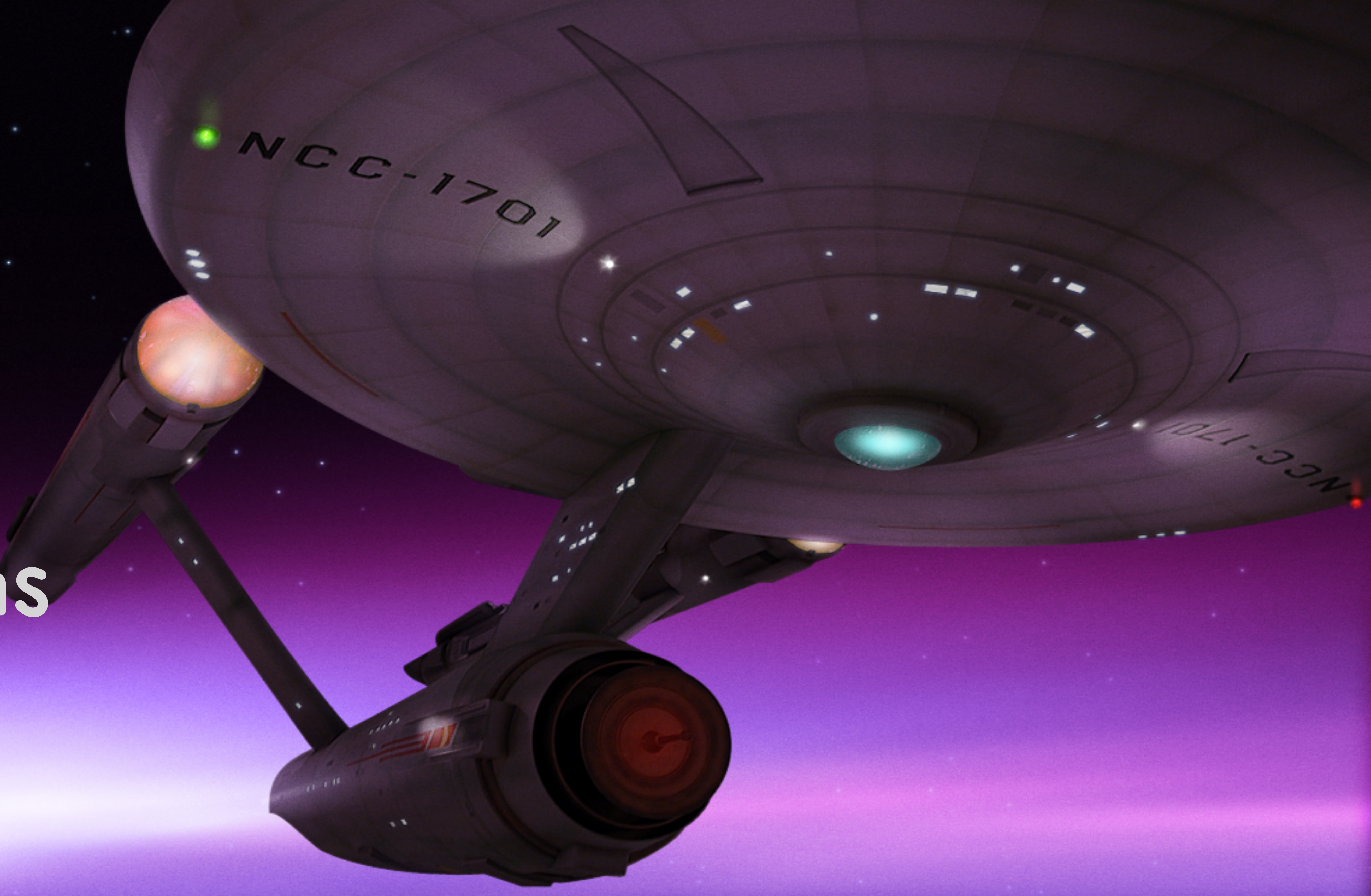
Chpt the 3rd

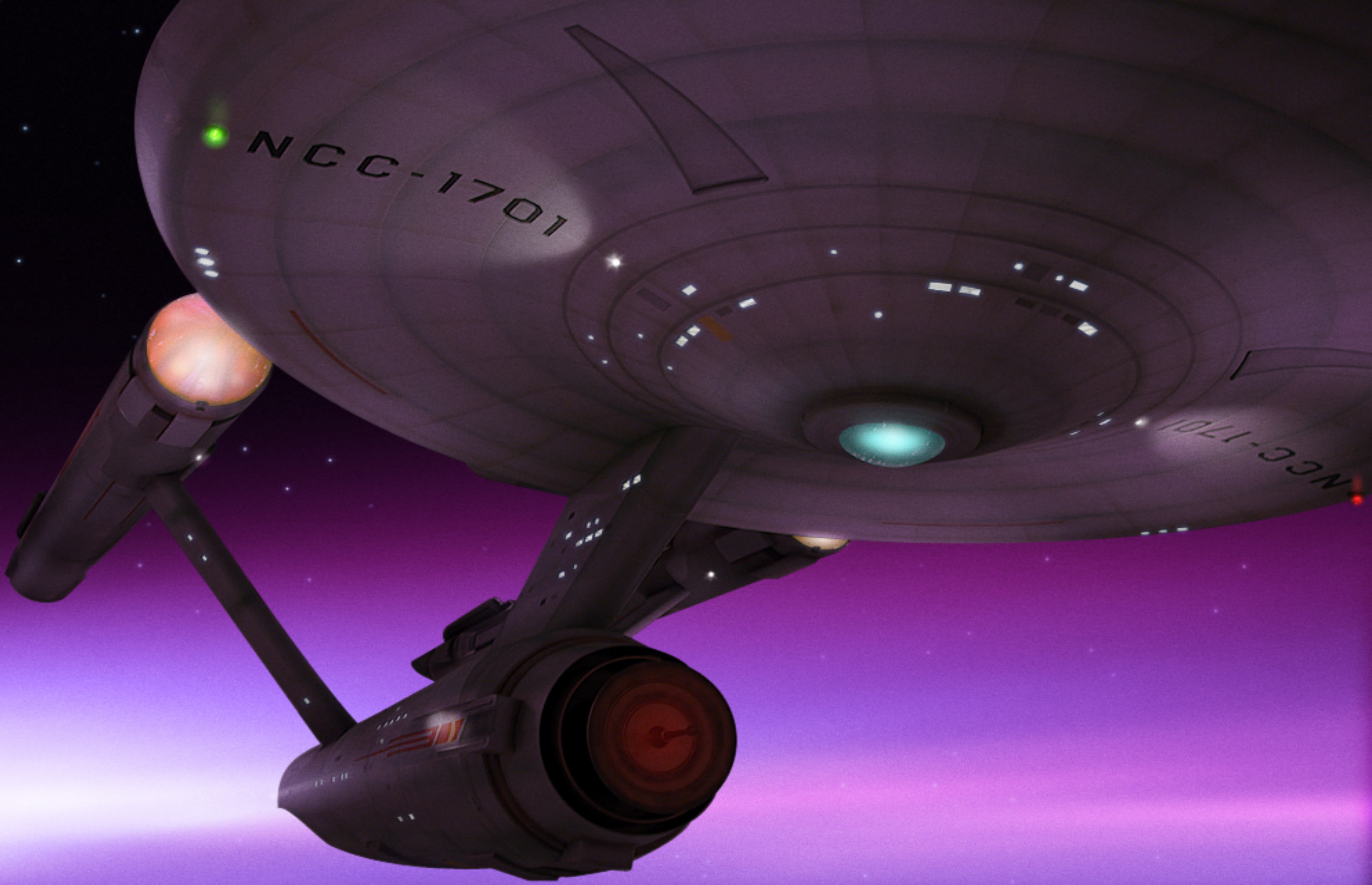
Exponential and Logarithmic Functions



3.4 Exponential and Logarithmic Equations

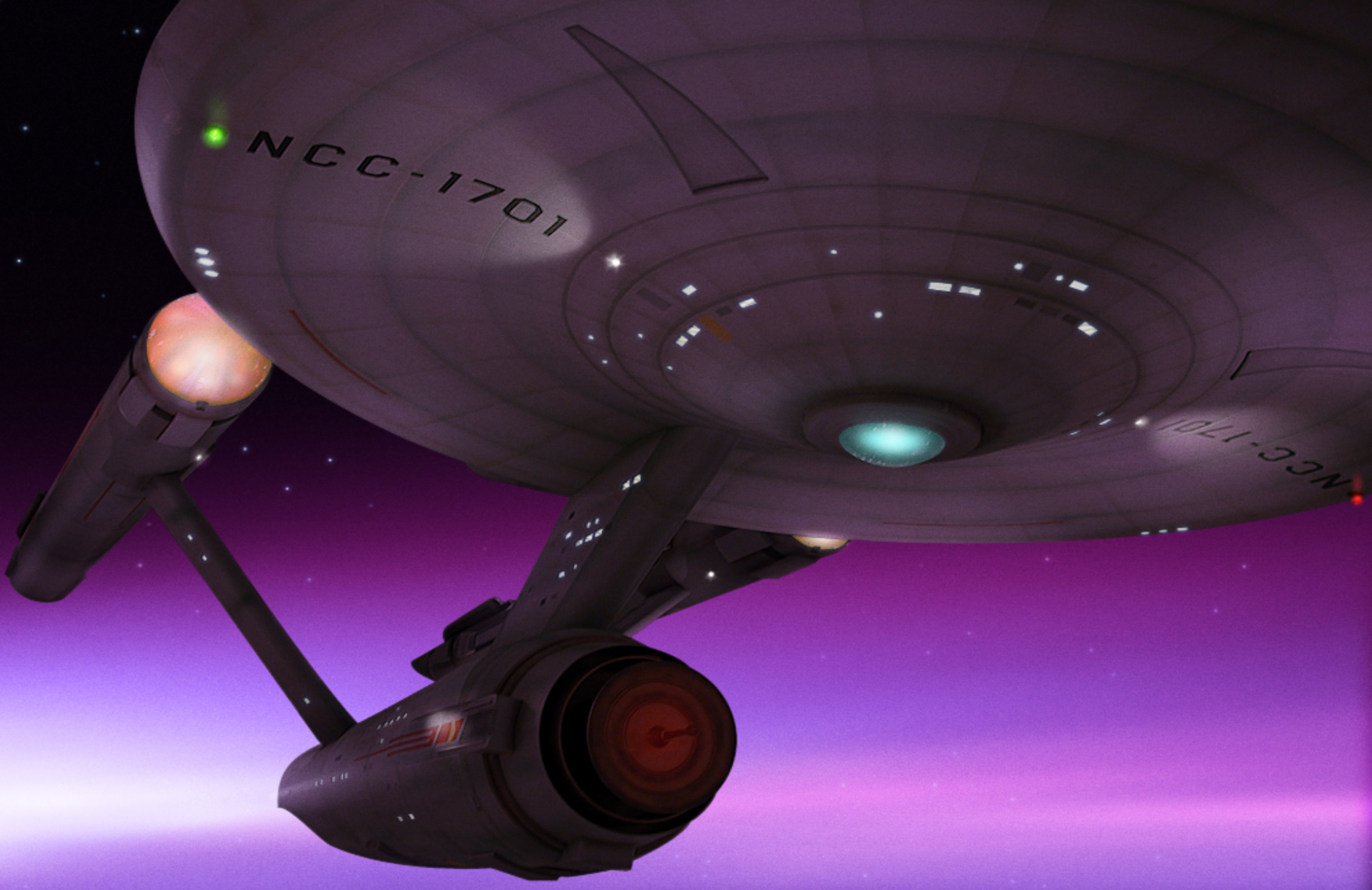
Chpt 3-4

- Homework



3-4 p253 21, 23, 29, 45, 53, 55, 59, 63, 69, 83, 85, 89, 94, 101, 117

Chpt 3-4



Objectives

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

Solving Expnential Equations

- There are two two primary methods for solving exponential equations:
  - 1. Isolate the exponential expression.
