Chapter 3

Exponential and Logarithmic Functions

3.2 Logarithmic Functions

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29, 33, 37, 41, 55, 63, 71, 85

Chapter 3-2

Objectives

Change from logarithmic to exponential form.
Change from exponential to logarithmic form.
Evaluate logarithms.
Use basic logarithmic properties.
Graph logarithmic functions.
Find the domain of a logarithmic function.
Use common logarithms.
Use natural logarithms.

Definition of the Logarithmic Function

- We can solve an equation when the variable is in the exponent by using an **inverse** operation. The inverse of an exponential function is a **logarithmic** function.

- The function given by $f(x) = \log_b x$, where $x > 0$, $b > 0$, and $b \neq 1$, is called the logarithmic function with base b.

- The logarithmic function with base b is the inverse of the exponential function with base b; thus $y = \log_b x$ if and only if $x = b^y$.

$$y = b^x$$

exponent (becomes logarithm)

base

Definition of the Logarithmic Function

- Since exponential functions are one-to-one functions, the inverse of an exponential function is also a function.
  - The inverse operation is called finding the logarithm. A **logarithm** is the **exponent** to which a specified **base** is raised to obtain a given value.
    - Let me repeat that…

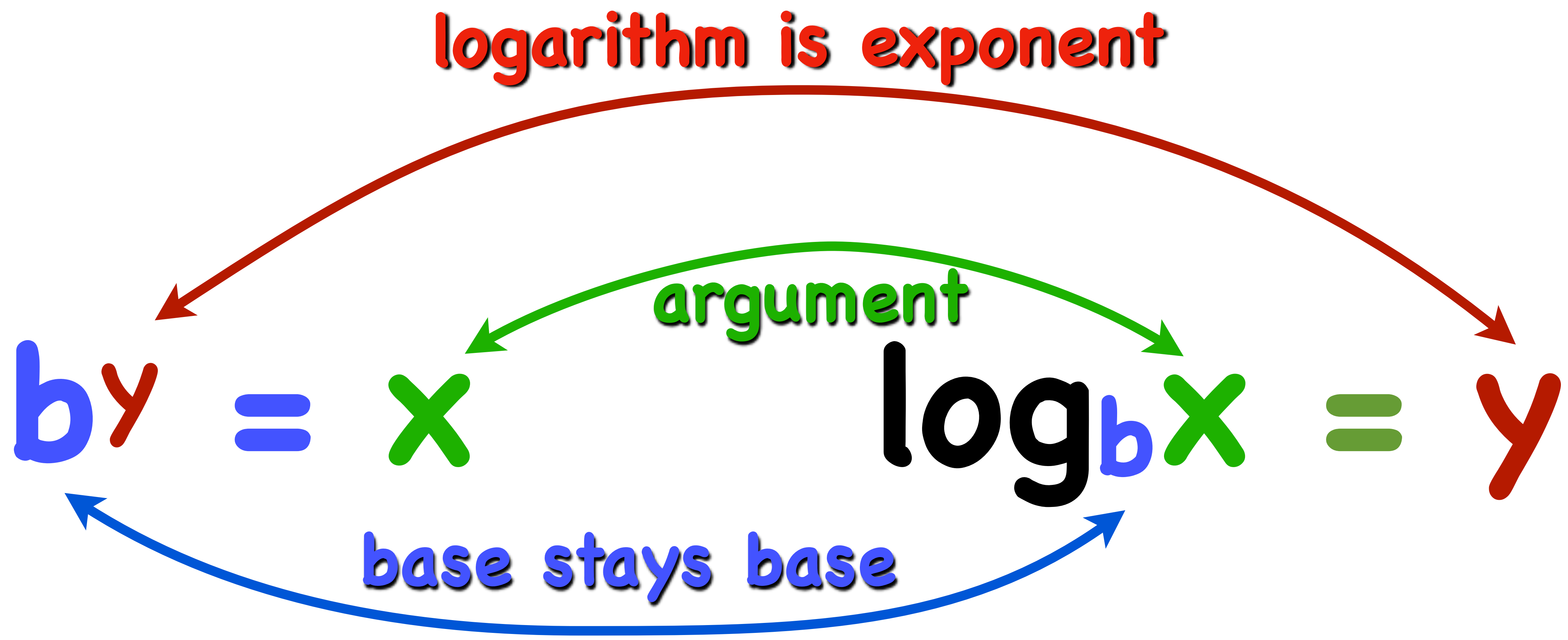
A **logarithm** is the **exponent** to which a specified **base** is raised to obtain a given value (x).

