Chapter 4

Trigonometric Functions

∡ 4.1 Angle and Radian Measure

Chapter 4.1

- Homework
  - 4.1, p290 1-95 odd

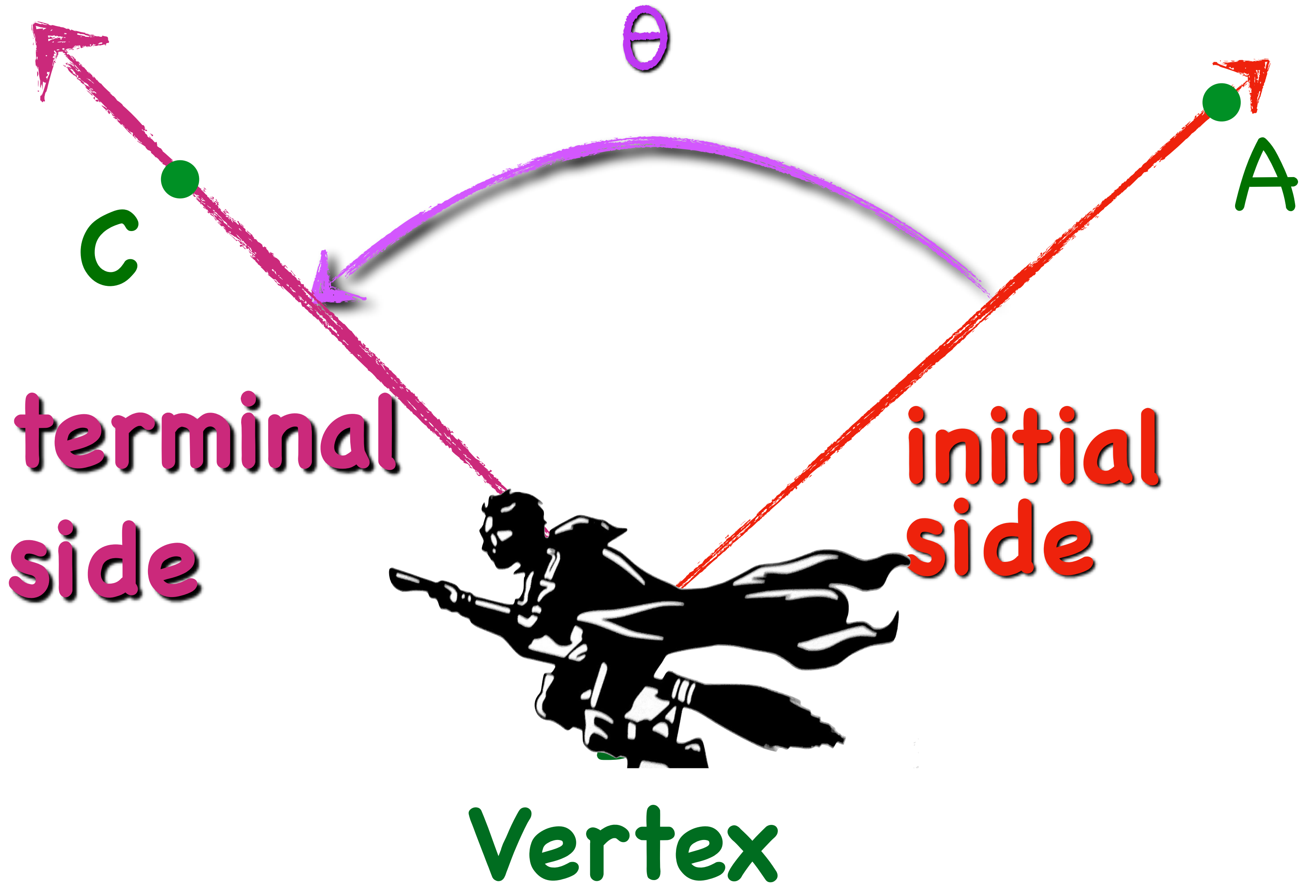
Chapter 4.1

Objectives

- Recognize and use the vocabulary of angles.
- Use degree measure.
- Use radian measure.
- Convert between degree and radian measures
- Draw angles in standard position
- Find coterminal angles
- Find the length of a circular arc
- Use linear and angular speed to describe motion on a circular path
- Find the area of a Sector

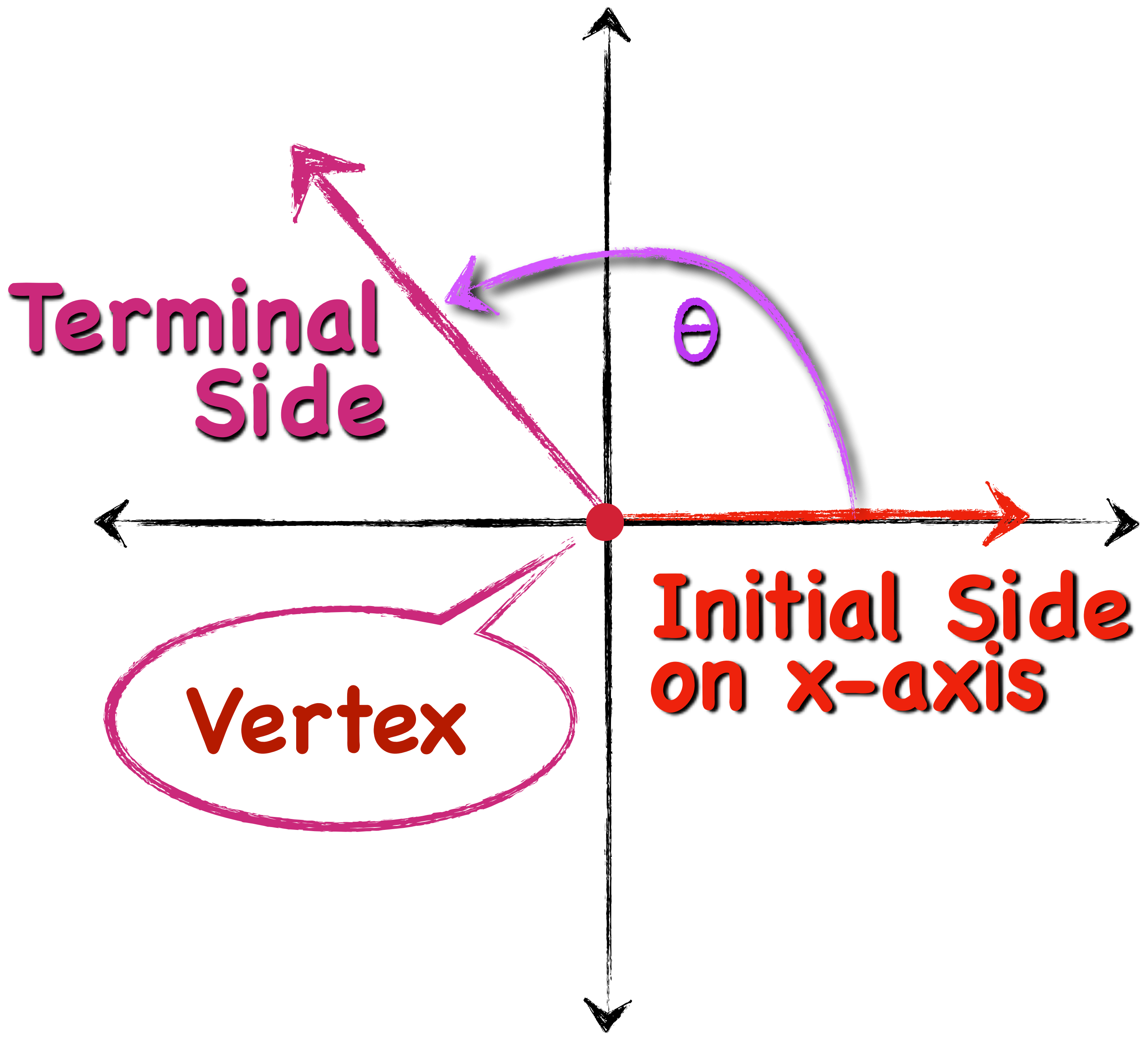
Angles

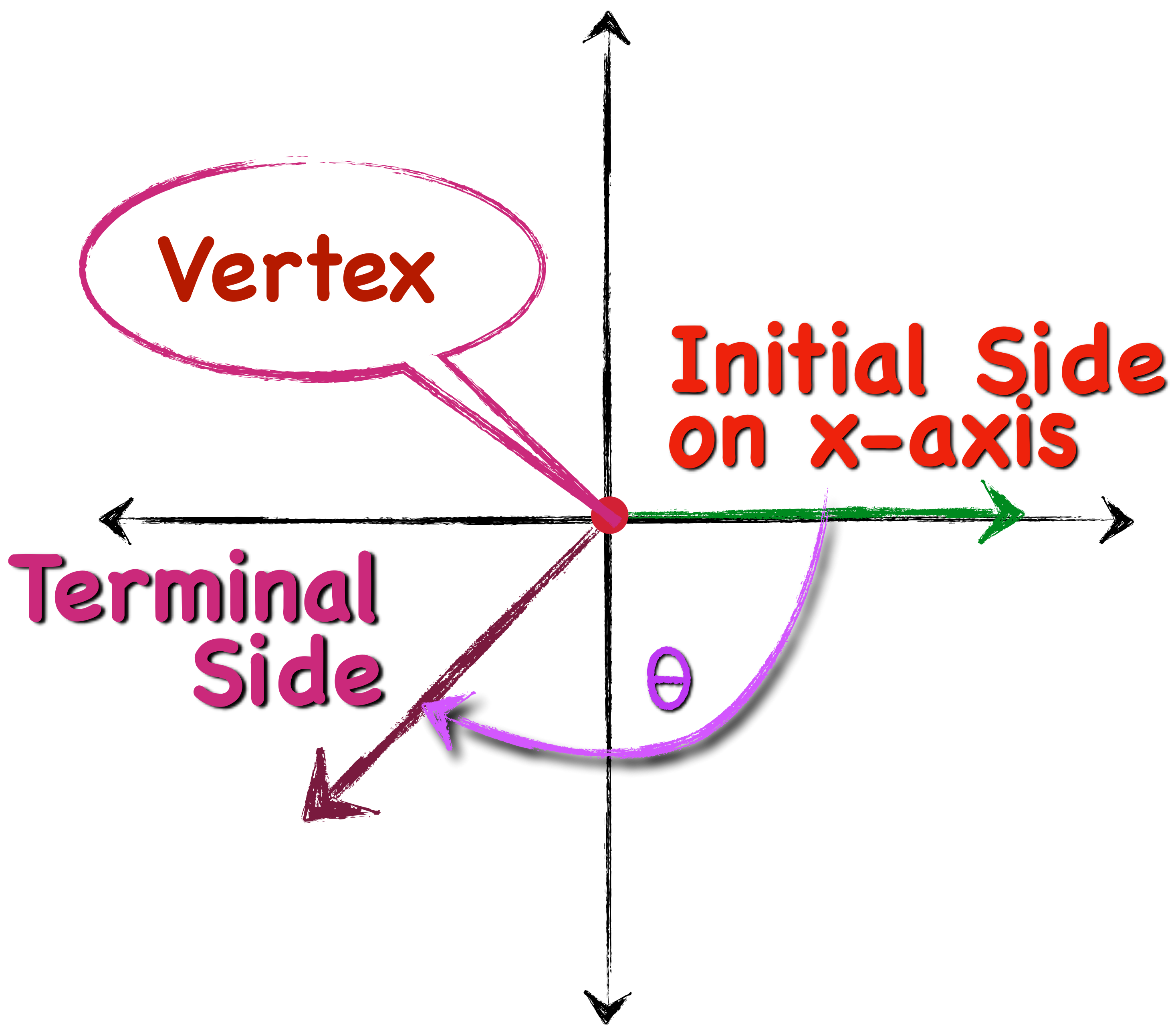
- An **angle** is formed by two rays that have a common endpoint.
- The common endpoint is called the **vertex**.
- One **ray** is called the **initial side** and the other the **terminal side**.



Angles

An angle is in **standard position** if its vertex is at the origin of a rectangular coordinate system and its initial side lies along the positive x-axis.





Angles

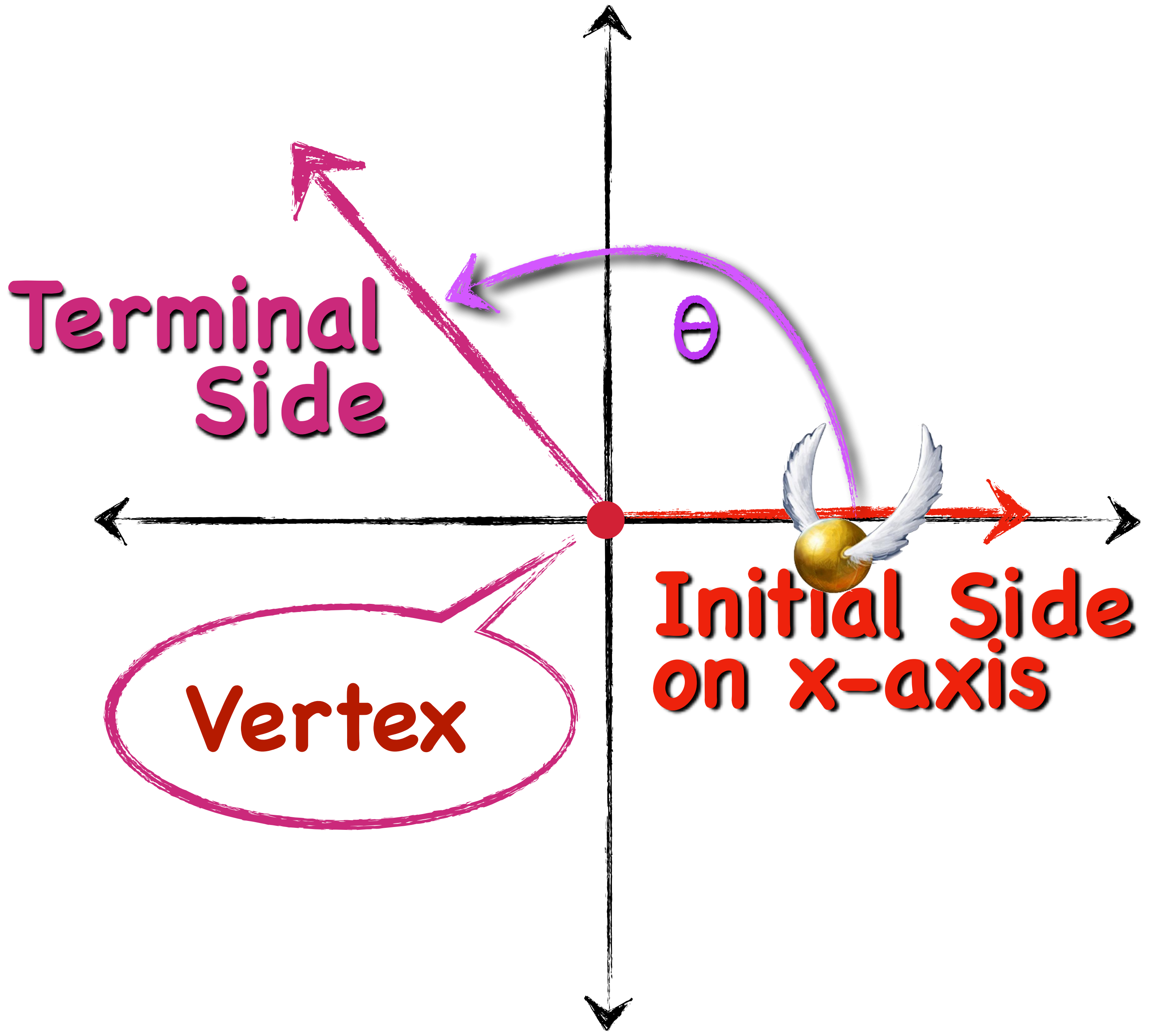
- When we see an initial side and a terminal side in place, there are two kinds of rotations that could have generated the angle.