  - 2. Use the one-to-one property.

- There are also two primary ways of solving logarithmic equations.
  - 1. Isolate the logarithm, then write the equation in equivalent exponential form.
  - 2. Get a single logarithmic expression with the same base on each side of the equation; then use the one-to-one property.

- One to one properties
  - $a^x = a^y$ if and only if $x = y$.
  - $\log_a x = \log_a y$ if and only if $x = y$.

Note: In both cases the base must be the same.

Solving Exponential Equations by Expressing Each Side as a Power of the Same Base

- Solving Using Exponential Equations with common base.
  - If $b^m = b^n$ then $m = n$.

1. Rewrite the equation in the form $b^m = b^n$.
   - Note; the base (b) is the same in both expressions.
2. Set $m = n$.
3. Solve the equation.

Example: Solving Exponential Equations

Solve:

$$5^{3x-6} = 125$$
$$5^{3x-6} = 5^3$$
$$3x - 6 = 3$$
$$x = 3$$

The solution set is {3}

$$8^{x+2} = 4^{x-3}$$
$$(2^3)^{x+2} = (2^2)^{x-3}$$
$$2^{3(x+2)} = 2^{2(x-3)}$$
$$3(x + 2) = 2(x - 3)$$
$$3x + 6 = 2x - 6$$
$$x = -12$$