Definition of the Logarithmic Function

For $x > 0$ and $b > 0$, $b \neq 1$,

$y = \log_b x$ is equivalent to $b^y = x$.

The function $f(x) = \log_b x$ is the logarithmic function with **base b**.



$b > 0$, $b \neq 1$

Changing from Logarithmic to Exponential Form

- Write each equation in its equivalent exponential form:
  - $y = \log_b x$ is equivalent to $b^y = x$.

$3 = \log_7 x$ is equivalent to $7^3 = x$

$2 = \log_b 25$ is equivalent to $b^2 = 25$

$\log_4 26 = y$ is equivalent to $4^y = 26$

- Write each equation in its equivalent logarithmic form:
  - $y = \log_b x$ is equivalent to $b^y = x$.

$2^5 = x$ is equivalent to $5 = \log_2 x$

$b^3 = 27$ is equivalent to $3 = \log_b 27$

$e^y = 33$ is equivalent to $\log_e 33 = y$

Example: Evaluating Logarithms

Evaluate: $\log_{10} 100$ means 10 to what power gives 100?

$\log_{10} 100 = y$ $\quad 10^y = 100$ $\quad 10^2 = 100$ $\quad y = 2$ $\quad \log_{10} 100 = 2$

Evaluate: $\log_5 \frac{1}{125}$ means 5 to what power gives $\frac{1}{125}$?

$$\frac{1}{125} = \frac{1}{5^3} = 5^{-3}$$

$\log_5 \frac{1}{125} = y$ $\quad 5^y = \frac{1}{125} = 5^{-3}$ $\quad \log_5 \frac{1}{125} = -3$

The log is (=) the exponent.

Example: Evaluating Logarithms

Evaluate: $\log_{36} 6$ means 36 to what power equals 6?

$$\log_{36} 6 = y \qquad 36^y = 6 \qquad 6 = \sqrt{36} = 36^{\frac{1}{2}}$$

$$\log_{36} 6 = \log_{36} 36^{\frac{1}{2}} = y$$

$$36^y = 36^{\frac{1}{2}} \qquad y = \frac{1}{2} \qquad \log_{36} 6 = \frac{1}{2}$$

The log is (=) the exponent.

Example: Evaluating Logarithms

Evaluate: $\log_3 \sqrt[7]{3}$ means 3 to what power gives $\sqrt[7]{3}$?

$$\log_3 \sqrt[7]{3} = y \qquad 3^y = \sqrt[7]{3} \qquad \sqrt[7]{3} = 3^{\frac{1}{7}}$$

$$\log_3 \sqrt[7]{3} = \log_3 3^{\frac{1}{7}} = y \qquad 3^y = 3^{\frac{1}{7}} \qquad y = \frac{1}{7}$$

$$\log_3 \sqrt[7]{3} = \frac{1}{7}$$

The log is (=) the exponent.

Basic Log Properties Involving One

1. $\log_b b = 1$ because 1 is the exponent to which b must be raised to obtain b. $(b^1 = b)$

2. $\log_b 1 = 0$ because 0 is the exponent to which b must be raised to obtain 1. $(b^0 = 1)$

   Evaluate: $\log_9 9$ Because $\log_b b = 1$, we conclude $\log_9 9 = 1$

   Evaluate: $\log_8 1$ Because $\log_b 1 = 0$, we conclude $\log_8 1 = 0$.

3. $\log_b x = \log_b y$ iff $x = y$.

Inverse Properties of Logarithms

For b > 0 and b ≠ 1,

$$\log_b b^x = x \quad \text{and} \quad b^{\log_b x} = x$$

Note: In both cases, the bases are the same!

Evaluate: $\log_7 7^8$ $\qquad$ $\log_7 7^8 = 8$

Evaluate: $3^{\log_3 17}$ $\qquad$ $3^{\log_3 17} = 17$

Graphs of Exponential and Logarithmic Functions

Graph $f(x) = 3^x$ and $g(x) = \log_3 x$ in the same rectangular coordinate system.

We can first set up a table of values for $f(x) = 3^x$.