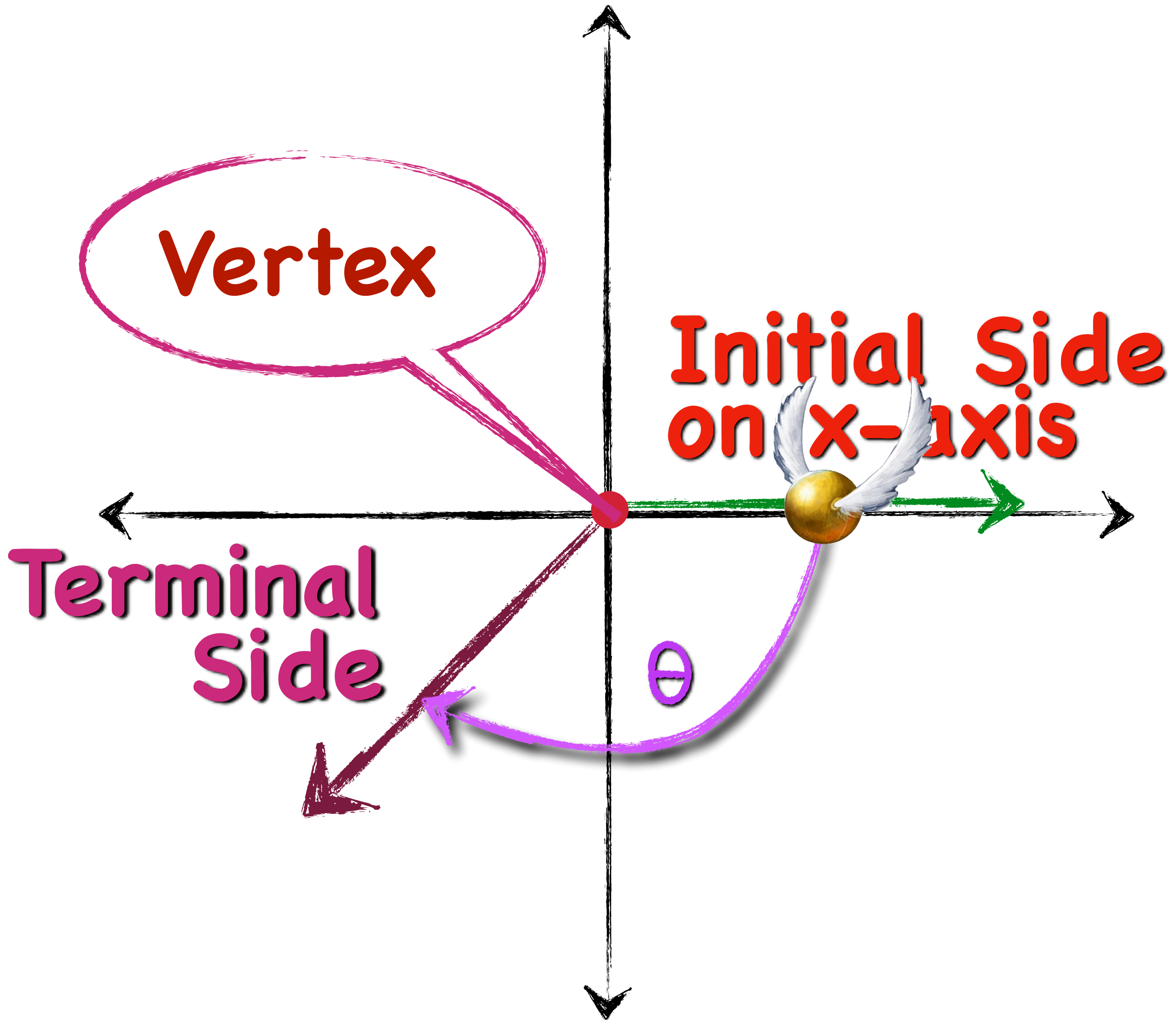
Positive angles are generated by counterclockwise rotation.

Thus, angle α is positive.

Negative angles are generated by clockwise rotation.

Thus, angle θ is negative.

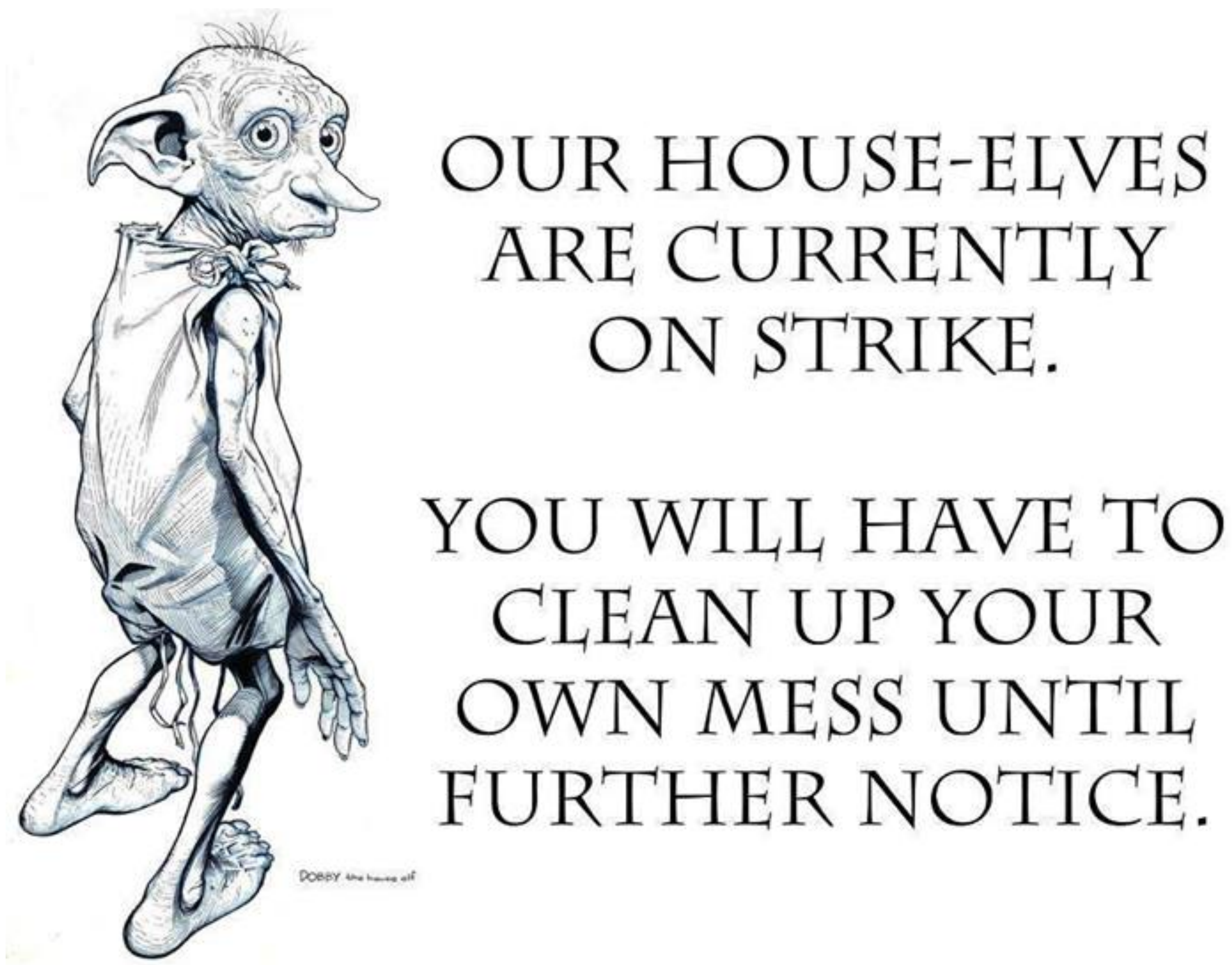


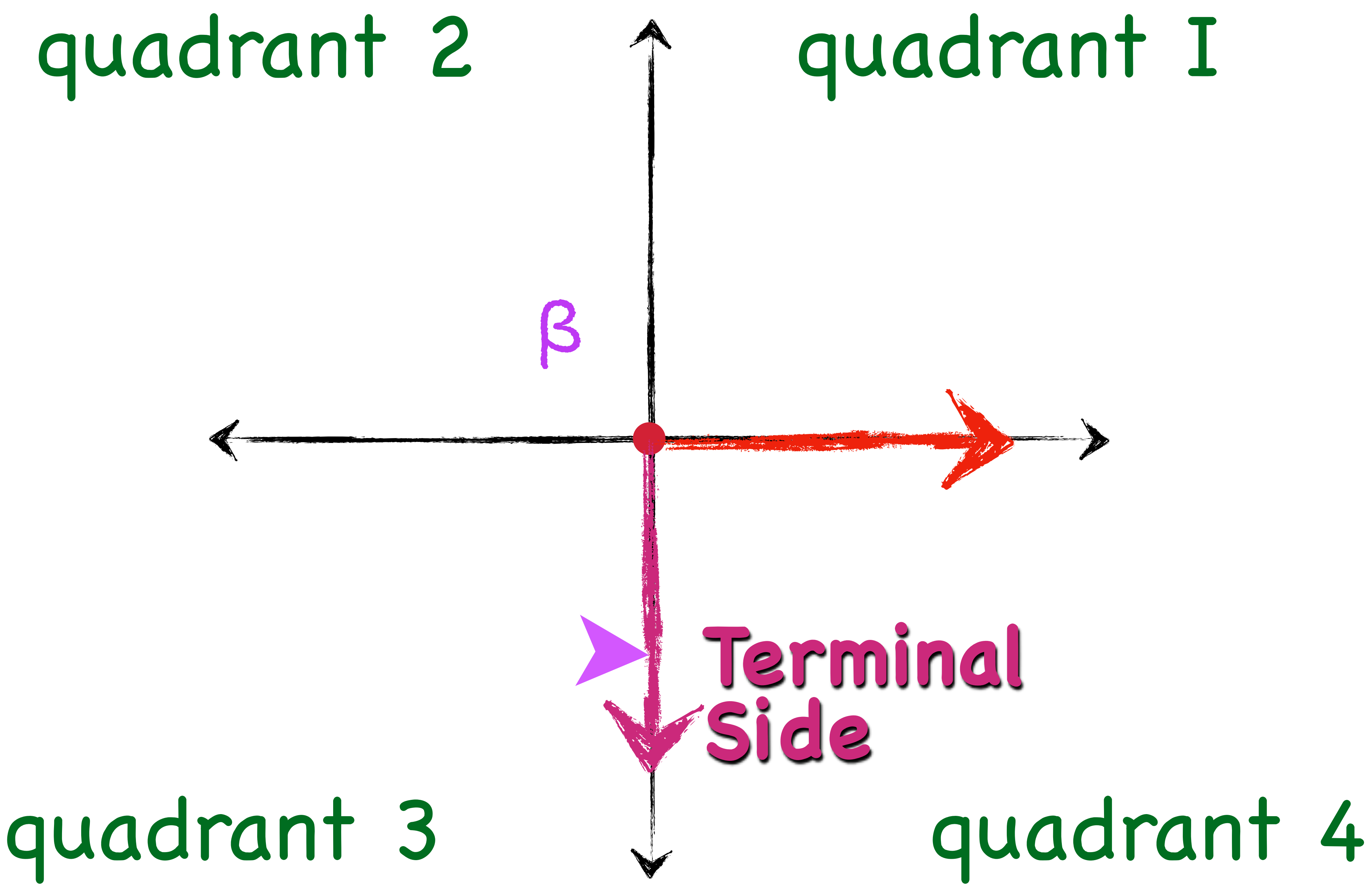


Angles

- An angle is called a **quadrantal angle** if its terminal side lies along an axis.

  Angle β is an example of a quadrantal angle.





Measuring Angles in Degrees

- Angles are measured by determining the amount of rotation from the initial side to the terminal side.

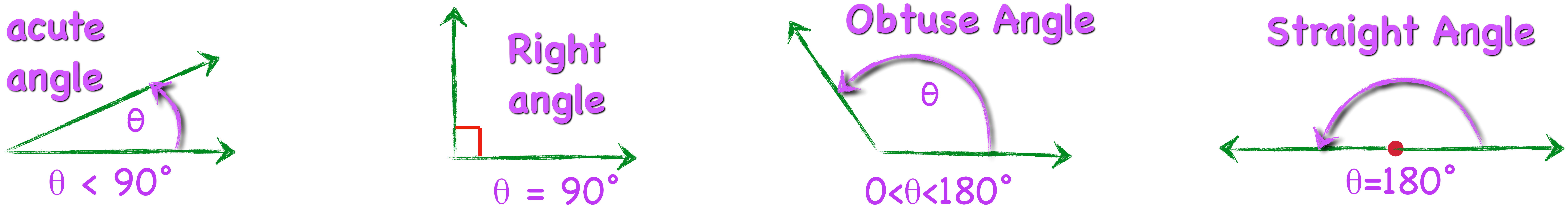
A complete rotation of the circle is 360 degrees, or 360°.