The solution set is {-12}

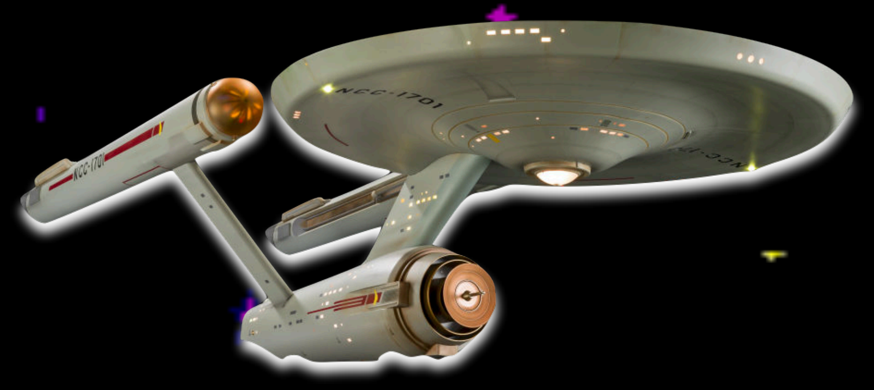
$$e^{x+1} = \frac{1}{e}$$
$$e^{x+1} = e^{-1}$$
$$x + 1 = -1$$

The solution set is {-2}

$$9^x = \frac{1}{\sqrt[3]{3}}$$
$$9^x = \frac{1}{3^{\frac{1}{3}}}$$
$$9^x = 3^{-\frac{1}{3}}$$
$$\left(3^2\right)^x = 3^{-\frac{1}{3}}$$
$$3^{2x} = 3^{-\frac{1}{3}}$$
$$2x = -\frac{1}{3}$$

The solution set is $\left\{-\frac{1}{6}\right\}$

Using Logarithms to Solve Exponential Equations



Solving by taking logarithm of both sides.

1. Isolate the exponential expression.

2. It the base of the exponent is 10, take the common logarithm of both sides of the equation. Use the natural logarithm on both sides of the equation for bases other than 10.

3. Simplify using: $\ln b^x = x\ln b$ or $\ln e^x = x$ or $\log 10^x = x$.

4. Solve for the variable.

STUDY TIP

Remember that the natural logarithmic function has a base of *e*.

Solving Exponential Equations

- Solve:

$$5^x = 134$$

$$\ln 5^x = \ln 134$$

$$x \ln 5 = \ln 134$$

$$x = \frac{\ln 134}{\ln 5} \approx 3.043$$

- The solution set is $\left\{\frac{\ln 134}{\ln 5}\right\}$

$$10^x = 8000$$

$$\log 10^x = \log 8000$$

$$x \log 10 = \log 8000$$

$$x = \log 8000$$

$$x \approx 3.903$$

- The solution set is log8000.

Solving an Exponential Equation

Solve:

$$3^{2x-1} = 7^{x+1}$$

$$\ln 3^{2x-1} = \ln 7^{x+1}$$

$$(2x-1)\ln 3 = (x+1)\ln 7$$

$$2x\ln 3 - \ln 3 = x\ln 7 + \ln 7$$

$$2x\ln 3 - x\ln 7 = \ln 3 + \ln 7$$

$$x(2\ln 3 - \ln 7) = \ln 3 + \ln 7$$

$$x = \frac{\ln 3 + \ln 7}{2\ln 3 - \ln 7}$$

$$x = \frac{\ln 21}{\ln 9 - \ln 7} = \frac{\ln 21}{\ln \frac{9}{7}} \approx 12.1144$$

The solution set is $\left\{ \frac{\ln 21}{\ln \frac{9}{7}} \right\}$

Solving Exponential Equations

Solve:

$$4e^{2x} = 16$$

$$e^{2x} = 4$$

$$\ln e^{2x} = \ln 4$$

$$2x = \ln 4$$

$$x = \frac{\ln 4}{2} \approx .693$$

$$5e^{2+x} - 8 = 14$$

$$5e^{2+x} = 22$$

$$e^{2+x} = \frac{22}{5}$$

$$\ln e^{x+2} = \ln \frac{22}{5}$$

$$x + 2 = \ln \frac{22}{5}$$

$$x = \ln \frac{22}{5} - 2 \approx -0.518$$

Solving Exponential Equations

- **Solve:**

$5(4^{3x-7}) - 6 = 15 \qquad 5(4^{3x-7}) = 21 \qquad 4^{3x-7} = \frac{21}{5}$

- From this point you have a couple of choices.