Reverse the coordinates for the inverse function $g(x) = \log_3 x$

x	$f(x)=3^x$
-2	1/9
-1	1/3
0	1
1	3
2	9

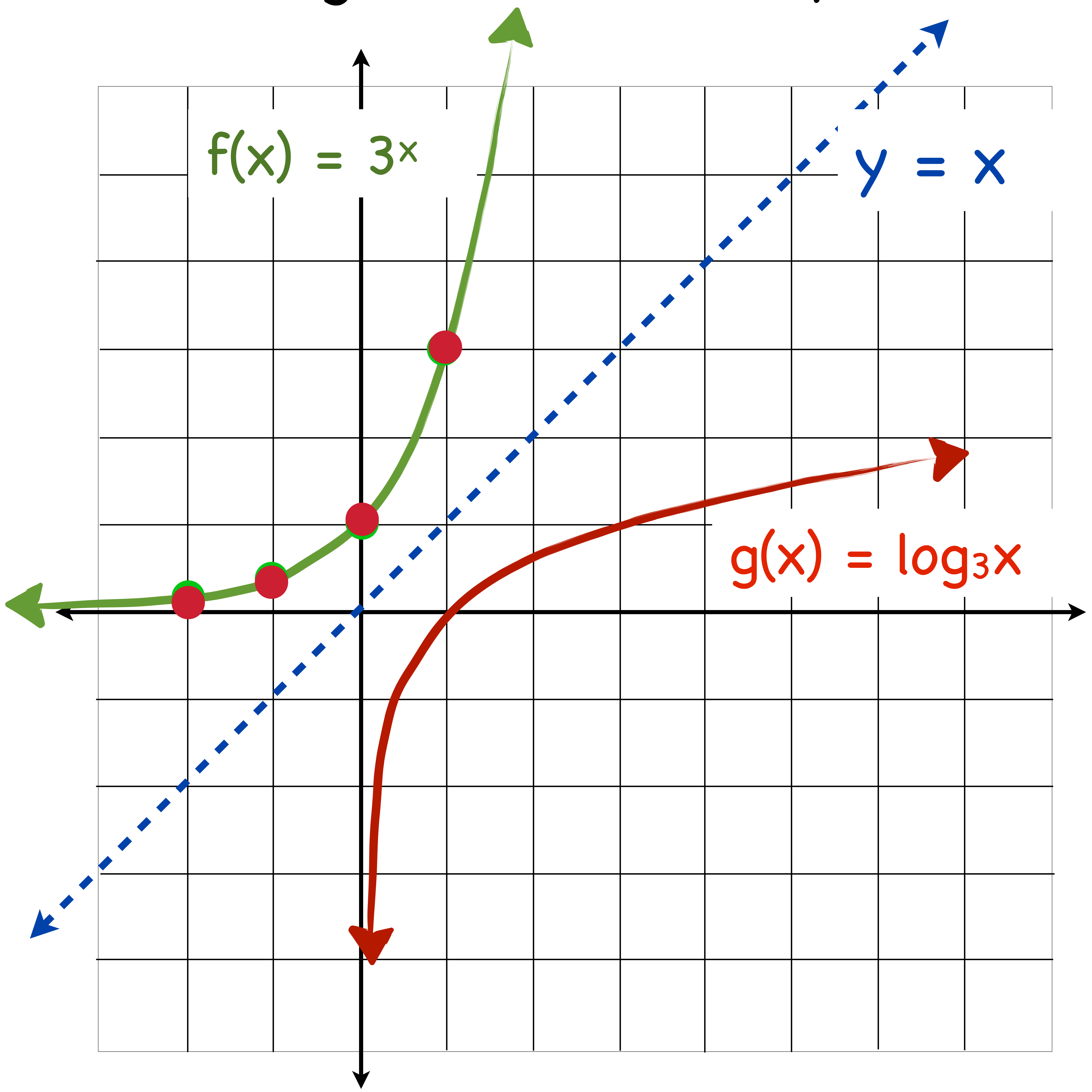
x	$g(x)=\log_3 x$
1/9	-2
1/3	-1
1	0
3	1
9	2

Graphs of Exponential and Logarithmic Functions

We now graph $f(x) = 3^x$ and $g(x) = \log_3 x$ the same rectangular coordinate system.

x	$f(x)=3^x$
-2	1/9
-1	1/3
0	1
1	3
2	9

x	$g(x)=\log_3 x$
1/9	-2
1/3	-1
1	0
3	1
9	2



$g(x) = \log_3 x$ is $f(x) = 3^x$
reflected across $y = x$.

Characteristics of Logarithmic functions of the Form $f(x) = \log_b x$

- The domain of $f(x) = \log_b x$ consists of all **positive** real numbers: $(0, \infty)$
- The range of $f(x) = \log_b x$ consists of all real numbers: $(-\infty, \infty)$
- The graphs of all log functions of the form $f(x) = \log_b x$ pass through the point (1,0).
- If $b > 1$, $f(x) = \log_b x$ is an increasing function and has a graph that goes up to the right
- If $0 < b < 1$, $f(x) = \log_b x$ is a decreasing function and has a graph that goes down to the right
- The graph of $f(x) = \log_b x$ is asymptotic to the y-axis. (The y-axis is an asymptote of $f(x) = \log_b x$).