An **acute angle** measures **less than 90°**.

A **right angle** measures **90°**.

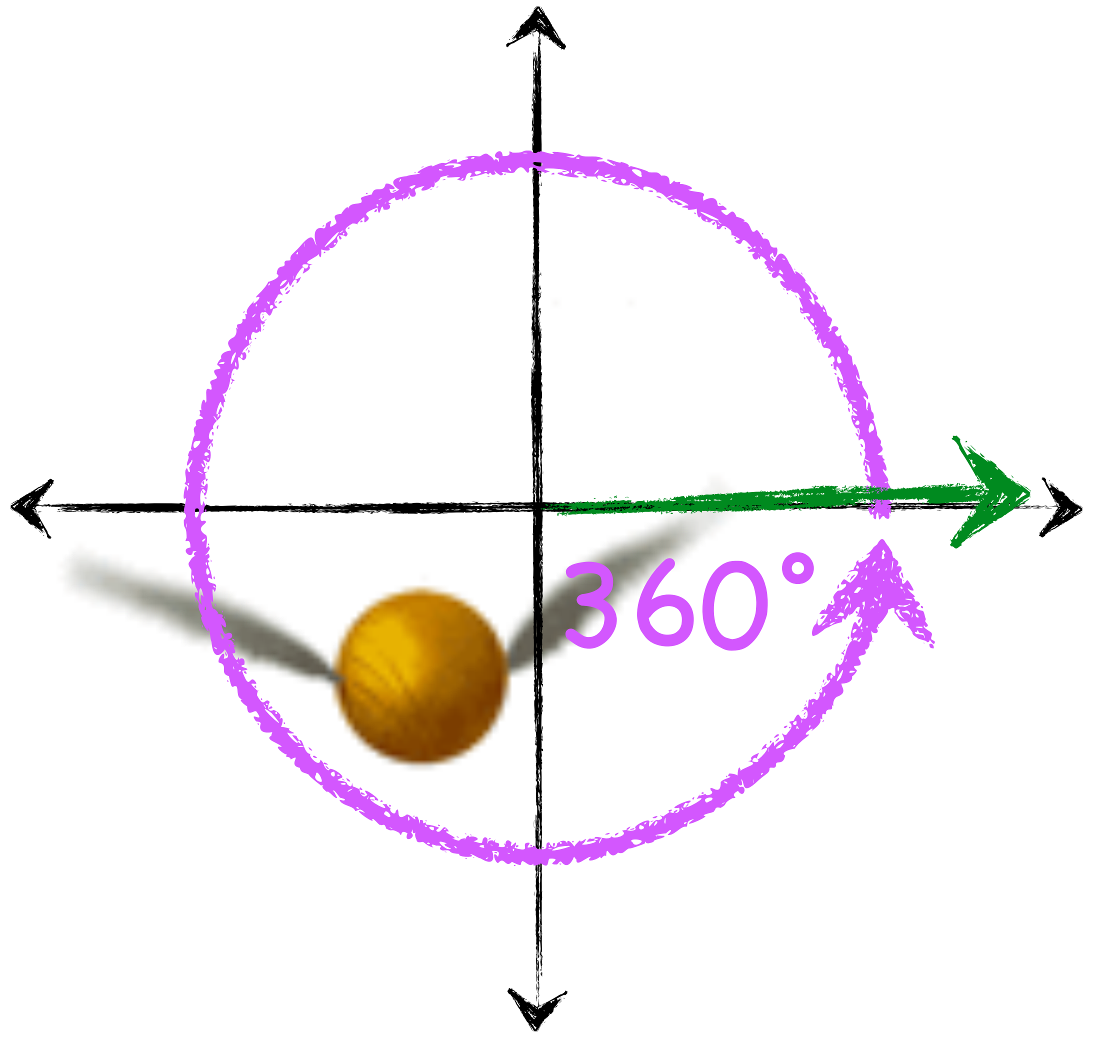
An **obtuse angle** measures **more than 90° but less than 180°**.

A **straight angle** measures **180°**.



Measuring Angles Using Degrees

- A complete rotation of a circle is 360°.
  - One degree (1°) is 1/360th of a complete rotation.
  - An angle is measured in degrees, minutes, seconds.
  - One degree = 60 minutes, one minute = 60 seconds.
  - Or one second = 1/3600th degree, one minute = 1/60th degree.



49°32′58″ = 49 degrees 32 minutes 58 seconds.

Converting to decimal

- Be cautious when converting degrees into decimal form.
  - 36.5° = 36 degrees, 30 minutes.
  - 48 degrees, 20 minutes = 48.333... degrees.

∡ 39°45′ = 39 45/60 degrees = 39.75 degrees

∡ 39°28′13″ = 39 + 28/60 + 13/3600 degrees

= 39 1693/3600 degrees ≈ 39.4702778 degrees

TI-84

You can convert decimal degrees to degrees, minutes, seconds on the TI-84.

- Ensure the calculator is in **Degree Mode**.

  MODE ↓ RADIAN DEGREE ENTER

- Back to home screen and enter degree measure in decimal form, then

  2nd APPS ↓ 4: ➤DMS ENTER

TI-84

- You can also convert degrees, minutes, seconds to decimal degrees but it takes a few steps.

  - Ensure the calculator is in **Degree Mode**.
  - Enter the degrees only, then

    2nd APPS ⇓ 1: ° ENTER

  - Enter the minutes next, then

    2nd APPS ⇓ 2: ‘ ENTER

  - Enter the seconds, then

    ALPHA + ENTER

  - Now when you hit ENTER the calculator will convert to decimal.

Converting Degrees to Radians

- From this point forward you need to be careful to ensure the calculator is in the **correct mode** (Degree or Radian). Get in the habit of checking for degrees or radians before you start working.

  - Most of our work (and calculus) is done in radians so I keep my calculator in radians and try to remember to switch to degrees when needed.

  - To convert degrees to radians.

    - Make certain the calculator is in Radian Mode.

    - Enter the degrees, then, **before** hitting ENTER

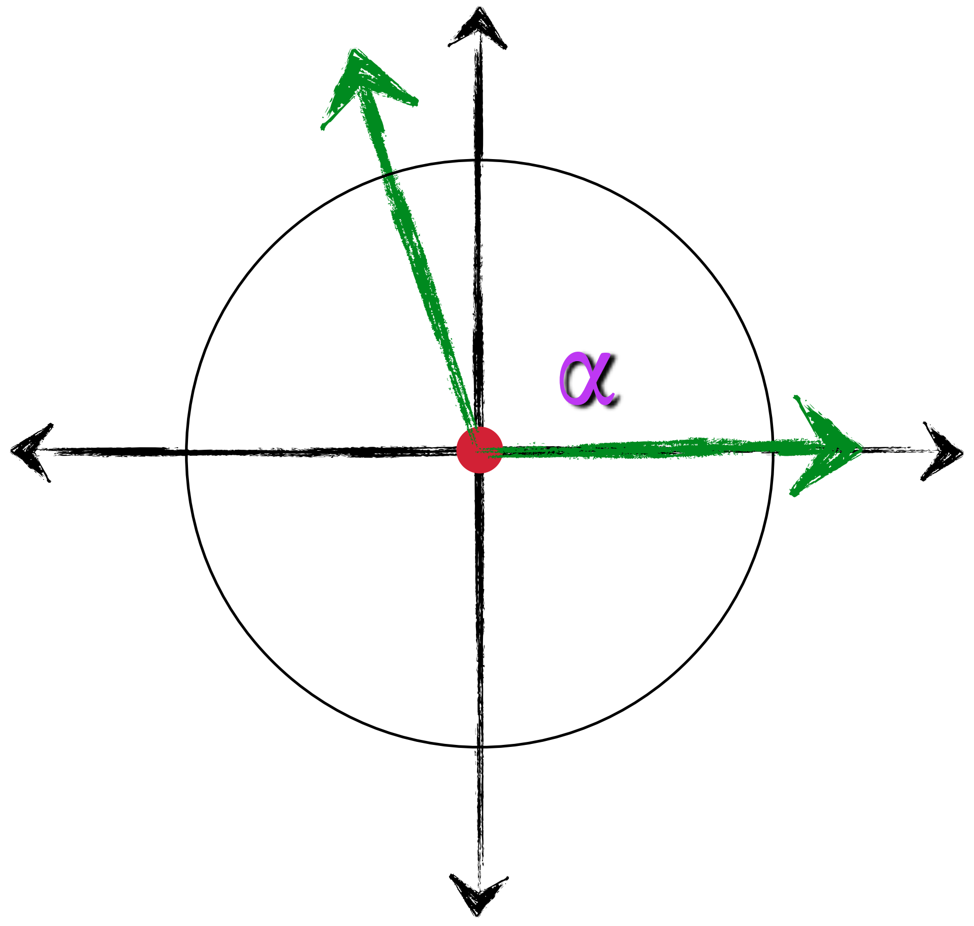
      2nd APPS ↓ 1: ° ENTER

    - Aaaaaah Radians

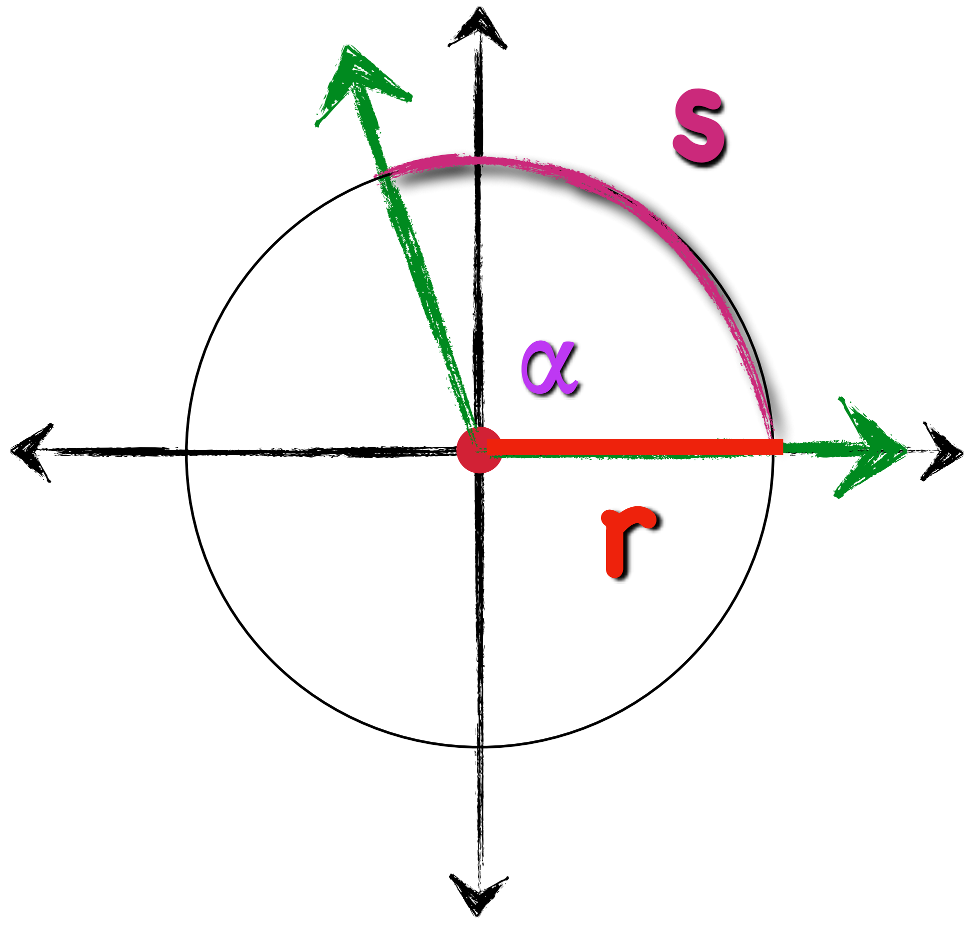
Measuring Angles in Radians

An angle whose vertex is at the center of a circle is called a **central angle**.

The **radian measure** of any central angle of a circle is the **length of the intercepted arc divided by the length of the circle's radius**.



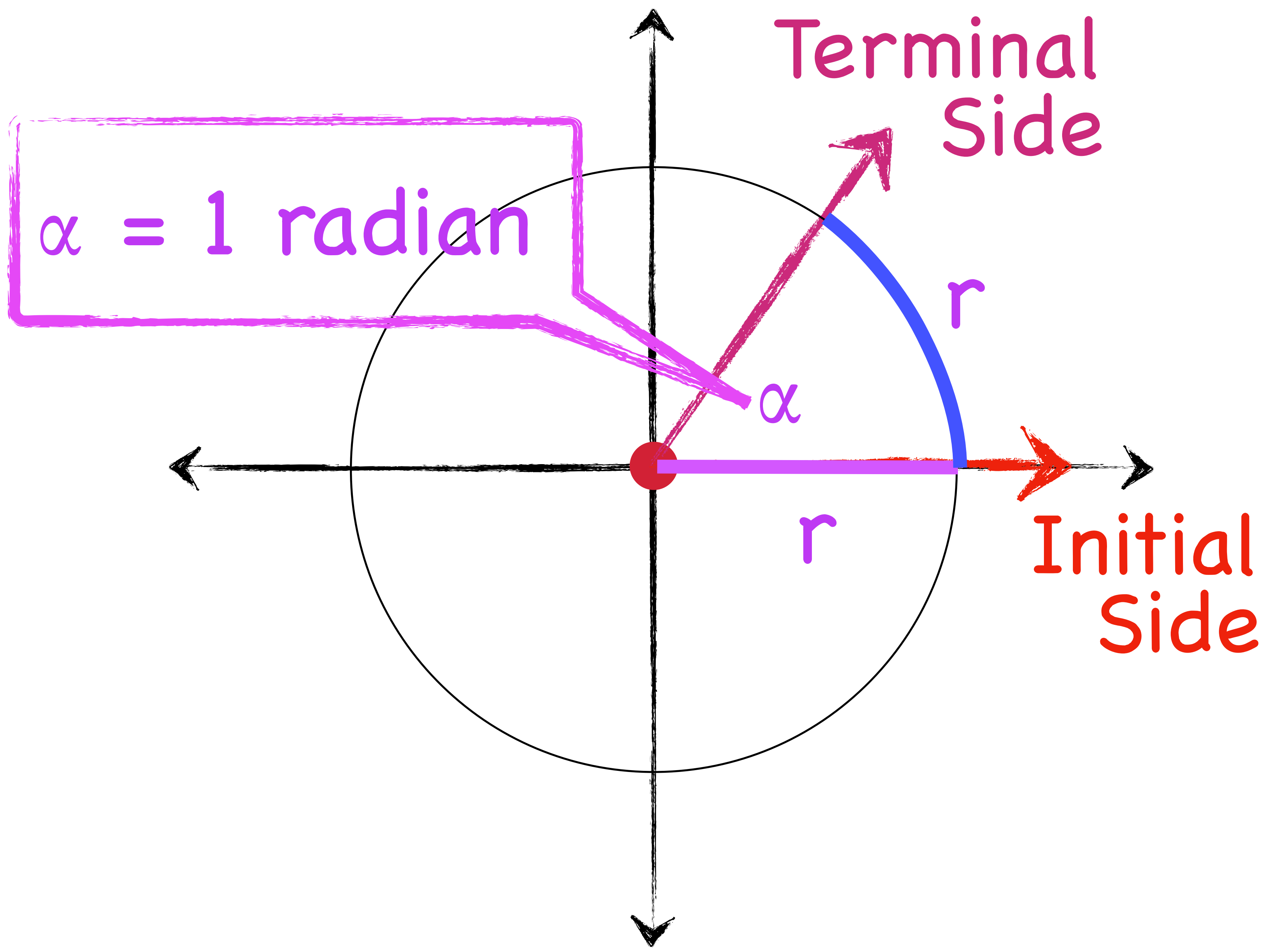
Or to put that another way, the **radian measure** of any central angle of a circle is the **length of the intercepted arc in the number of radii.**



$$\#\ radians = \frac{length\ \text{of intercepted arc}\ (s)}{\text{length of radias}\ (r)}$$

Definition of a Radian

- One **radian** is the measure of the central angle of a circle that intercepts an **arc equal in length to the radius** of the circle.