$$\log_4 \frac{21}{5} = 3x - 7$$

$$3x = \log_4 \frac{21}{5} + 7$$

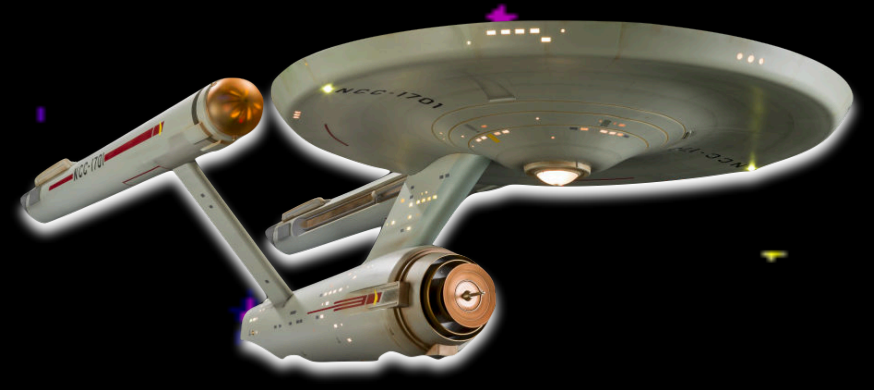
$$x = \frac{1}{3}\left(\log_4 \frac{21}{5} + 7\right) \qquad x = \frac{1}{3}\left(\frac{\ln \frac{21}{5}}{\ln 4} + 7\right)$$

$$\ln 4^{3x-7} = \ln \frac{21}{5}$$

$$(3x - 7)\ln 4 = \ln \frac{21}{5}$$

$$3x - 7 = \frac{\ln \frac{21}{5}}{\ln 4} \qquad x = \frac{1}{3}\left(\frac{\ln \frac{21}{5}}{\ln 4} + 7\right)$$

Quadratic Form of the Exponential



- Some exponential equations are in the form of a quadratic equation.

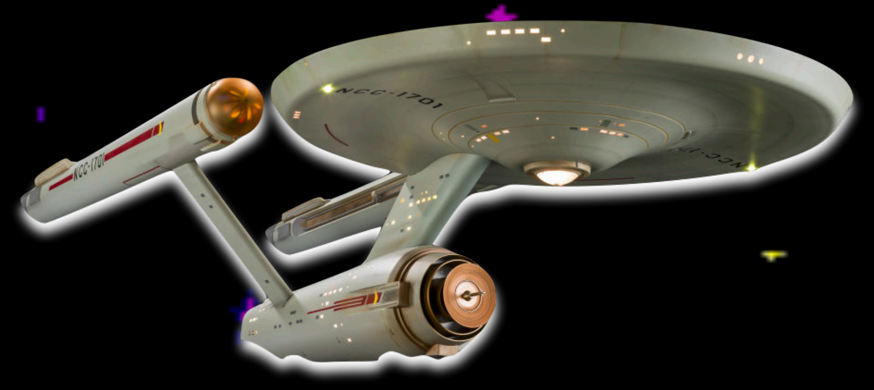
   1. Substitute u for the exponential term (u substitution).

