\infty, \infty)$.

    The range is $\{y \mid y > 0\}$ $(0, \infty)$



x	−2	−1	0	1	2	3
f(x)	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Graphing an Exponential Function TI-84

- Graph: $f(x) = 2^x,\ 2^{x+2}\ ,\ 2^x+2$

$y = 2^x$

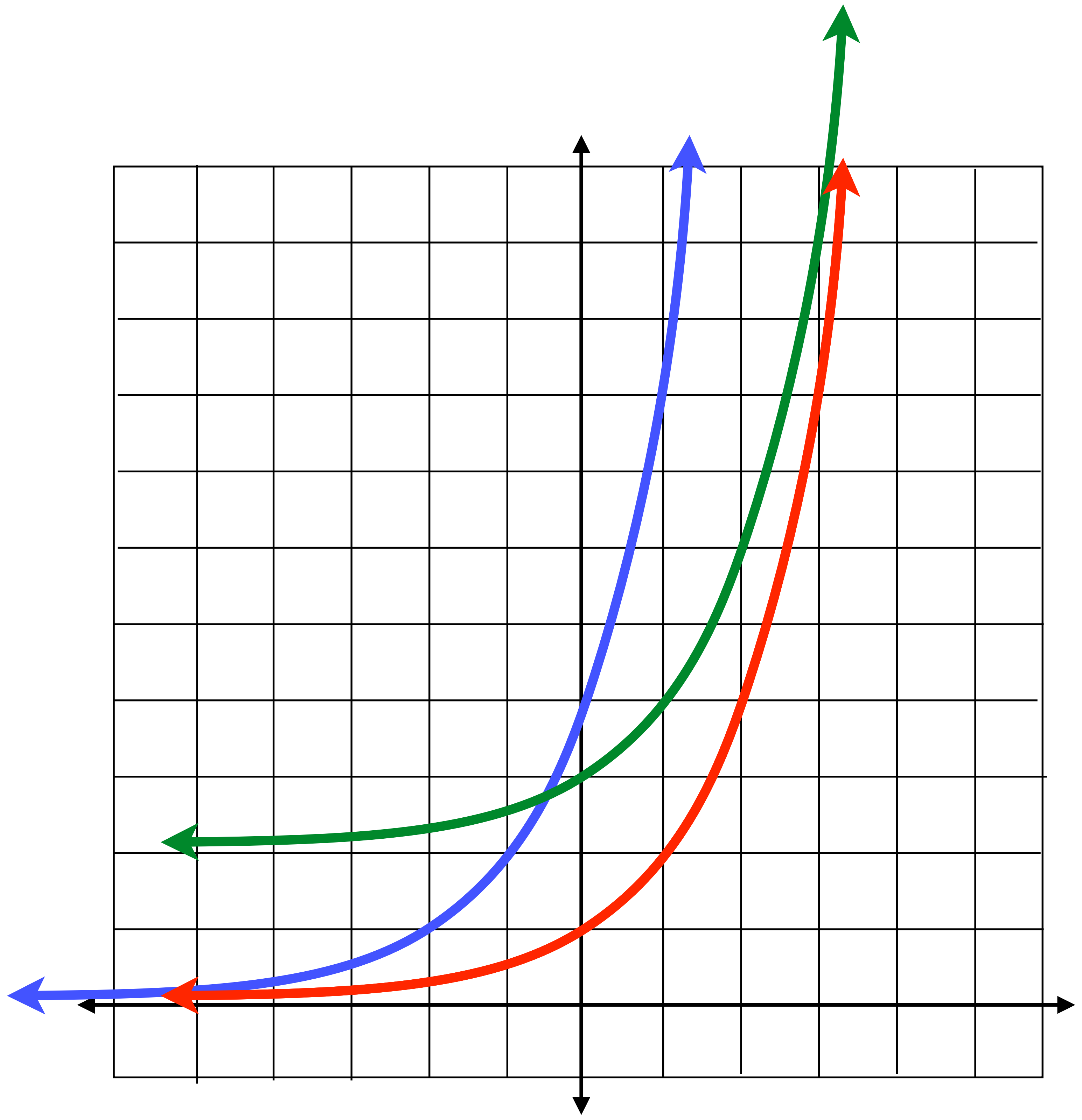
x	-2	-1	0	1	2	3
f(x)	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$y = 2^{x+2}$

x	-4	-3	-2	-1	0	1
f(x)	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$y = 2^x + 2$

x	-2	-1	0	1	2	3
f(x)	$2\frac{1}{4}$	$2\frac{1}{2}$	3	4	6	10



Characteristics of Exponential Functions of Form $f(x) = b^x$.