Table Repeat

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.
2. $\log_a a = 1$ because $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$ — Inverse Properties
4. If $\log_a x = \log_a y$, then $x = y$. — One-to-One Property

Transformations of Log Function

Table 4.4 Transformations Involving Logarithmic Functions

In each case, c represents a positive real number.

Transformation	Equation	Description
Vertical translation	$g(x) = \log_b x + c$ $g(x) = \log_b x - c$	• Shifts the graph of $f(x) = \log_b x$ upward c units. • Shifts the graph of $f(x) = \log_b x$ downward c units.
Horizontal translation	$g(x) = \log_b(x + c)$ $g(x) = \log_b(x - c)$	• Shifts the graph of $f(x) = \log_b x$ to the left c units. Vertical asymptote: $x = -c$ • Shifts the graph of $f(x) = \log_b x$ to the right c units. Vertical asymptote: $x = c$
Reflection	$g(x) = -\log_b x$ $g(x) = \log_b(-x)$	• Reflects the graph of $f(x) = \log_b x$ about the x-axis. • Reflects the graph of $f(x) = \log_b x$ about the y-axis.
Vertical stretching or shrinking	$g(x) = c \log_b x$	• Vertically stretches the graph of $f(x) = \log_b x$ if $c > 1$. • Vertically shrinks the graph of $f(x) = \log_b x$ if $0 < c < 1$.
Horizontal stretching or shrinking	$g(x) = \log_b(cx)$	• Horizontally shrinks the graph of $f(x) = \log_b x$ if $c > 1$. • Horizontally stretches the graph of $f(x) = \log_b x$ if $0 < c < 1$.

The Domain of a Logarithmic Function

- The domain of an exponential function of the form $f(x) = b^x$ includes all real numbers and its range is the set of positive real numbers.

- Because the logarithmic function is the inverse of the exponential function, the logarithmic function reverses the domain and the range of the exponential function.

- **The domain of a logarithmic function of the form** $f(x) = \log_b x$ is the set of all positive real numbers. In general, **the domain of** $f(x) = \log_b[g(x)]$ **consists of all x for which** $g(x) > 0$.

Finding the Domain of a Logarithmic Function

Find the domain of **$f(x) = \log_4(x - 5)$**.

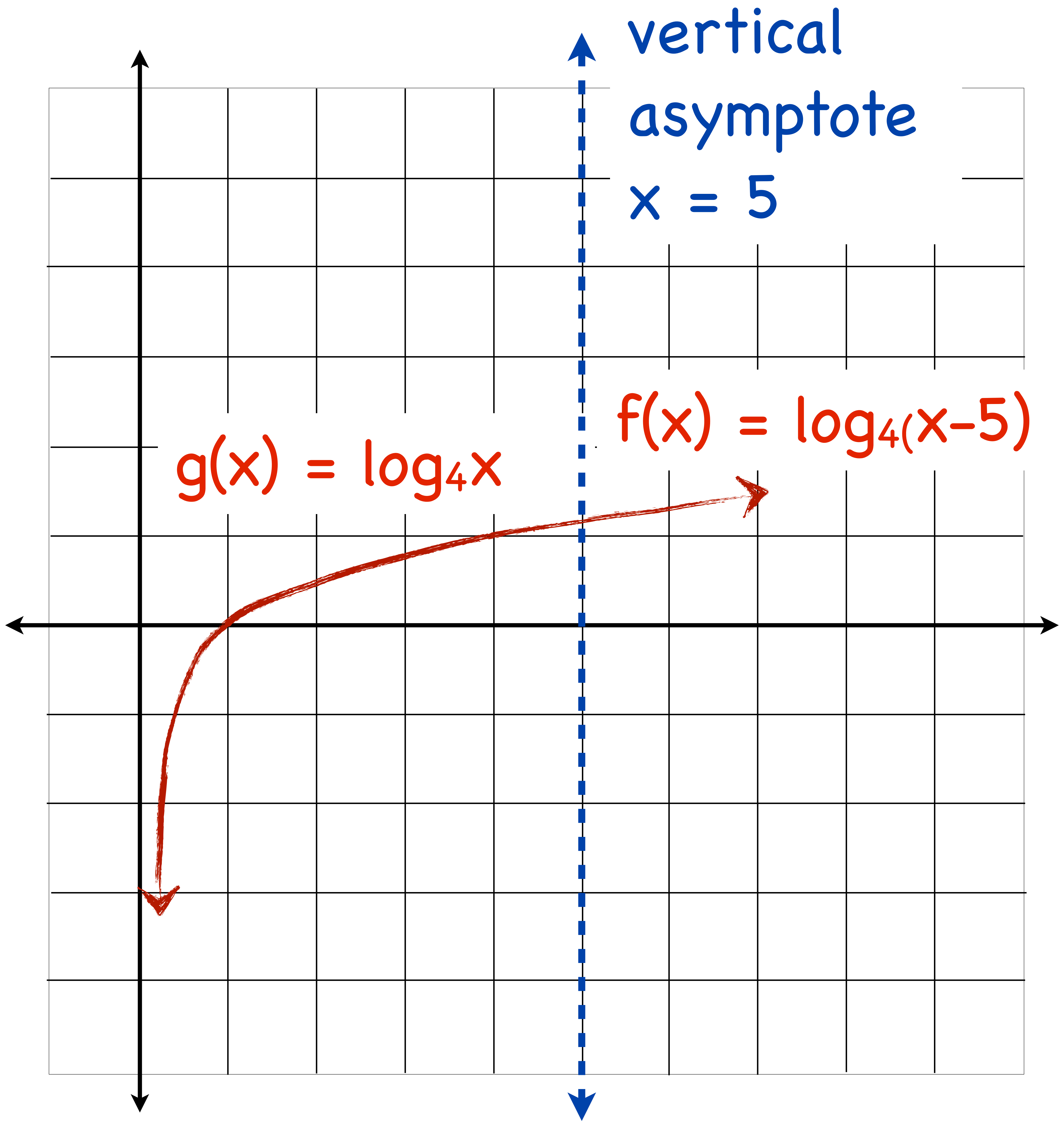
The domain of f consists of all x for which $x - 5 > 0$.