   2. Solve the quadratic equation for u.

   3. Substitute the exponential for u.

   4. Solve for the variable.

Quadratic Form of the Exponential



- Solve: $e^{2x} - 8e^{x} + 7 = 0$

Let $u = e^{x}$

Then $u^2 = (e^{x})^2 = e^{2x}$

$e^{2x} - 8e^{x} + 7 = 0 \Rightarrow u^2 - 8u + 7 = 0$

$u^2 - 8u + 7 = 0$

$(u - 7)(u - 1) = 0$

$u = 7$ or $u = 1$

$e^{x} = 7$ or $e^{x} = 1$

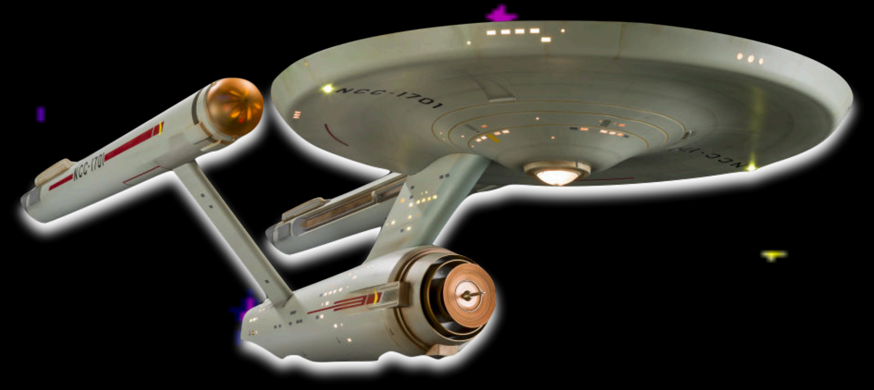
$\ln e^{x} = \ln 7 \quad x = \ln 7$

or

$\ln e^{x} = \ln 1 = 0 \quad x = 0$

- The solution set is $\{0, \ln 7\}$

Quadratic Form of the Exponential



Solve: $2^{2x} + 2^x - 12 = 0$

Let $u = 2^x$

Then $u^2 = (2^x)^2 = 2^{2x}$

$2^{2x} + 2^x - 12 = 0 \Rightarrow u^2 + u - 12 = 0$

$u^2 + u - 12 = 0$

$(u - 3)(u + 4) = 0$

$u = 3$ or $u = -4$

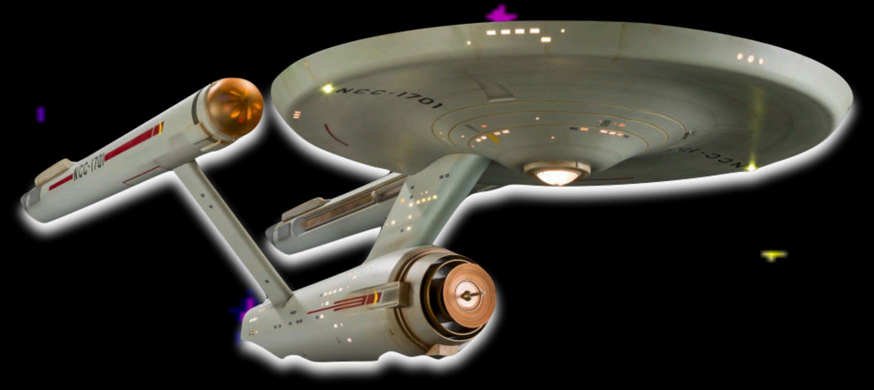
$2^x = 3$ or ~~$2^x = -4$~~

$\ln 2^x = \ln 3 \qquad x\ln 2 = \ln 3$

$$x = \frac{\ln 3}{\ln 2} \approx 1.5850$$

The solution set is $\left\{ \frac{\ln 3}{\ln 2} \right\}$

TI-84



Graph $y = e^{2x} - 8e^{x} + 7$ and find the zeros.

2nd | TRACE CALC | ↓ | 2:Zero | Set your left boundary by moving the cursor. | ENTER | Set your right boundary by moving the cursor. | ENTER | Guess? | ENTER

- Here is your problem. I will insist on exact solutions. Your calculator suggests a zero at about 1.94591.

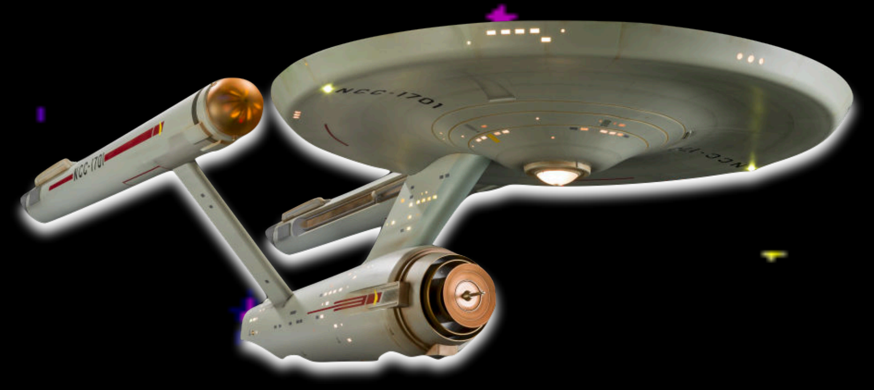
- Calculate ln(7)

- I expect you to be able to solve both graphically and algebraically.

Check your Results

- Logarithmic expressions are defined only for logs of positive real numbers.
  - Always check proposed solutions of a logarithmic equation in the original equation.
  - Exclude from the solution set any proposed solution that produces the logarithm of a negative number or the logarithm of 0 (Extraneous solutions).

Using the Definition of a Logarithm to Solve Logarithmic Equations



- Sometimes you can convert a log equation into an exponential equation that is easier to solve.