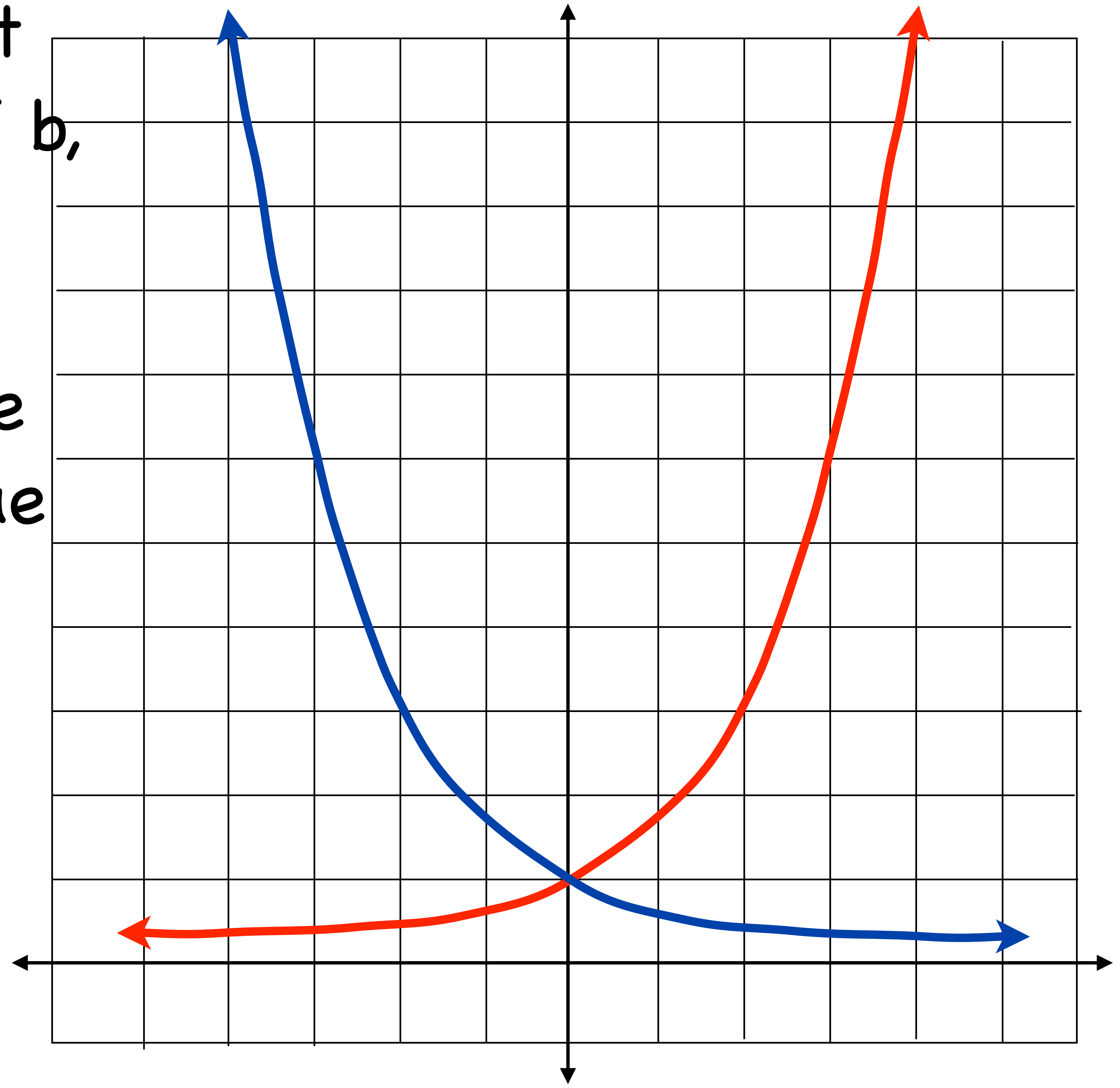
- The domain of $f(x) = b^x$ consists of all real numbers $(-\infty, \infty)$
- The range of $f(x) = b^x$ consists of all positive real numbers $(0, \infty)$
  - The graph of $f(x) = b^x$ passes through $(0, 1)$ as $f(0) = b^0 = 1$.
    - Thus the y-intercept is 1.
  - The graph of $f(x) = b^x$ is asymptotic to the x-axis.
    - Thus there is no x-intercept.

Characteristics of Exponential Functions of Form $f(x) = b^x$.