STUDY TIP

One revolution around a circle of radius r corresponds to an angle of 2π radians because

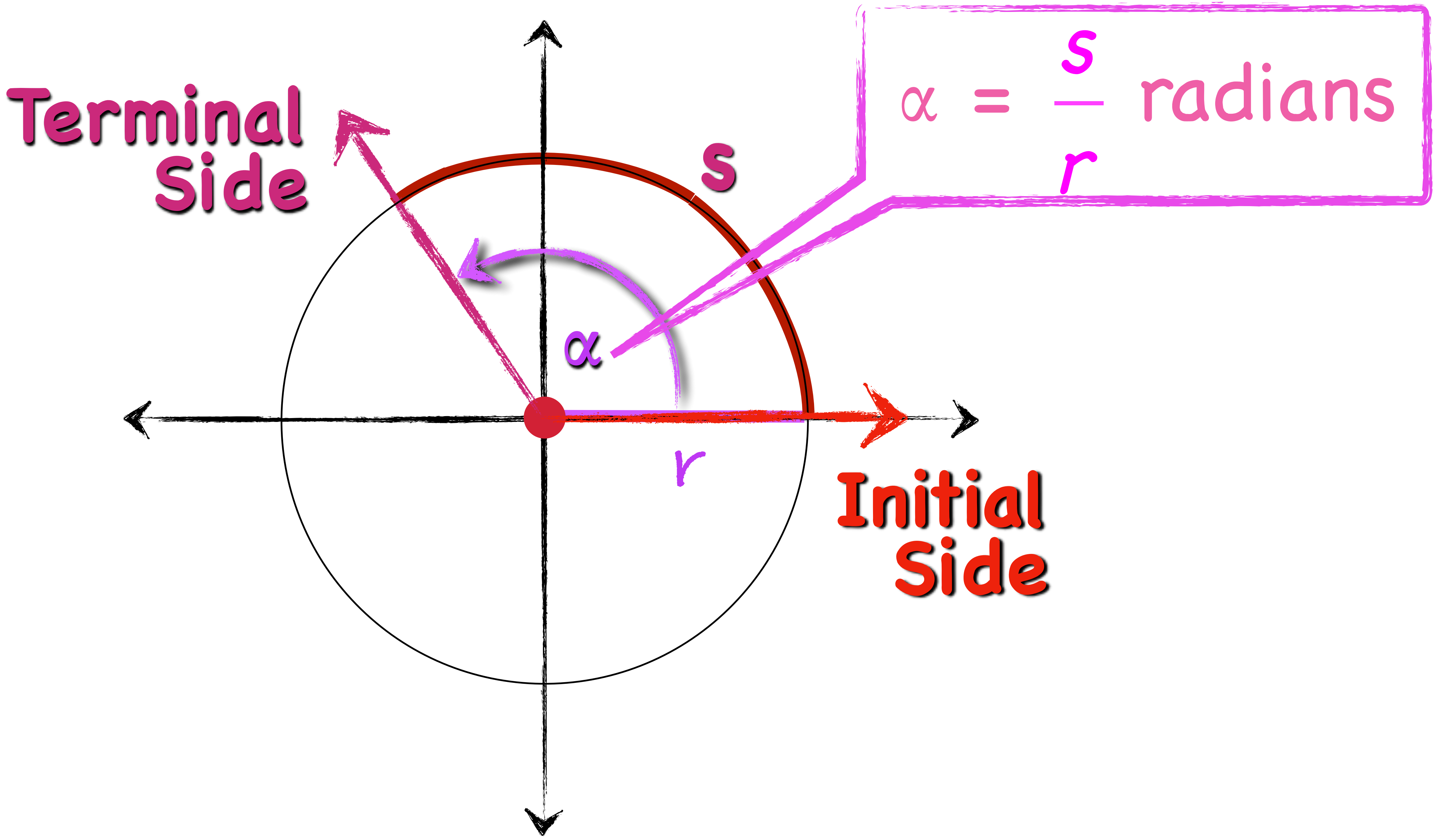
$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ radians.}$$

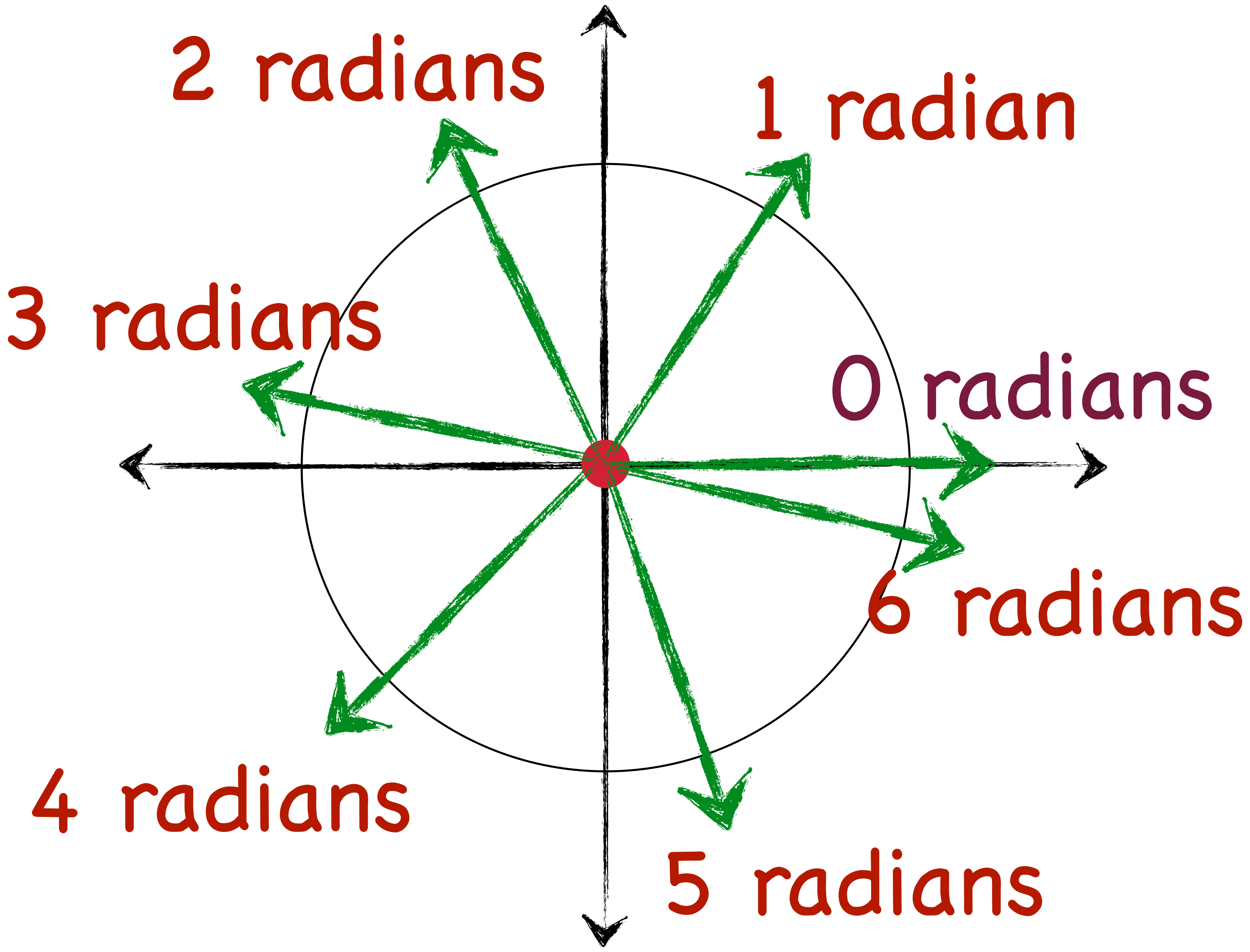
A Little Vocabulary

- An angle intercepts an arc when the initial and terminal sides of an angle intersect a circle. The arc between the angle sides is the intercepted arc.
  - Chords of a circle also intercept an arc
- An angle is subtended by an arc when the initial and terminal sides of an angle are at ends of the arc.
  - Any object that forms an angle from its extremities is said to subtend an angle.

Radian Measure

- A central angle that intercepts an **arc of length s**, on a circle with radius r, has a measure of $\frac{s}{r}$ radians.

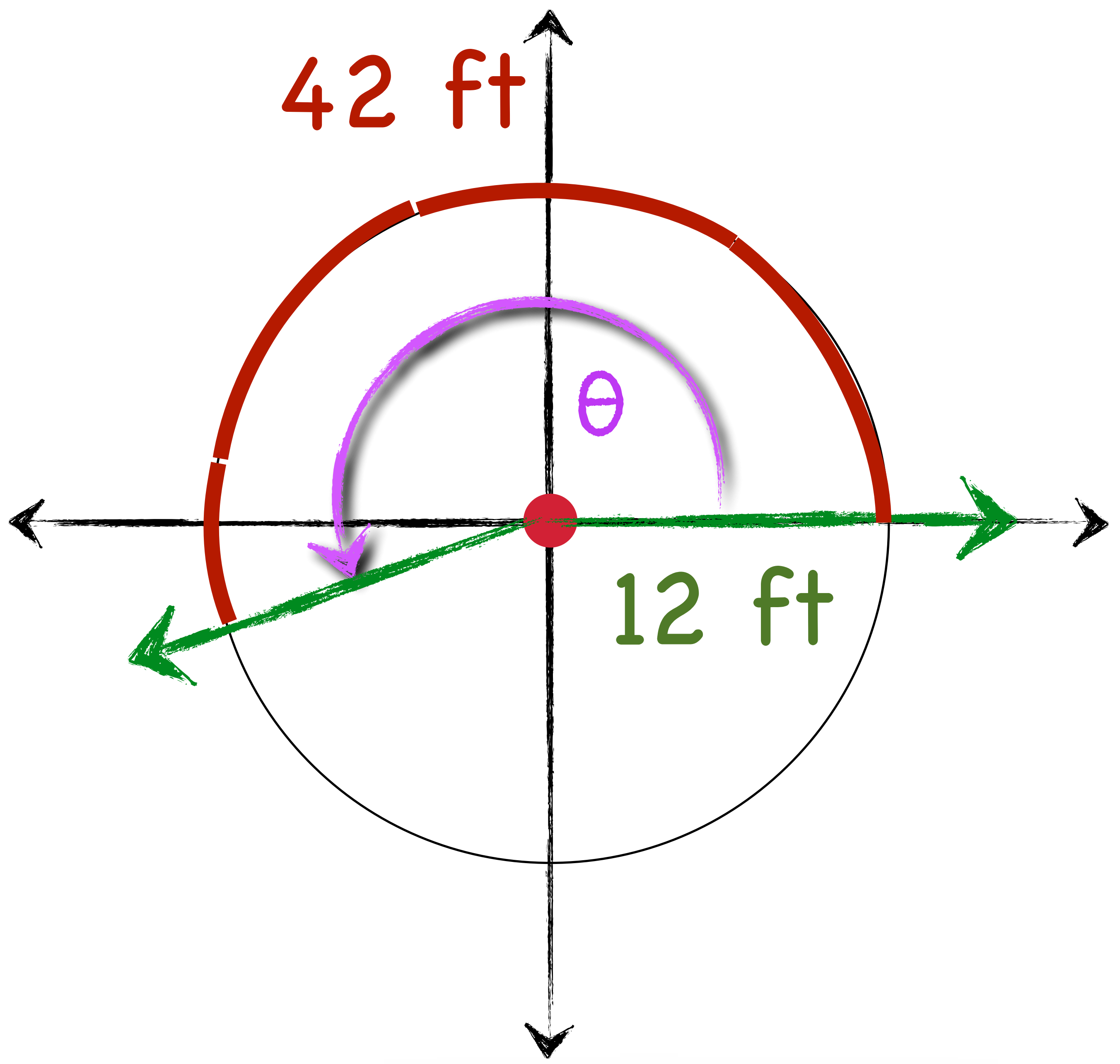




Computing Radian Measure

- A central angle θ, in a circle of radius 12 feet intercepts an arc of length 42 feet. What is the radian measure of θ?