1. Express the equation in the form $\log_b M = x$.

2. Use the definition of a log to rewrite the equation in exponential form.

$$\log_b M = x \Rightarrow b^x = M$$

3. Solve for the variable.

4. Check proposed solutions in the original equation. Include in the solution set only values for which $M > 0$.

Example: Solving Logarithmic Equations

- Solve:

$$\log_2(x-4) = 3$$

$$2^3 = (x-4)$$

$$x = 12$$

$$\log_2(12-4) = 3$$

- The solution set is {12}.

$$4\ln 3x = 8$$

$$\ln 3x = 2$$

$$e^2 = 3x$$

$$x = \frac{e^2}{3}$$

$$4\ln\left(3\left(\frac{e^2}{3}\right)\right) = 4\ln e^2 = 4(2) = 8$$

- The solution set is $\left\{\frac{e^2}{3}\right\}$

Example: Solving Logarithmic Equations

- Solve:

$\log x + \log(x-3) = 1$

$\log x(x-3) = 1$

$x(x-3) = 10$

$x^2-3x-10 = 0$

$(x-5)(x+2) = 0$

$x = 5$ or $x = -2$

~~$\log(-2) + \log(-2-3) = 1$~~

$\log 5 + \log(5-3) = 1$

$\log 5 + \log 2 = 1$

$\log 10 = 1$

- The solution set is $\{5\}$.

Example: Solving Logarithmic Equations

Solve:

$$\log(5x+1) = \log(2x+3) + \log 2$$

$$\log(5x+1) = \log 2(2x+3)$$

$$5x+1 = 4x+6$$

$$x = 5$$

$$\log(26) = \log(13) + \log 2$$

The solution set is {5}.

$$\log_2(x-3) + \log_2(x) - \log_2(x+2) = 2$$

$$\log_2\left(\frac{x(x-3)}{x+2}\right) = 2$$

$$\frac{x(x-3)}{x+2} = 4$$

$$x(x-3) = 4(x+2)$$

$$x^2 - 3x = 4x + 8$$

$$x^2 - 7x - 8 = 0$$

$$(x-8)(x+1) = 0$$

$x = 8$ or ~~$x = -1$~~

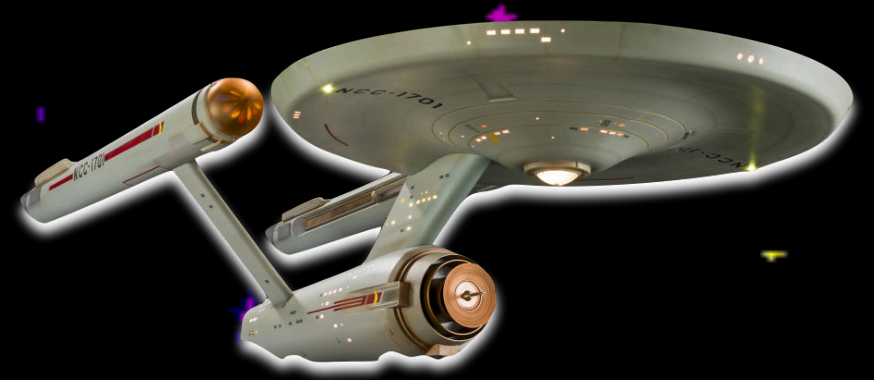
$$\log_2 5 + \log_2 8 - \log_2 10 = 2$$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

- Writing the equation as equal logarithms.

  1. Express the equation in the form $\log_b M = \log_b N$.
  2. Use the one-to-one property to rewrite the equation without logs.
     If $\log_b M = \log_b N$, then $M = N$.
  3. Solve for the variable
  4. Check the proposed solutions in the original equation. Include in the solution set only value from which $M>0$ and $N>0$.

Example: Solving a Logarithmic Equation



- **Solve:**

$$\log_2(x-1)-\log_2(x+3)=\log_2\left(\frac{1}{x}\right)$$

$$\log_2\left(\frac{x-1}{x+3}\right)=\log_2\left(\frac{1}{x}\right)$$

$$\frac{x-1}{x+3}=\frac{1}{x}$$

$$x(x-1)=x+3$$

$$x^2-x=x+3$$

$$x^2-2x-3=0$$

$$(x+1)(x-3)=0$$

$$x=3 \text{ or } \cancel{x=-1}$$

$$\log_2(2)-\log_2(6)=\log_2\left(\frac{1}{3}\right)$$

Example: Solving a Logarithmic Equation

- Solve:

$$\ln(x-3) = \ln(7x-23) - \ln(x+1)$$

$$\ln(x-3) = \ln\frac{7x-23}{x+1}$$

$$x-3 = \frac{7x-23}{x+1}$$

$$(x-3)(x+1) = 7x-23$$

$$x^2-2x-3 = 7x-23$$

$$x^2-9x+20 = 0$$

$$(x-5)(x-4) = 0 \qquad x=5 \text{ or } x=4$$

$$\ln(5-3) = \ln(7(5)-23) - \ln(5+1)$$

$$\ln 2 = \ln 12 - \ln 6$$

$$\ln(4-3) = \ln(7(4)-23) - \ln(4+1)$$

$$\ln 1 = \ln 5 - \ln 5$$

- The solution set is $\{4,5\}$.