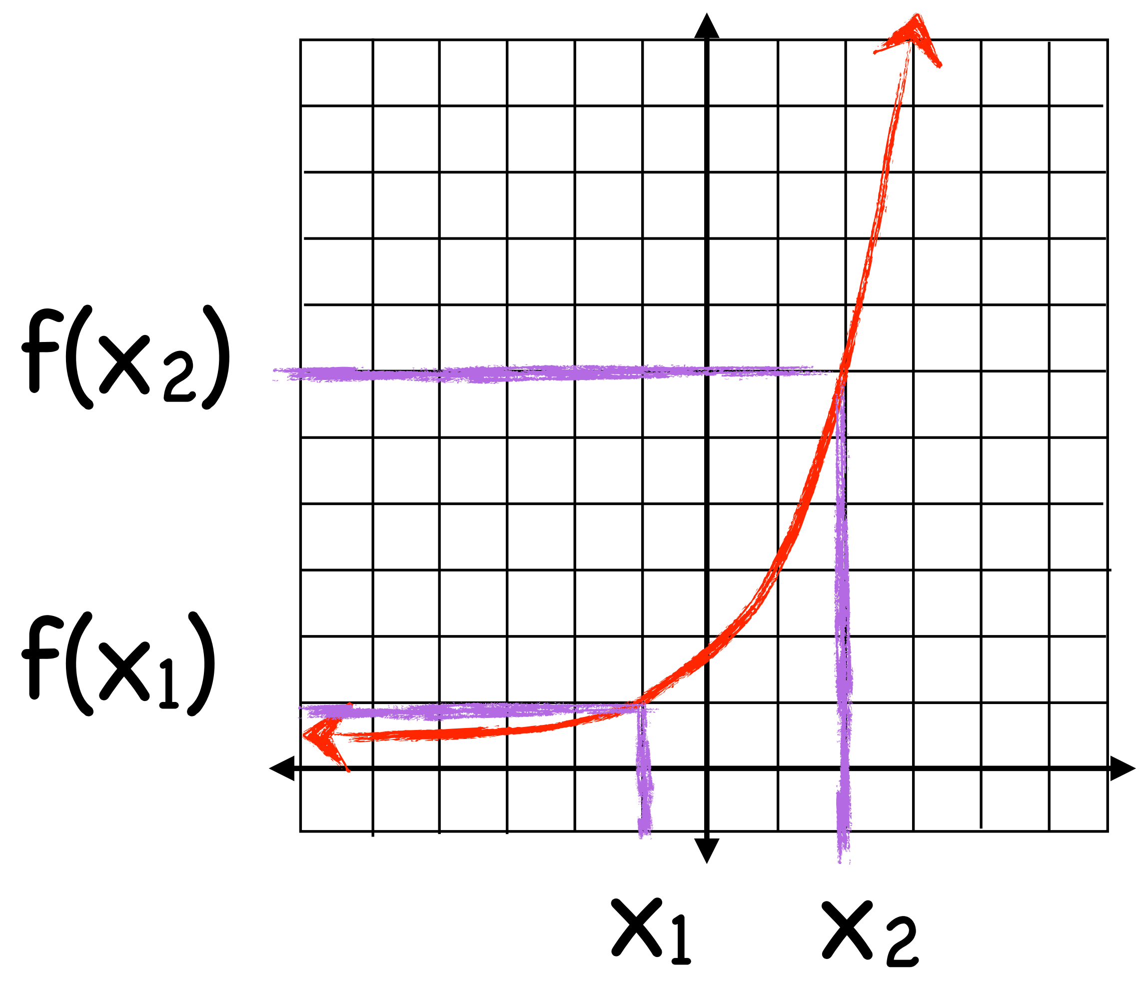
- If $b > 1$, $f(x) = b^x$ has a graph that goes up to the right and is an increasing function. The greater the value of b, the steeper the increase.

- If $0 < b < 1$, $f(x) = b^x$ has a graph that goes down to the right and is a decreasing function. The smaller the value of b, the steeper the decrease.

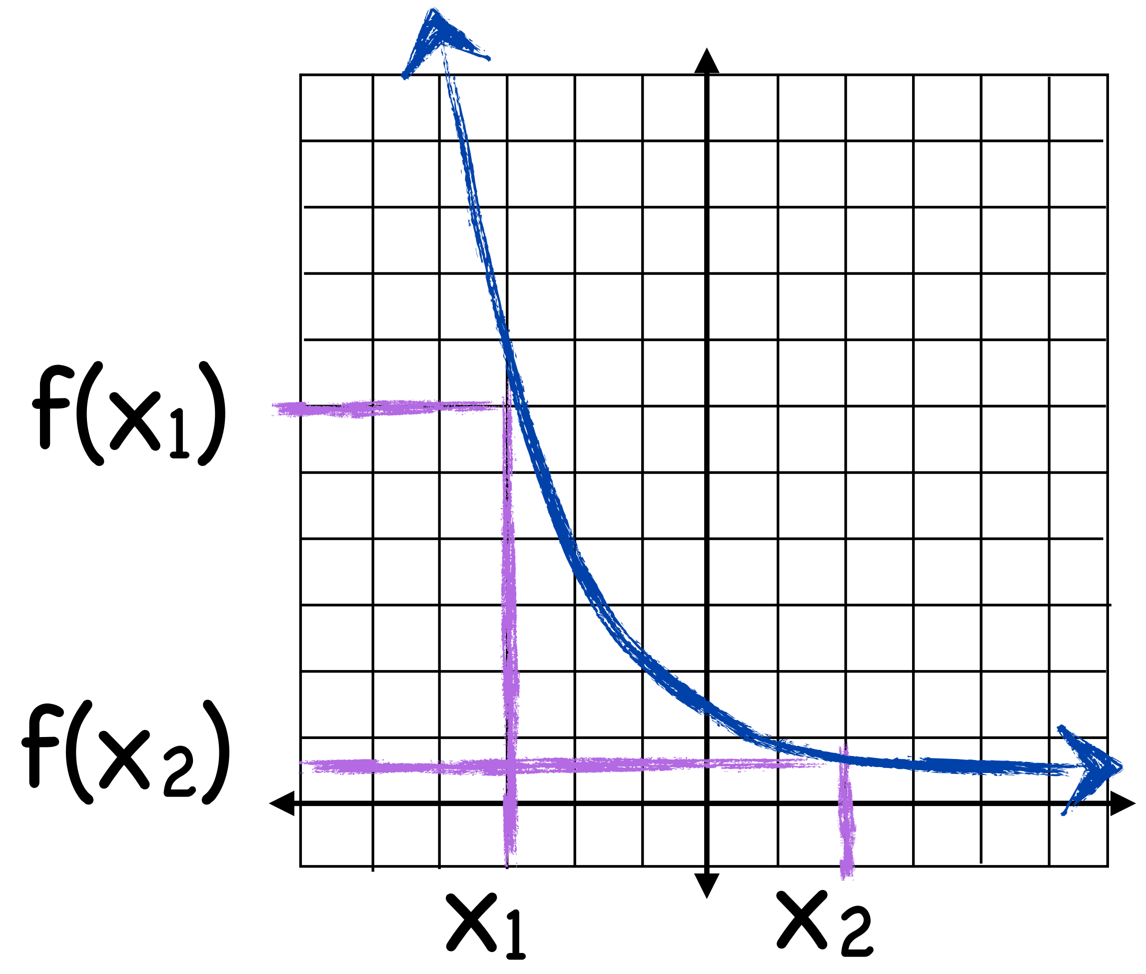
- $f(x) = b^x$ is a one-to-one function, thus the inverse of f is also a function.