$$\theta = \frac{42ft}{12ft} = \frac{21ft}{6ft} = 3.5\, radians$$

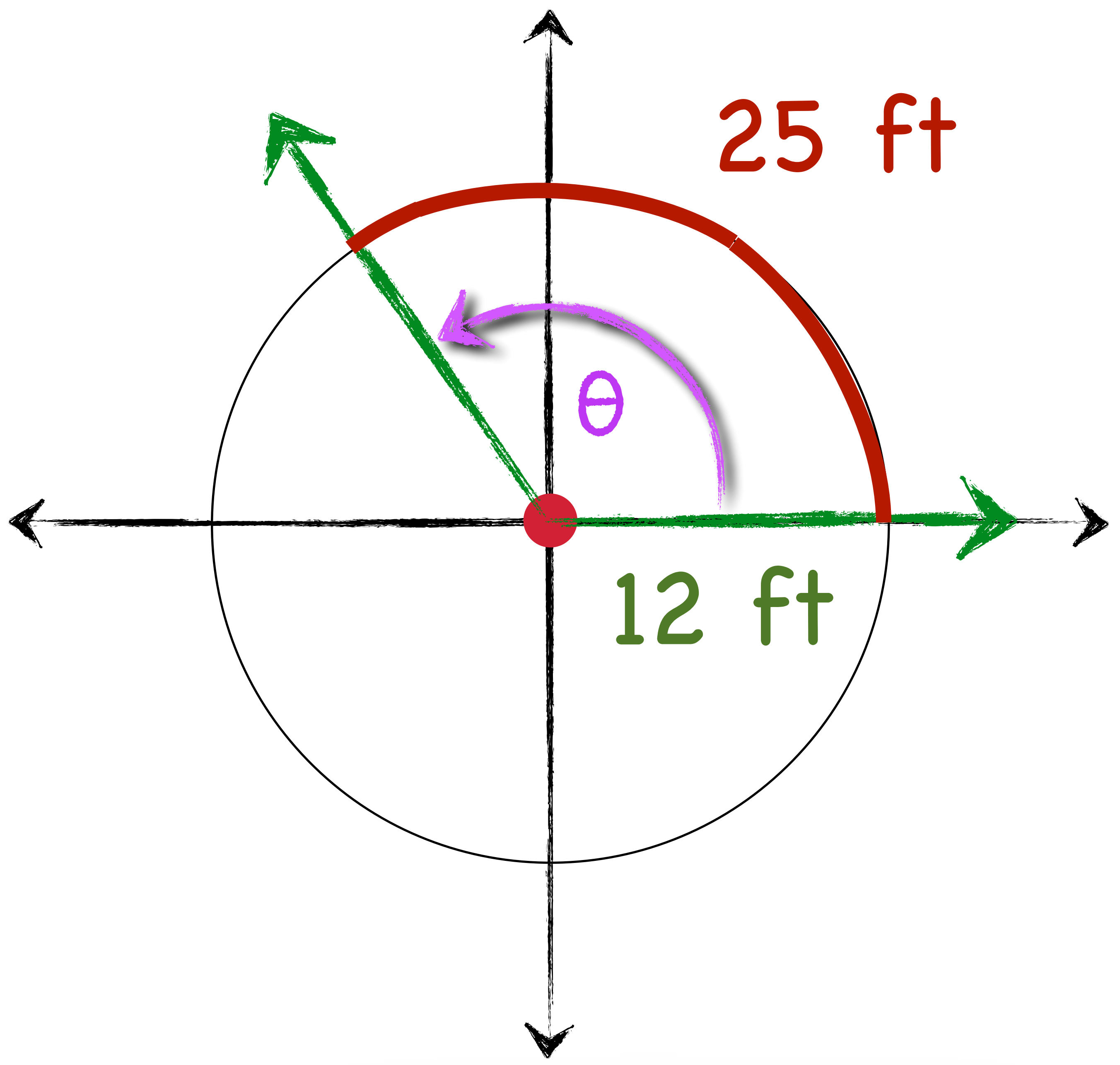


Note: radians have no unit of measure other than simply, radians.

Computing Radian Measure

- A central angle θ, in a circle of radius 12 feet intercepts an arc of length 25 feet. What is the radian measure of θ?

$$\theta = \frac{25ft}{12ft} = 2.083\,radians$$



Note: radians have no unit of measure other than simply, radians.

Radian Measure

Recall the definition of π.

$$\pi = \frac{Circumference}{Diameter} = \frac{Circumference}{2 \times radius}$$

A little algebra

$$2\pi r = Circumference$$

$$2\pi \ radians = Circumference$$

A little substitution

$$2\pi \ radians = 360^\circ$$

$$\pi \ radians = 180^\circ$$

Conversion between Degrees and Radians

- To convert degrees to radians or radians to degrees remember a circle has 360° and a circumference of $2\pi r$ (or 2π radians)

  - $2\pi r = 360°$, solving for r,

$$1\ radian = \frac{360°}{2\pi} = \frac{180°}{\pi}$$

  - To convert degrees to radians divide the degrees by the number of degrees for 1 radian:

$$radians = \frac{degrees}{\frac{180°}{\pi}} = degrees \times \frac{\pi}{180°}$$

  - To convert radians to degrees multiply the radians by the number of degrees for 1 radian:

$$degrees = radians \times \frac{180°}{\pi}$$

Converting from Degrees to Radians

Convert each angle in degrees to radians:

$$radians = \frac{degrees}{\frac{180^\circ}{\pi}} = degrees \times \frac{\pi}{180^\circ}$$

a. 60°

$$60^\circ x \frac{\pi}{180^\circ} = \frac{60^\circ \bullet \pi}{180^\circ} = \frac{\pi}{3} \, radians$$

b. 270°

$$270^\circ x \frac{\pi}{180^\circ} = \frac{270^\circ \bullet \pi}{180^\circ} = \frac{3\pi}{2} \, radians$$

c. −300°

$$-300^\circ x \frac{\pi}{180^\circ} = \frac{-300^\circ \bullet \pi}{180^\circ} = -\frac{5\pi}{3} \, radians$$

Converting from Radians to Degrees

Convert each angle in radians to degrees:

$$\text{degrees} = \text{radians} \times \frac{180^\circ}{\pi}$$

a. $\frac{\pi}{4}$ radians $\qquad \frac{\pi}{4}\text{ radians} \times \frac{180^\circ}{\pi\text{ radians}} = 45^\circ$

b. $-\frac{4\pi}{3}$ radians $\qquad -\frac{4\pi}{3}\text{ radians} \times \frac{180^\circ}{\pi\text{ radians}} = \frac{-4 \bullet 180^\circ}{3} = -240^\circ$

c. 6 radians $\qquad 6\text{ radians} \times \frac{180^\circ}{\pi\text{ radians}} = \frac{6 \bullet 180^\circ}{\pi} = \frac{1080^\circ}{\pi}$

Sums of Angles

Complementary angles.

- Two positive angles whose **sum is 90°** $\left(\frac{\pi}{2}\,radians\right)$ are said to be **complementary** angles.
- Find the complements of 38°, and $\frac{\pi}{3}\,radians$
  - $38° + x = 90°$
    - $x = 52°$

$$\frac{\pi}{3}\,radians + x = \frac{\pi}{2}\,radians$$

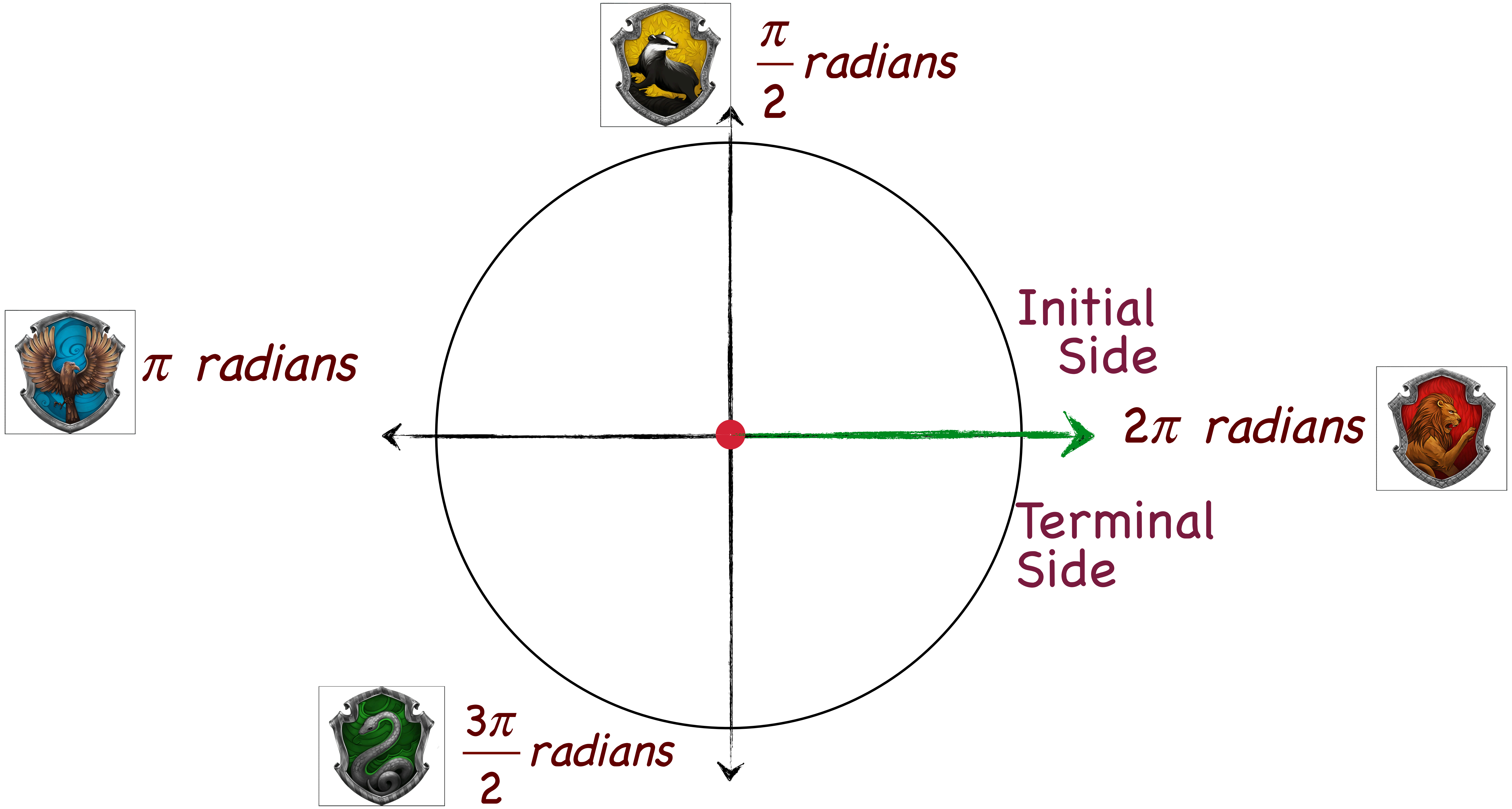
$$x = \frac{\pi}{6}\,radians$$

Supplementary angles.

- Two positive angles whose **sum is 180°** $(\pi\,radians)$ are said to be **supplementary** angles.

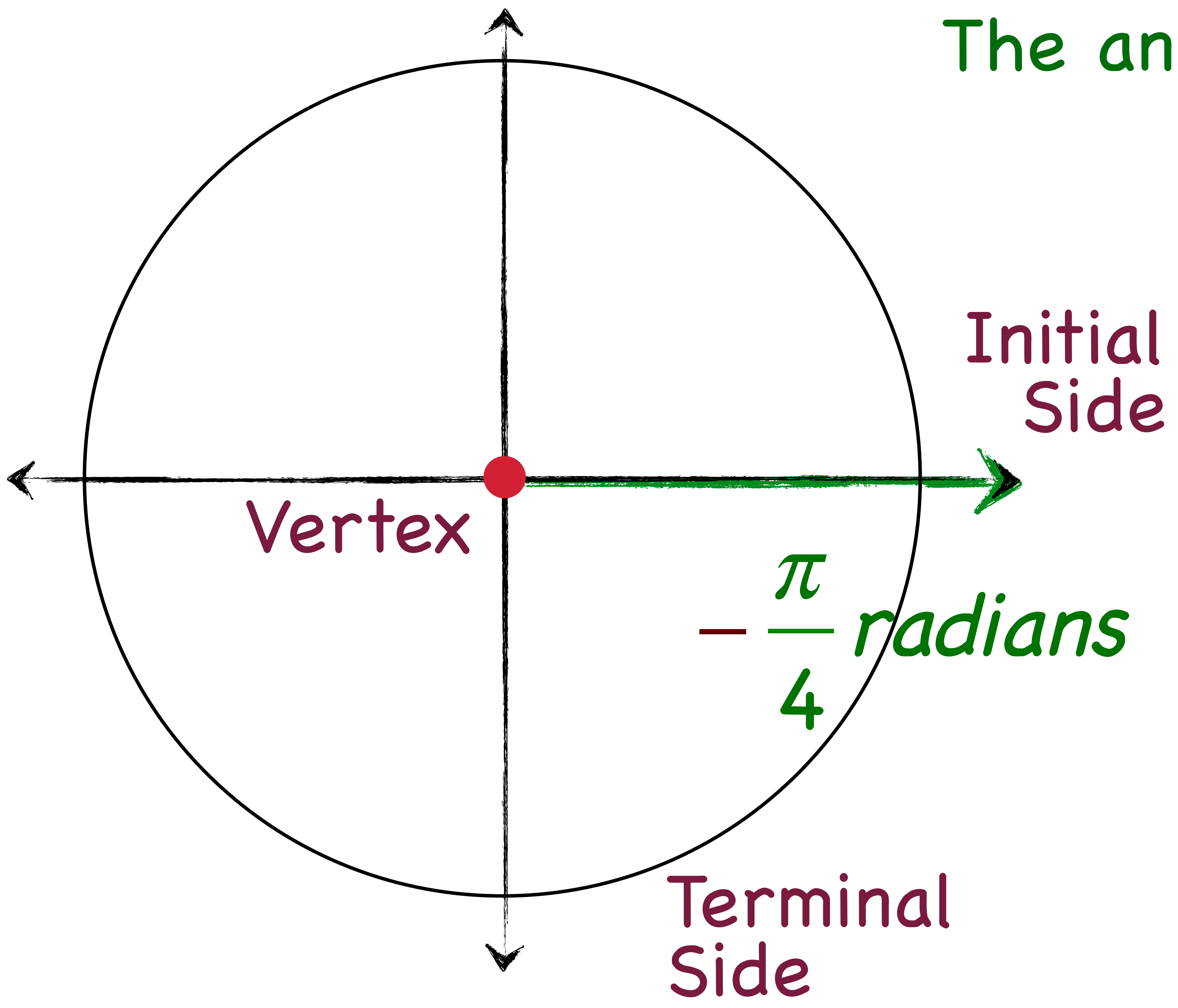
Drawing Angles in Standard Position

- The figure illustrates that when the terminal side makes one full revolution, it forms an angle whose radian measure is 2π. The figure shows the quadrantal angles formed by 3/4, 1/2, and 1/4 of a revolution.