Solving Exponential Equations

Summary

1. Rewrite the equation in the form $b^m = b^n$.
   - Note; the bases (b) are equal.

$$4^{5x-2} = 8^{2x+3} \Rightarrow 2^{2(5x-2)} = 2^{3(2x+3)}$$

$$2(5x-2) = 3(2x+3)$$

2. Take the ln (or log) of both sides of the equation.

$$6^{2x} = 1024 \Rightarrow \ln 6^{2x} = \ln 1024$$

$$2x \ln 6 = \ln 1024$$

Solving Logarithmic Equations

Summary

1. Rewrite the logarithmic equation in exponential form.

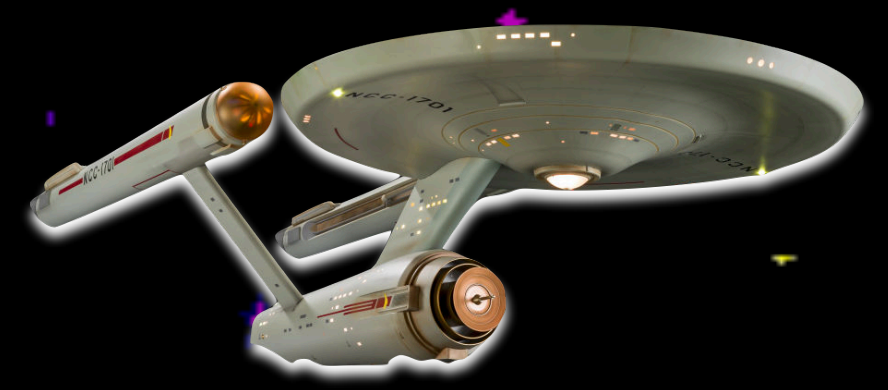
$$\log_7(2x-5) = 11 \Rightarrow 7^{11} = 2x-5$$

2. Simplify the logarithmic equation to $\log_b m = \log_b n$.

$$\ln x + \ln(2x-1) = \ln(5x-3) - \ln 2x$$

$$\Downarrow$$

$$\ln \frac{x}{2x-1} = \ln \frac{5x-3}{2x}$$



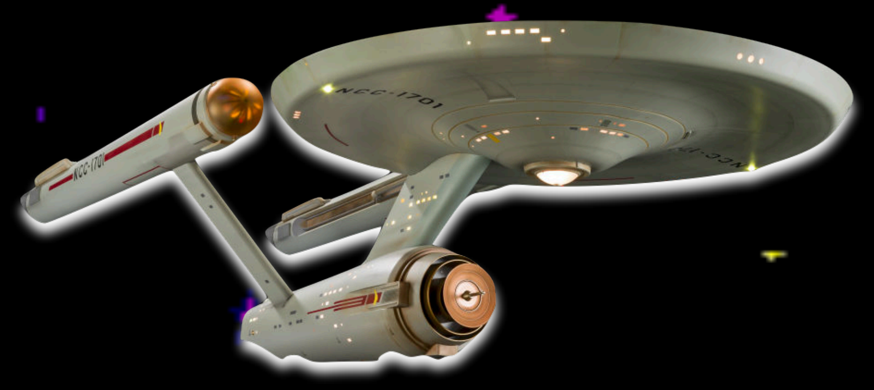
Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.

2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.

3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Example: Application



- How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$3600 = 1000\left(1+\frac{.08}{4}\right)^{4t}$$

$$3.6 = \left(1+.02\right)^{4t}$$

$$\ln 3.6 = \ln\left(1.02\right)^{4t}$$

$$\ln 3.6 = 4t\ln\left(1.02\right)$$

$$4t = \frac{\ln 3.6}{\ln 1.02}$$

$$t = \frac{\ln 3.6}{4\ln 1.02} \approx 16.17$$

It will take about 16.2 years for \$1000 to become \$3600.

Example: Application

- The percentage of adult height attained by a girl who is x years old can be modeled by $f(x)=62+35\log(x-4)$ where x represents the girl's age (from 5 to 15) and f(x) represents the percentage of her adult height. At what age has a girl attained 97% of her adult height?

$$f(x)=62+35\log(x-4)$$

$$97=62+35\log(x-4)$$

$$35=35\log(x-4)$$

$$1=\log(x-4)$$

$$x-4=10$$

$$x=14$$

She will reach 97% of her height at age 14.