- A function is said to be increasing in the interval $[x_1, x_2]$ if for every value in the interval, if $a > b$, then $f(a) > f(b)$



- A function is said to be decreasing in the interval $[x_1, x_2]$ if for every value in the interval, if $a > b$, then $f(a) < f(b)$



Transformations of exponential functions.

Transformation	Equation	Description
Vertical translation	$g(x) = b^x + c$ $g(x) = b^x - c$	• Shifts the graph of $f(x) = b^x$ upward c units. • Shifts the graph of $f(x) = b^x$ downward c units.
Horizontal translation	$g(x) = b^{x+c}$ $g(x) = b^{x-c}$	• Shifts the graph of $f(x) = b^x$ to the left c units. • Shifts the graph of $f(x) = b^x$ to the right c units.
Reflection	$g(x) = -b^x$ $g(x) = b^{-x}$	• Reflects the graph of $f(x) = b^x$ about the x-axis. • Reflects the graph of $f(x) = b^x$ about the y-axis.
Vertical stretching or shrinking	$g(x) = cb^x$	• Vertically stretches the graph of $f(x) = b^x$ if $c > 1$. • Vertically shrinks the graph of $f(x) = b^x$ if $0 < c < 1$.
Horizontal stretching or shrinking	$g(x) = b^{cx}$	• Horizontally shrinks the graph of $f(x) = b^x$ if $c > 1$ • Horizontally stretches the graph of $f(x) = b^x$ if $0 < c < 1$.

Vertical Shift $g(x) = b^x + c$

$$f(x) = 1.5^x$$

x	-2	-1	0	1	2
f(x)	4/9	2/3	1	3/2	9/4

$$g(x) = 1.5^x + 2$$

x	-2	-1	0	1	2
g(x)	2 4/9	2 2/3	3	2 3/2	4 1/4

- To transform $f(x) = 1.5^x$ into $g(x) = 1.5^x + 2$ we added 2 to each y value and the graph shifts up 2 units.

Horizontal Shift $g(x) = b^{x+c}$

$f(x) = 1.5^x$

x	-2	-1	0	1	2
f(x)	4/9	2/3	1	3/2	9/4

$g(x) = 1.5^{x+2}$

x	-4	-3	-2	-1	0
g(x)	4/9	2/3	1	3/2	9/4

- To transform $f(x) = 1.5^x$ into $g(x) = 1.5^{x+2}$ we subtracted 2 from each x value and the graph shifts **left** 2 units.

Vertical Reflection $g(x) = -b^x$

$$f(x) = 1.5^x$$

x	-2	-1	0	1	2
f(x)	4/9	2/3	1	3/2	9/4

$$g(x) = -1.5^x$$

x	-2	-1	0	1	2
g(x)	-4/9	-2/3	-1	-3/2	-9/4

- To transform $f(x) = 1.5^x$ into $g(x) = -1.5^x$ we multiply each y value by −1 and the graph is **reflected** across the x-axis.

Horizontal Reflection $g(x) = b^{-x}$

$$f(x) = 1.5^x$$

x	-2	-1	0	1	2
f(x)	4/9	2/3	1	3/2	9/4

$$g(x) = 1.5^{-x}$$

x	−2	−1	0	1	2
g(x)	9/4	3/2	1	2/3	4/9

- To transform $f(x) = 1.5^x$ into $g(x) = 1.5^{-x}$ we multiply each x value by −1 and the graph is **reflected** across the y-axis.