Drawing Angles in Standard Position

Draw and label the angle $-\frac{\pi}{4}$ ***radians*** in standard position:



The angle is negative so we rotate the terminal side clockwise.

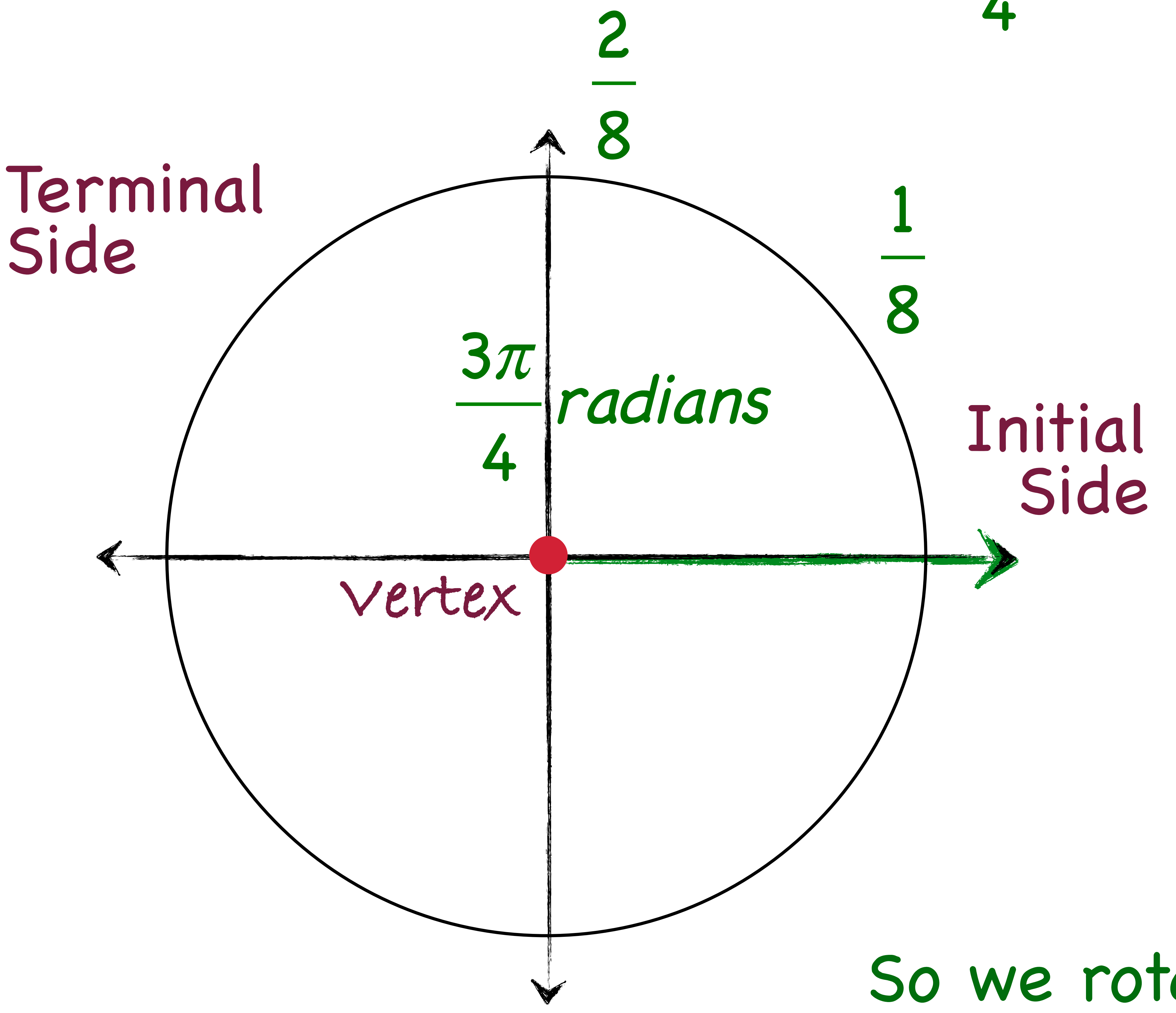
One full rotation is -2π radians

$$\frac{-\frac{\pi}{4}}{-2\pi} = \frac{1}{8} \textit{ rotations}$$

So we rotate the terminal side clockwise $\frac{1}{8}$ revolution

Drawing Angles in Standard Position

Draw and label the angle $\frac{3\pi}{4}$ *radians* in standard position:



The angle is positive so we rotate the terminal side counter-clockwise.

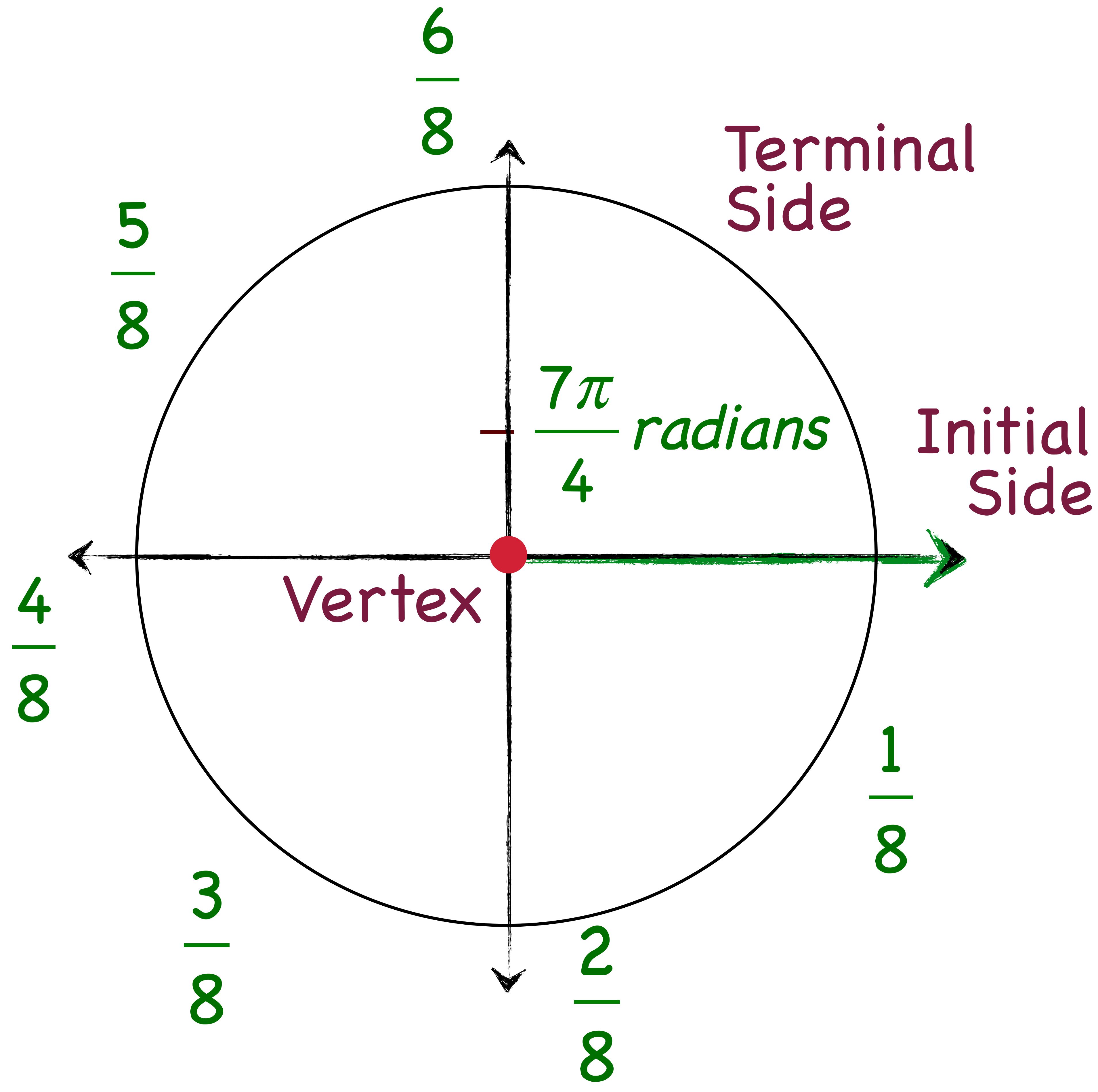
One full rotation is 2π radians

$$\frac{\frac{3\pi}{4}}{2\pi} = \frac{3}{8} \text{ rotations}$$

So we rotate the terminal side counter-clockwise $\frac{3}{8}$ revolution.

Drawing Angles in Standard Position

Draw and label the angle $-\dfrac{7\pi}{4}$ *radians* in standard position:



The angle is negative so we rotate the terminal side clockwise.

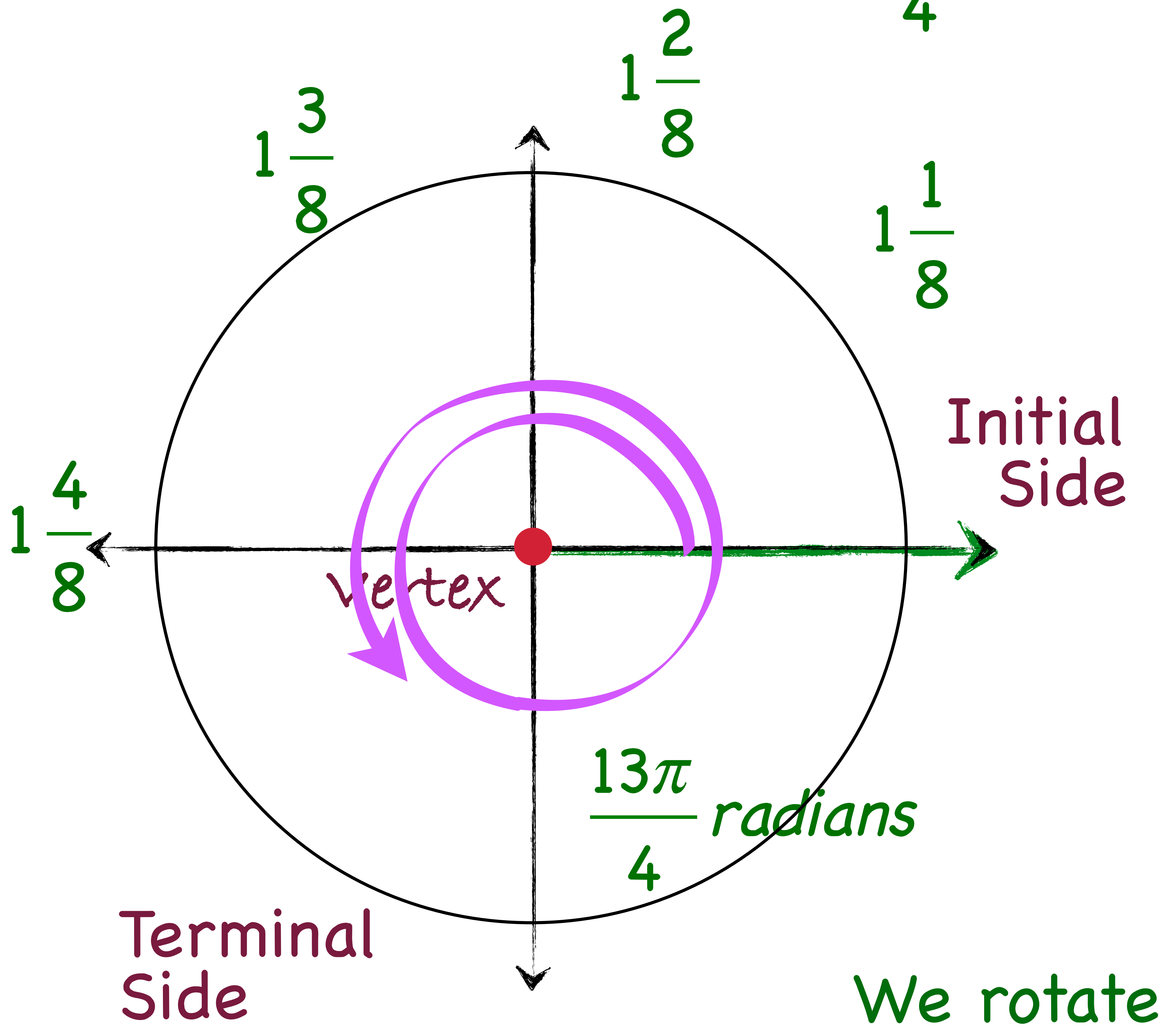
One full rotation is -2π radians

$$-\frac{7\pi}{4}\bullet\frac{1}{-2\pi}=\frac{7}{8}\ \textit{rotations}$$

So we rotate the terminal side clockwise $\dfrac{7}{8}$ revolution.