Thus, the domain of f is $(5, \infty)$

If we graph $f(x) = \log_4(x - 5)$ we can see the domain.

$f(x) = \log_4(x - 5)$ is $g(x) = \log_4 x$ shifted 5 units right.

All point of f(x) = log4(x-5) have x coordinates > 5.



Common Logarithms

- The logarithmic function with **base 10** is called the **common logarithmic function**.

- The function $f(x) = \log_{10}x$ is usually expressed $\mathbf{f(x) = \log x}$.

- When the base is not given in the expression it is assumed to be 10.

- The TI-84 has a LOG button for **common log**. You should not need instruction on how to calculate the value of a **common log** on the TI.

Example: Application

The percentage of adult height attained by a boy **x years old** can be modeled by **$f(x) = 29 + 48.8\log(x + 1)$** where **x** represents the boy's age (from 5 to 15) and **$f(x)$** represents the percentage of his adult height. Approximately what percentage of his adult height has a boy attained at age ten?

$$f(x) = 29 + 48.8\log(x + 1)$$

$$f(10) = 29 + 48.8\log(10 + 1)$$

$$f(10) = 29 + 48.8\log(11)$$

$$f(10) \approx 29 + 48.8(1.04139)$$

$$f(10) \approx 79.82$$

A ten-year old boy has attained approximately 79.8% of his adult height.

Natural Logarithms

- The logarithmic function with **base e** is called the **natural logarithmic function**.
  - The function $f(x) = \log_e x$ is usually expressed $f(x) = \ln x$.
  - The properties of the natural logarithm are identical to the properties for logarithms of any other base.

$$\ln e = 1 \ (e^1 = e) \qquad \ln 1 = 0 \ (e^0 = 1)$$

  - The domain of $f(x) = \ln x$ consists of all positive real numbers: $(0, \infty)$
  - The range of $f(x) = \ln x$ consists of all real numbers: $(-\infty, \infty)$
  - The graph of $f(x) = \ln x$ is asymptotic to the y-axis.
  - The graph of $f(x) = \ln x$ passes through the point (1,0).
- The TI-84 has a LN button for **natural logs**. You should not need instruction on how to calculate the value of a **natural log** on the TI.

Example: Application

When the outside air temperature is anywhere from 72° to 96° Fahrenheit, the temperature in an enclosed vehicle climbs by 43° in the first hour. The function

$$f(x) = 13.4\ln x - 11.6$$

models the temperature increase, f(x), in degrees Fahrenheit, after x minutes. Use the function to find the temperature increase, to the nearest degree, after 30 minutes.

$$f(x) = 13.4\ln x - 11.6$$

$$f(30) = 13.4\ln 30 - 11.6$$

$$f(30) \approx 13.4(3.4012) - 11.6$$

$$f(30) \approx 33.98^\circ$$

The temperature will increase by approximately 34° after 30 minutes.