Vertical Stretch $g(x) = ab^x$

$f(x) = 1.5^x$

x	-2	-1	0	1	2
f(x)	4/9	2/3	1	3/2	9/4

$g(x) = 2(1.5)^x$

x	−2	−1	0	1	2
g(x)	8/9	4/3	2	3	9/2

- To transform $f(x) = 1.5^x$ into $g(x) = 2(1.5)^x$ we multiply each y value by 2 and the graph is **stretched** vertically.

Vertical Compression $g(x) = ab^x$

$$f(x) = 1.5^x$$

x	-2	-1	0	1	2
f(x)	4/9	2/3	1	3/2	9/4

$$g(x) = \frac{1}{2}(1.5)^x$$

x	2	1	0	-1	-2
g(x)	2/9	1/3	1/2	3/4	9/8

- To transform $f(x) = 1.5^x$ into $g(x) = 1/2(1.5)^x$ we simply multiply each y value by 1/2 and the graph is **compressed** vertically.

Horizontal Stretch $g(x) = b^{ax}$

$$f(x) = 1.5^x$$

x	-2	-1	0	1	2
f(x)	4/9	2/3	1	3/2	9/4

$$g(x) = 1.5^{\frac{1}{2}x}$$

x	-4	-2	0	2	4
g(x)	4/9	2/3	1	3/2	9/4

- To transform $f(x) = 1.5^x$ into $g(x) = 1.5^{1/2x}$ we simply multiply each x value by 2 and the graph is **stretched** horizontally by 2.

Horizontal Compression $g(x) = b^{ax}$, $a > 1$.

$$f(x) = 1.5^x$$

x	-2	-1	0	1	2
f(x)	4/9	2/3	1	3/2	9/4

$$g(x) = 1.5^{2x}$$

x	-1	-1/2	0	1/2	1
g(x)	4/9	2/3	1	3/2	9/4

- To transform $f(x) = 1.5^x$ into $g(x) = 1.5^{2x}$ we simply multiply each x value by 1/2 and the graph is **compressed** horizontally by 1/2.

Caution: Horizontal Shift w/ Compression $g(x) = b^{ax+c}$

$$f(x) = 1.5^x$$

x	-2	-1	0	1	2
f(x)	4/9	2/3	1	3/2	9/4

$$g(x) = 1.5^{2x+1}$$

x	-3/2	-3/4	-1/2	0	1/2
g(x)	4/9	2/3	1	3/2	9/4

- To transform $f(x) = 1.5^x$ into $g(x) = 1.5^{2x+1}$ we subtract 1 from each x and multiply by 1/2, the graph is shifted left and compressed.

Order of Transformations

- Transformations can be combined within the same function so that one graph can be shifted, stretched, and reflected. If a function contains more than one transformation it may be graphed using the following order:

    1. Horizontal Translation
    2. Stretch or compress
    3. Reflect
    4. Vertical Translation

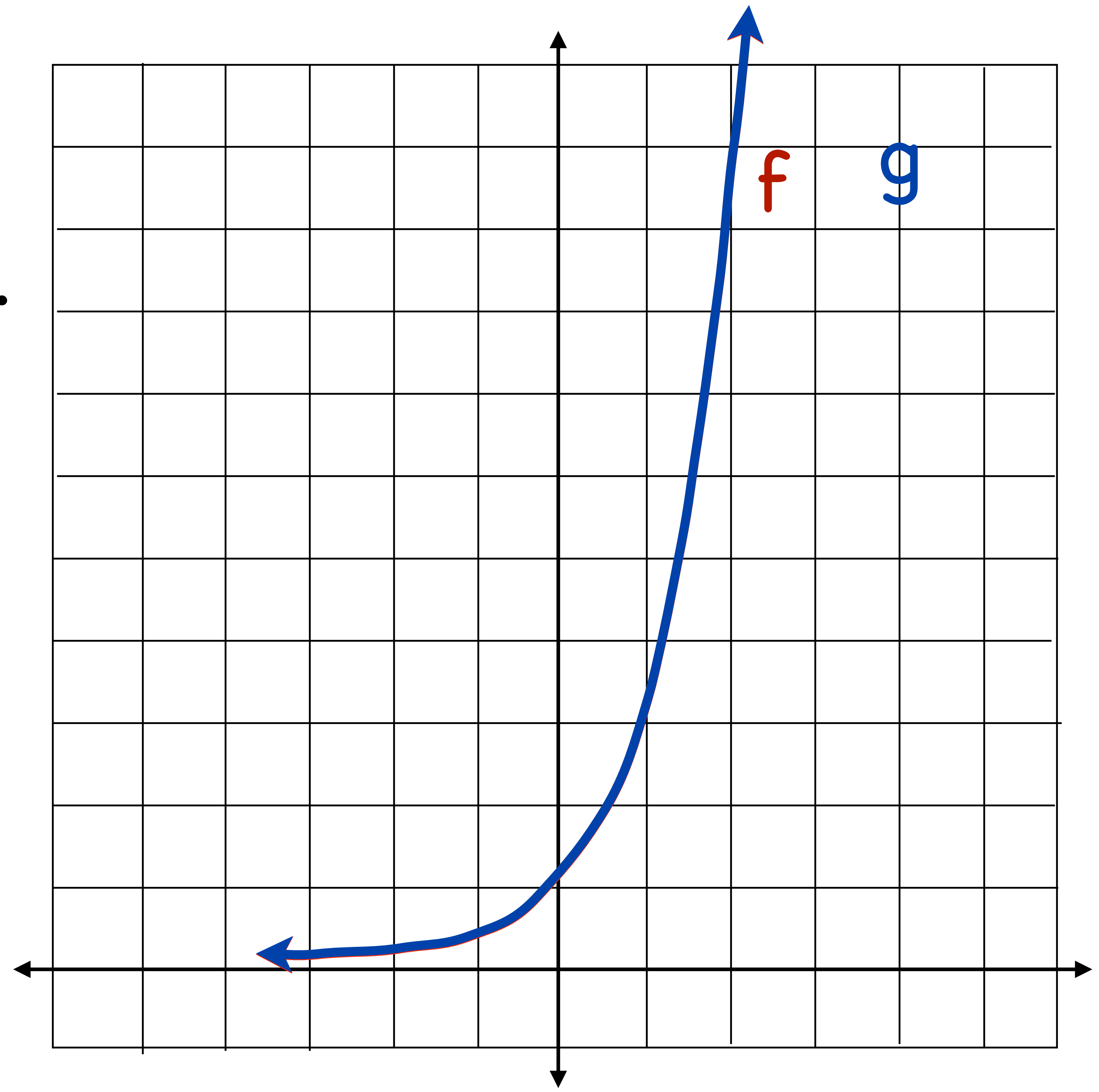
- Use the graph of $f(x) = 3^x$ to obtain the graph of $g(x) = 3^{x-1}$.