Drawing Angles in Standard Position

Draw and label the angle $\frac{13\pi}{4}$ *radians* in standard position:



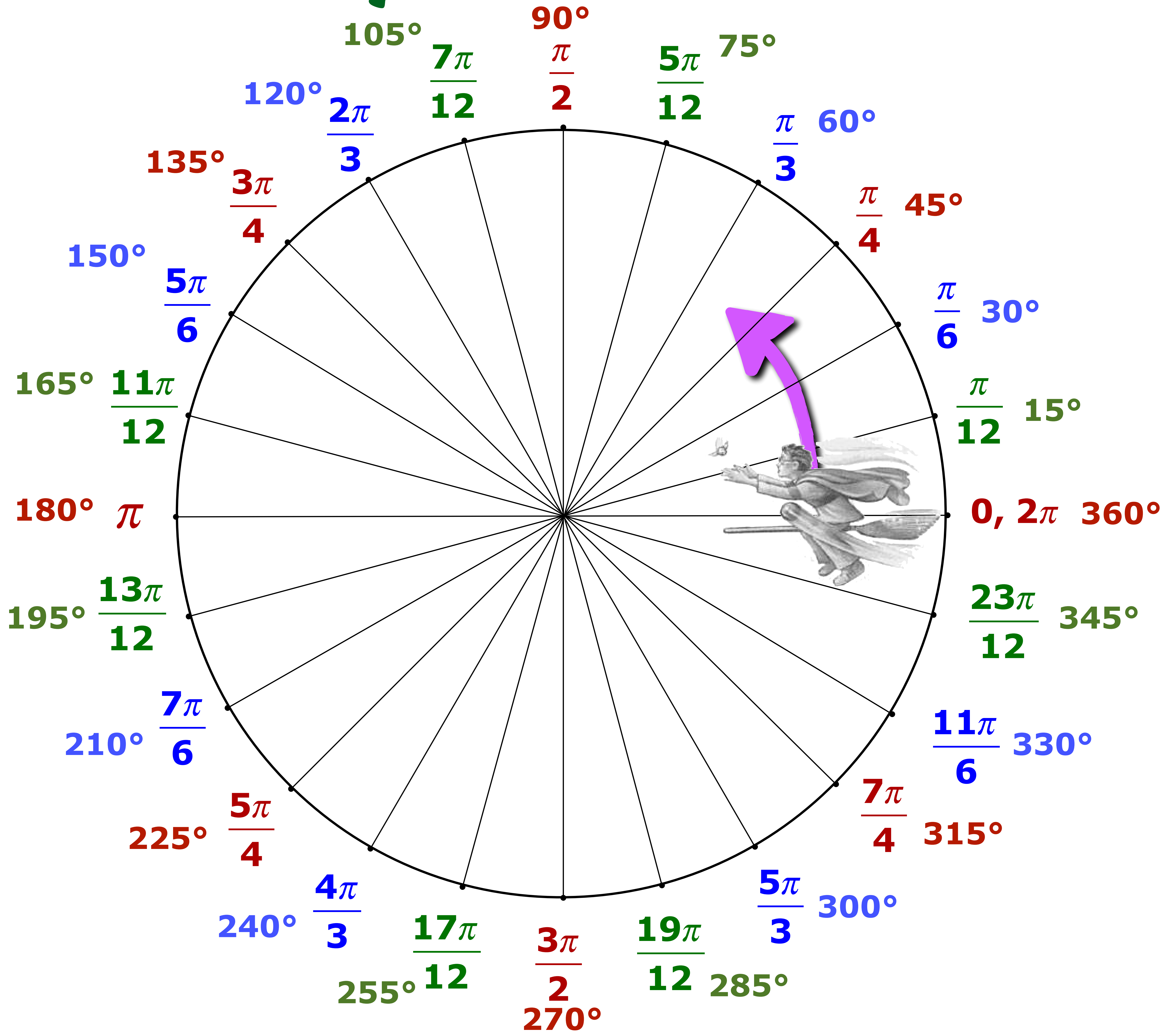
The angle is positive so we rotate the terminal side counter-clockwise.

One full rotation is 2π radians

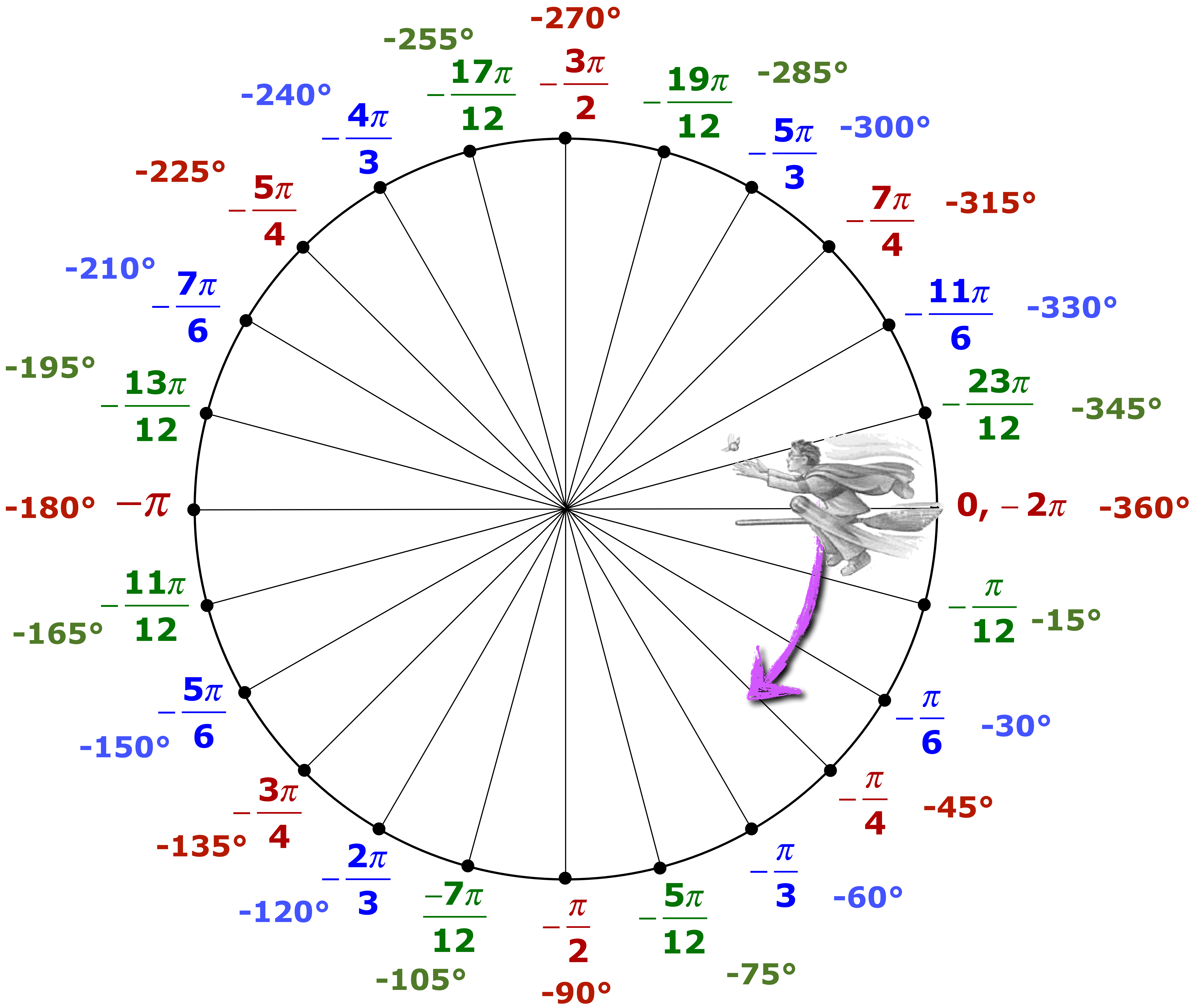
$$\frac{13\pi}{4} \bullet \frac{1}{2\pi} = 1\frac{5}{8} \textit{ rotations}$$

We rotate the terminal side counter-clockwise $1\frac{5}{8}$ revolutions.

Degree and Radian Measures of Angles Commonly Seen in Trigonometry



Degree and Radian Measures of Angles Commonly Seen in Trigonometry



Positive Angles in Revolutions of the Angle's Terminal

Terminal Side	Radian Measure of Angle	Degree Measure of Angle
$\frac{1}{12}$ revolution	$\frac{1}{12} \cdot 2\pi = \frac{\pi}{6}$	$\frac{1}{12} \cdot 360° = 30°$
$\frac{1}{8}$ revolution	$\frac{1}{8} \cdot 2\pi = \frac{\pi}{4}$	$\frac{1}{8} \cdot 360° = 45°$
$\frac{1}{6}$ revolution	$\frac{1}{6} \cdot 2\pi = \frac{\pi}{3}$	$\frac{1}{6} \cdot 360° = 60°$
$\frac{1}{4}$ revolution	$\frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$	$\frac{1}{4} \cdot 360° = 90°$
$\frac{1}{3}$ revolution	$\frac{1}{3} \cdot 2\pi = \frac{2\pi}{3}$	$\frac{1}{3} \cdot 360° = 120°$

Positive Angles in Revolutions of the Angle's Terminal

Terminal Side	Radian Measure of Angle	Degree Measure of Angle
$\frac{1}{2}$ revolution	$\frac{1}{2} \cdot 2\pi = \pi$	$\frac{1}{2} \cdot 360° = 180°$
$\frac{2}{3}$ revolution	$\frac{2}{3} \cdot 2\pi = \frac{4\pi}{3}$	$\frac{2}{3} \cdot 360° = 240°$
$\frac{3}{4}$ revolution	$\frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}$	$\frac{3}{4} \cdot 360° = 270°$
$\frac{7}{8}$ revolution	$\frac{7}{8} \cdot 2\pi = \frac{7\pi}{4}$	$\frac{7}{8} \cdot 360° = 315°$
1 revolution	$1 \cdot 2\pi = 2\pi$	$1 \cdot 360° = 360°$

Co-terminal Angles

Two angles with the **same initial and terminal sides** but possibly different rotations are called **co-terminal angles**.

Increasing or decreasing the degree measure of an angle in standard position by an integer multiple of 360° results in a co-terminal angle.

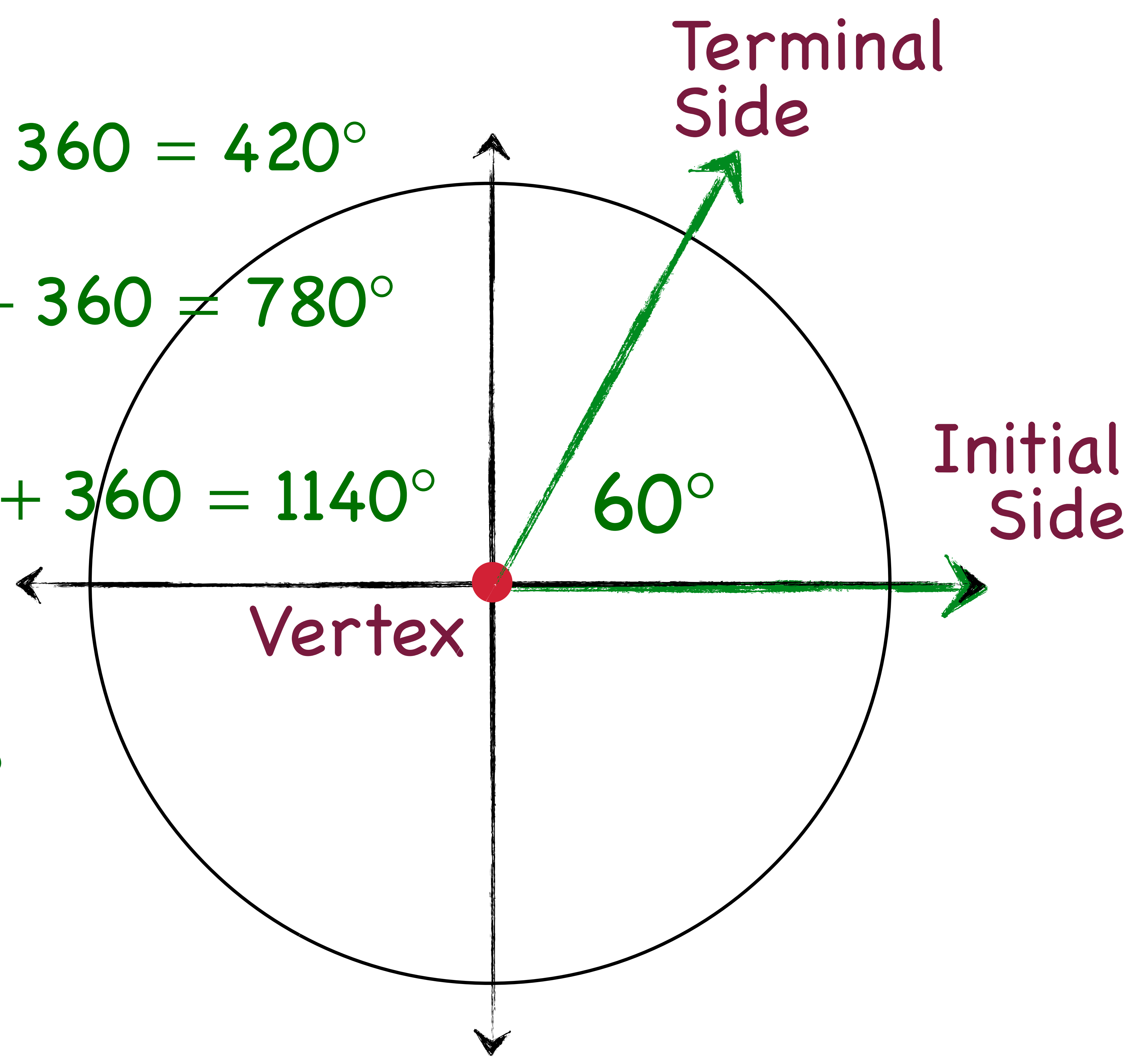
So, an angle of θ^0 is co-terminal with angles of θ^0 **± 360°k**, where k is an integer.

Also, an angle of A radians is co-terminal with angles of A radians **± 2πk radians**, where k is an integer.