$g(x) = f(x-1)$

g(x) is found by a horizontal shift of 1 unit to the right.

Of course there is always.

x	-1	0	1	2
g(x)	1/9	1/3	1	3



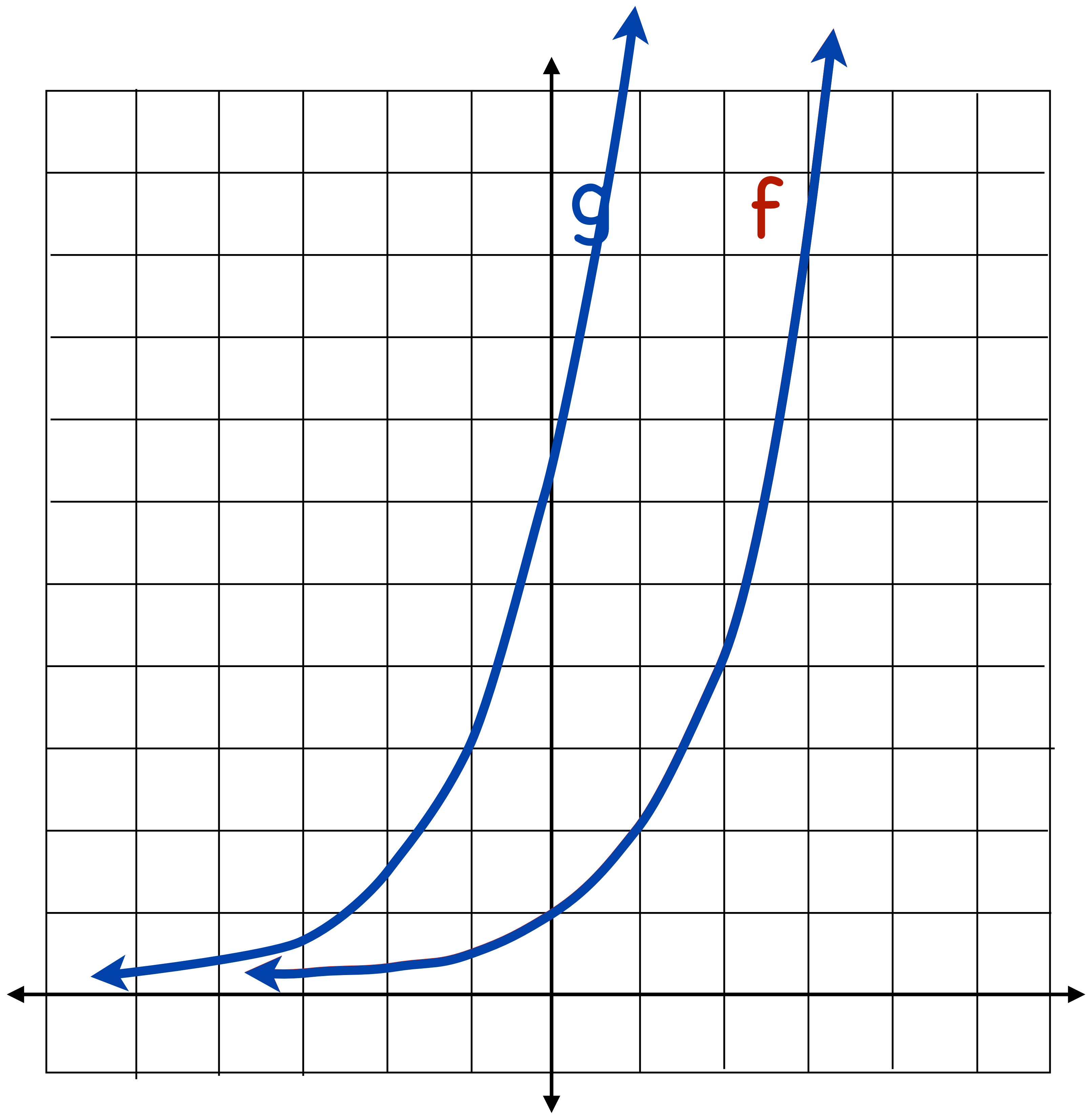
- Use the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 3(2^{x+1}) - 2$.

  $g(x) = 3f(x+1) - 2$.

  g(x) is found by a horizontal shift of 1 unit to the left, a vertical stretch of 3

  and a vertical shift down 2.

  Of course there is always.



x	-2	-1	0	1
g(x)	-1/2	1	4	10

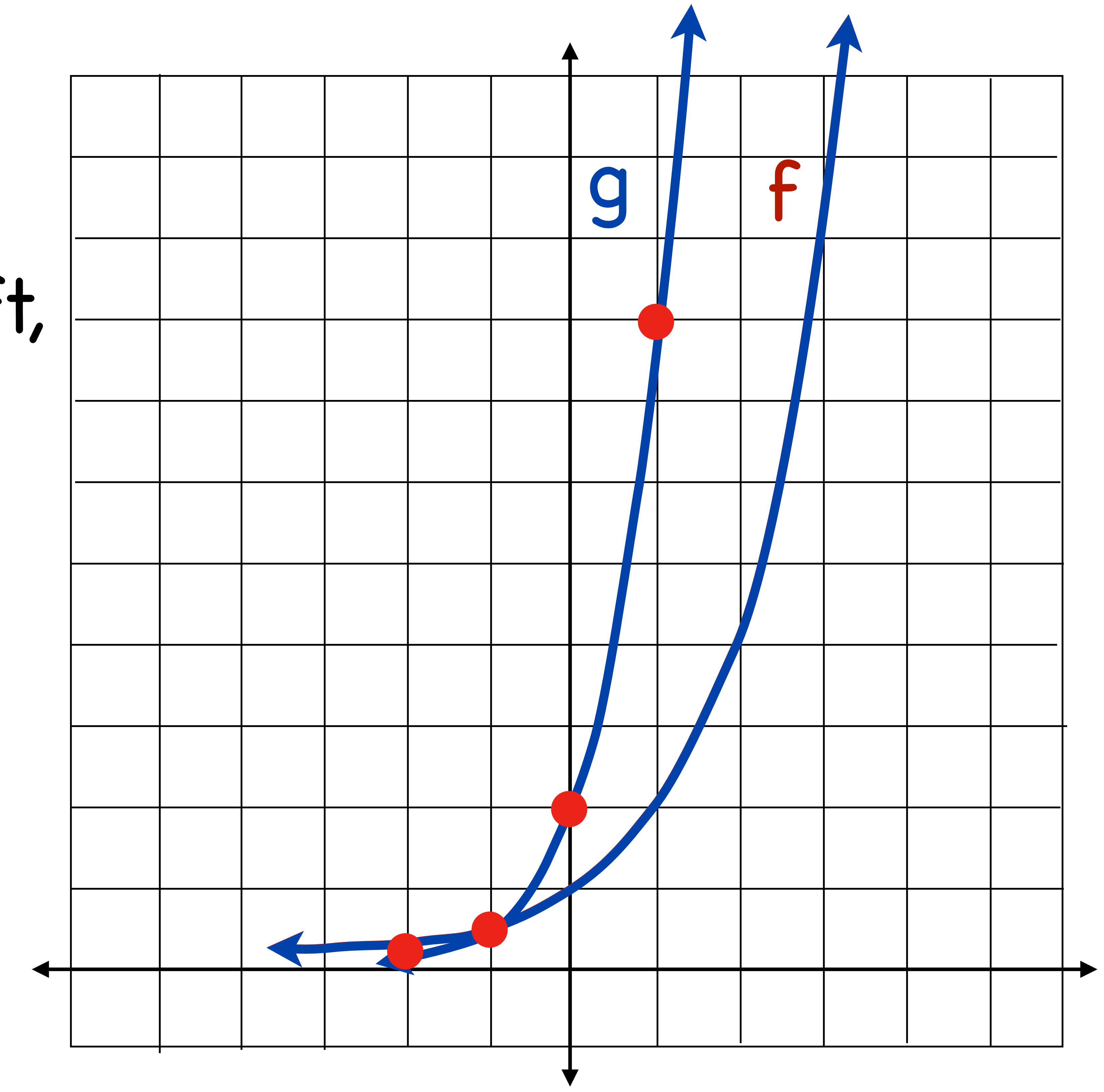
- Use the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 2^{2x+1}$.

$$g(x) = f(2x+1) = f\left(2\left(x+\frac{1}{2}\right)\right)$$

g(x) is found by a horizontal shift of 1/2 unit to the left, and a horizontal **compression** by a factor of 2.

Of course there is always.

x	-2	-1	0	1
g(x)	1/8	1/2	2	8



Ouch!

- If the annual rate of inflation averages 4% over the next 10 years, the approximate costs C of goods or services during any year in that decade will be modeled by $C(t) = P(1.04)^t$, where t is the time in years and P is the present cost. The price of an oil change for your car is currently 23.95. Estimate the price 10 years from now

$$C(t) = P(1.04)^t \qquad C(t) = 23.95(1.04)^{10} \approx 35.45$$

  - In 10 years an oil change is predicted to cost \$35.45.

Example

- In 2005, there were 180 inhabitants in a remote town. Population has increased by 12% every year. How many residents will there be in 15 years?

$$P(t) = P(1 + .12)^{t} \qquad P(15) = 180(1.12)^{15} \quad \approx 985.2418$$

- In 15 years the population is predicted to be about 985.

The Natural Base e

- The number e is defined as the value that $\left(1+\frac{1}{n}\right)^n$ approaches as n gets larger and larger. (As $n \to \infty$).

  - Break out the calculator and complete the table

x	1	10	100	1000	10,000	100,000	1,000,000
$\left(1+\frac{1}{x}\right)^x$	2	2.5937	2.7048	2.7169	2.7181	2.7183	2.7183

- Enter the function $y=\left(1+\frac{1}{x}\right)^x$ ZOOM 6

2nd WINDOW TBLSET

TblStart=1
ΔTbl=10
Indpnt: Ask
Depend: Auto

2nd GRAPH TABLE

Enter values for x in the table

The natural Base e

- The irrational number, **e**, approximately 2.718, is called the **natural base**. The function $f(x) = e^x$ is called the **natural exponential function**.

$$e \approx 2.718281827$$

- Graphing powers of **e** is the same as graphing other exponential functions.

  - Graph 2^x and 3^x and e^x on the TI-84.

  - Note the e button [2nd] [÷] (e)

  - Also note the e^x button [2nd] [ln] (e^x)

  - Looky there, e^x is between 2^x and 3^x, and very close to 3^x.

Example: Evaluating Functions with Base e

- The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, x years after 1978. Project the gray wolf's population in the recovery area in 2012.

2012 is 34 years after 1978, so $x = 34$.

$$f(x) = 1066e^{0.042x} \qquad f(x) = 1066e^{0.042(34)} \approx 4445.593255$$

- The model predicts the gray wolf's population to be approximately 4446.

Example: Evaluating Functions with Base e

- The number V of computers infected by a computer virus increases according to the model $V(t) = 100e^{4.6052t}$, where t is the time in hours. Find V(1), V(1.5), and V(2).

$$V(1) = 100e^{4.6052(1)} \approx 10,000.2981$$

$$V(1.5) = 100e^{4.6052(1.5)} \approx 100,004.4722$$

$$V(2) = 100e^{4.6052(2)} \approx 1,000,059.63$$

Formulas for Compound Interest

- After t years, the balance, A, in an account with principal P and annual interest rate r (in decimal form), for n compounding periods per year, is given by the following formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A = Amount accrued, P = Principal (original investment), r = annual percentage rate (APR), and n = n umber of compounding periods per year.

If interest is compounded continuously ($n \to \infty$)

$$A = Pe^{rt}$$

Using Compound Interest Formulas

- A sum of \$10,000 is invested at an annual rate of 8%. Find the balance in the account after 5 years subject to **quarterly** compounding.

We will use the formula for n compounding periods per year, with n = 4.

$$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt}} = 10,000\left(1+\frac{.08}{4}\right)^{4(5)} \approx 14,859.47$$

The balance of the account after 5 years subject to quarterly compounding will be about \$14,859.47.

- A sum of $10,000 is invested at an annual rate of 8%. Find the balance in the account after 5 years subject to **daily** compounding.

We will use the formula for n compounding periods per year, with n = 365.

$$\boxed{A = P\left(1 + \frac{r}{n}\right)^{nt}} = 10,000\left(1 + \frac{.08}{365}\right)^{365(5)} \approx 14,917.59$$

The balance of the account after 5 years subject to quarterly compounding will be about $14,917.59.

- A sum of \$10,000 is invested at an annual rate of 8%. Find the balance in the account after 5 years subject to **continuous** compounding.

We will use the formula for **continuous compounding**.

$$\boxed{A = Pe^{rt}} = 10,000e^{.08(5)} \approx 14,918.25$$

The balance in the account after 5 years subject to continuous compounding will be \$14,918.25.

Ewwww

- A strain of bacteria growing on your desktop grows at a rate given by;

  - $B(t) = B_0 e^{0.1386294361t}$

  - where t is the time in minutes. Assuming that you start with only one bacterium, how many bacteria could be present at the end of 56 minutes? (Note: B_0 is the bacteria count at time 0.)

$$\boxed{B = B_0 e^{0.1386294361t}} = 1e^{0.1386294361(56)} \approx 2352.5342$$

  - So if you get 1 at the start of the period, you will have 2353 by period end.