$60 + 360 = 420°$

$420 + 360 = 780°$

$780 + 360 = 1140°$



Example: Finding Coterminal Angles

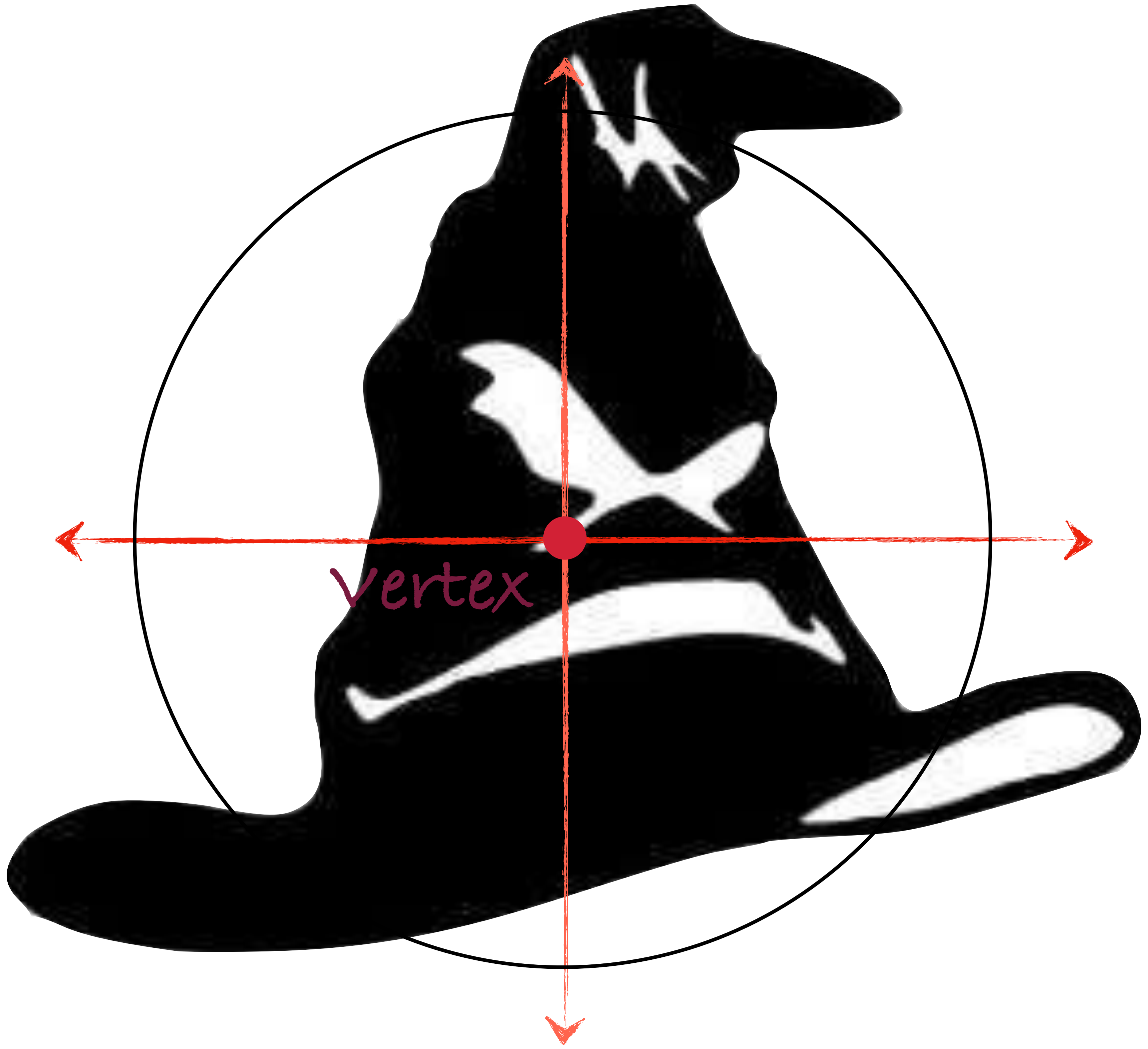
Assume the following angles are in standard position. Find a positive angle **less than 360°** that is co-terminal with each of the following:

a. a 400° angle $400° - 360° = 40°$

b. a −135° angle $-135° + 360° = 225°$

c. a 430° angle $430° - 360° = 70°$

d. a -40° angle $-40° + 360° = 320°$



Example: Finding Coterminal Angles

- Assume the following angles are in standard position. Find a positive angle **less than 2π** that is co-terminal with each of the following:

a. $\frac{13\pi}{5}$ radians $\qquad \frac{13\pi}{5} - 2\pi = \frac{13\pi}{5} - \frac{10\pi}{5} = \frac{3\pi}{5}$

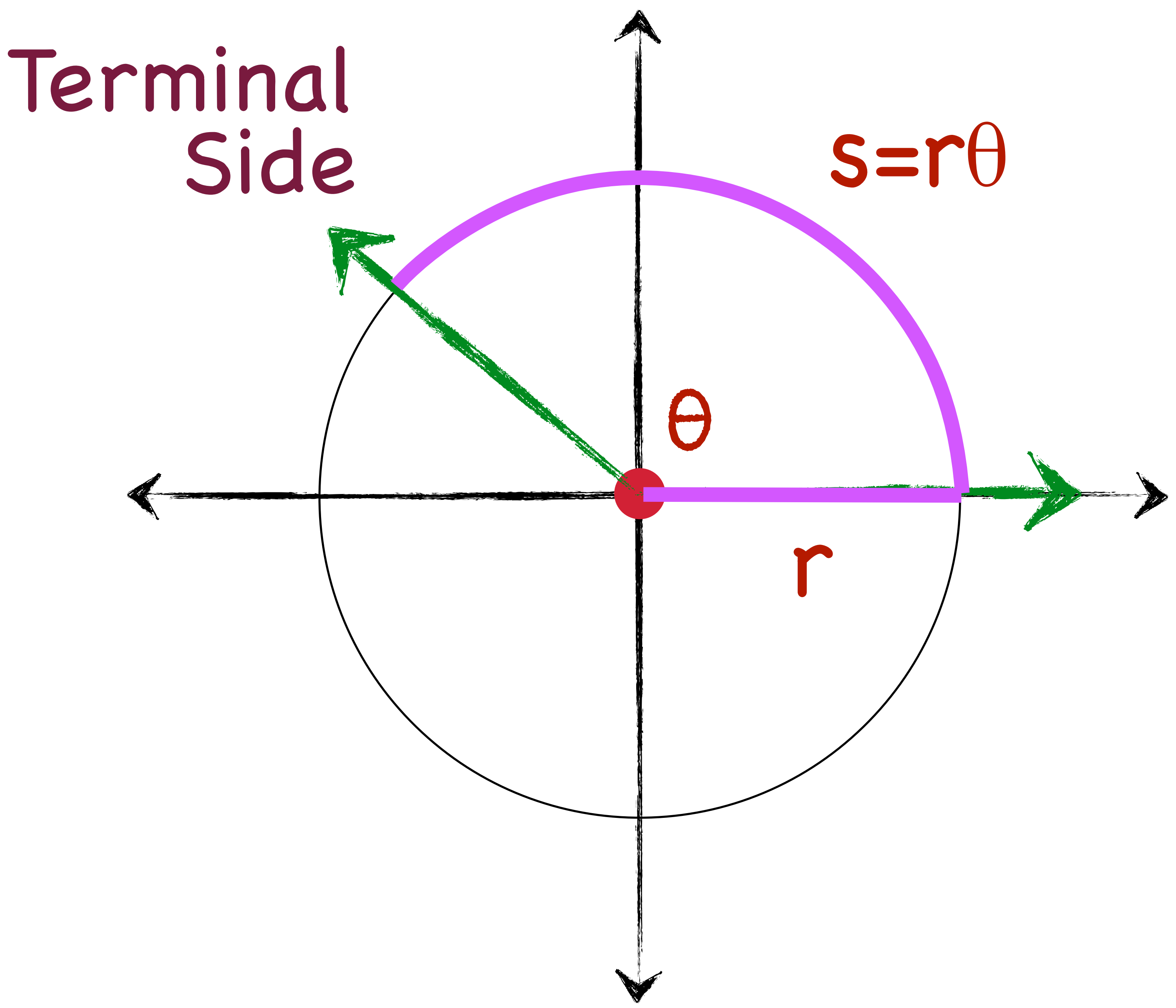
b. $-\frac{\pi}{15}$ radians $\qquad -\frac{\pi}{15} + 2\pi = -\frac{\pi}{5} + \frac{30\pi}{15} = \frac{29\pi}{15}$

c. $-\frac{37\pi}{6}$ radians $\qquad -\frac{37\pi}{6} + 8\pi = -\frac{37\pi}{6} + \frac{48\pi}{6} = \frac{11\pi}{6}$

The Length of a Circular Arc

- Assume a circle with radius **r** has positive central angle θ **radians**

  The length of the arc intercepted by the central angle is $s = r\theta$



Finding the Length of a Circular Arc

- A circle has a radius of 6 inches. Find the length of the arc intercepted by a central angle of 45°. Express arc length in terms of π. (Then approximate your answer to two decimal places.

First convert 45° to radians (We will calculate, but soon you should be able to convert 45° automatically):

$$45^\circ x \frac{\pi}{180^\circ} = \frac{45^\circ \bullet \pi}{180^\circ} = \frac{\pi}{4} radians$$

Then find the length of the arc.

$$s = \frac{\pi}{4} radians\ x \frac{6in}{radian} = \frac{3\pi}{2} in \approx 4.71in$$

Finding the Length of a Circular Arc

- A circle has a radius of 9 inches. Find the length of the arc intercepted by a central angle of 135°. Express arc length in terms of π. (Then approximate your answer to two decimal places.

First convert 135° to radians

$$135^\circ x \frac{\pi}{180^\circ} = \frac{135^\circ \bullet \pi}{180^\circ} = \frac{3\pi}{4} radians$$

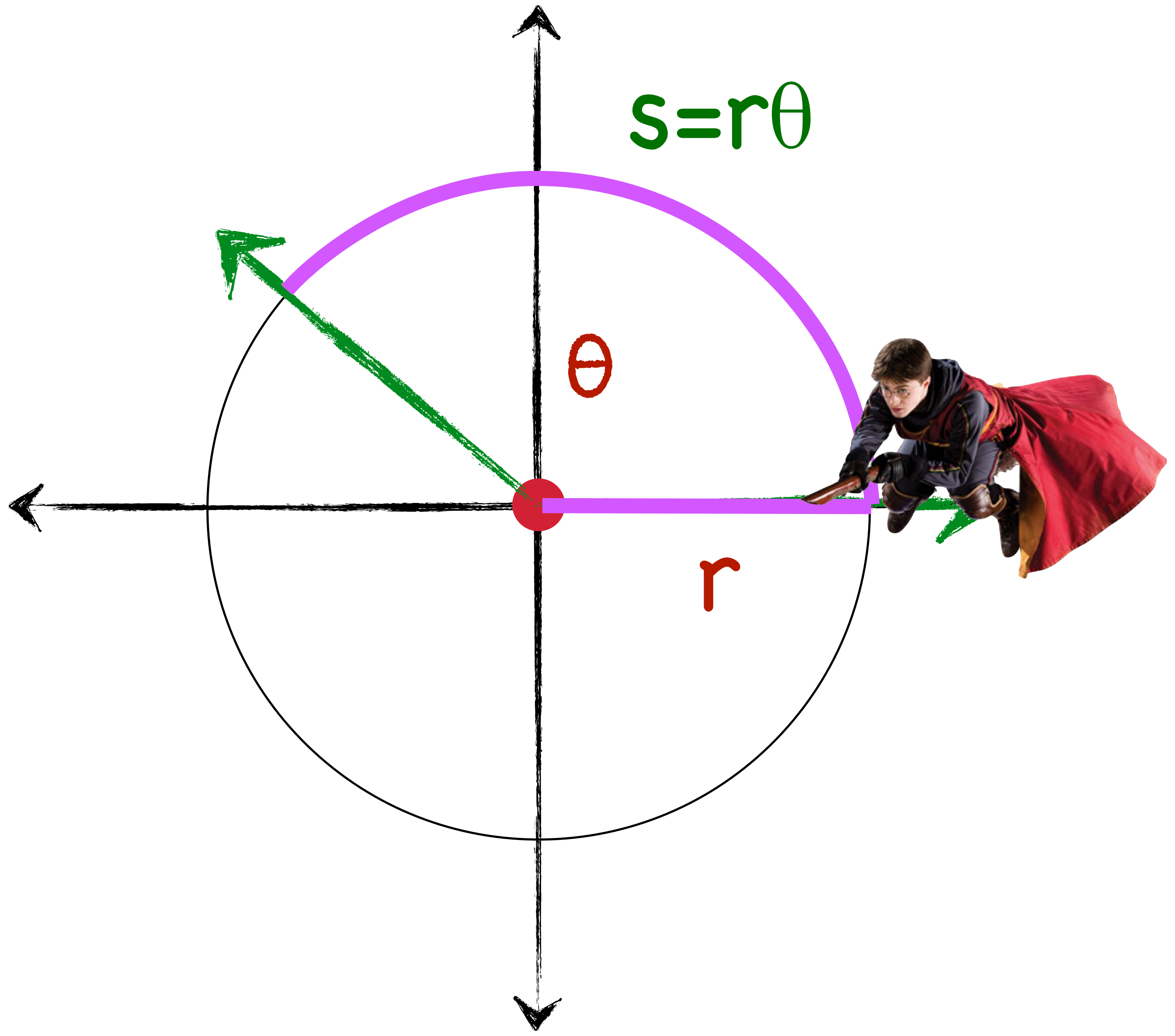
Find the length of the arc.

$$s = \frac{3\pi}{4} radians\ x \frac{9in}{radian} = \frac{27\pi}{4} in = 21.21in$$

Definitions of Linear and Angular Speed

- If a point is in motion on a circle of radius r through an angle of θ radians in time t, then the point's **linear speed** (how fast our little man is flying around the circle) is:

$$v = \frac{s}{t}$$ s is the arc length ($s = r\theta$)



The **angular speed** (spin rate in # revolutions/unit of time) is given by

$$\omega(omega) = \frac{\theta}{t}$$

Linear Speed in terms of Angular Speed

- The linear speed, v (velocity), of a point a distance r from the center of rotation is given by:

$$v = \frac{s}{t} = \frac{r\theta}{t} = r\frac{\theta}{t} = r\omega$$

$$v = r\omega$$

v is the **linear speed** of the point and
ω is the **angular speed** of the point.

STUDY TIP

Linear speed measures how fast the particle moves, and angular speed measures how fast the angle changes. By dividing the formula for arc length by t, you can establish a relationship between linear speed v and angular speed ω, as shown.

$$s = r\theta$$

$$\frac{s}{t} = \frac{r\theta}{t}$$

$$v = r\omega$$

Example: Finding Linear Speed

- The largest wind turbine has blades 88.4 meters long and the diameter of the rotating assembly is 180 meters. If one revolution takes 8 seconds, what is the linear velocity (m/s) of a point at the tip of a blade?

We are told it takes 8 seconds for one revolution, so the angular speed, ω, is 1/8 revolutions/second.

Before applying the formula v = rω, we must express ω in terms of radians per second:

$$\omega = \frac{1\,\text{revolution}}{8\,\text{seconds}} \times \frac{2\pi\ \textit{radians}}{1\,\text{revolution}} = \frac{2\pi\ \textit{radians}}{8\,\text{seconds}} = \frac{\pi\ \textit{radians}}{4\,\text{seconds}}$$

$$\omega \approx .7854\,\frac{\textit{radians}}{\text{sec}}$$

Example: Finding Linear Speed

- The largest wind turbine has blades 88.4 meters long and the diameter of the rotating assembly is 180 meters. If one revolution takes 7.5 seconds, what is the linear velocity (m/sec) of a point at the tip of a blade?

$$\omega \approx .7854 \frac{radians}{\text{sec}}$$

$$v = r\omega$$

The radius of the rotating assembly is 90 meters.

$$v = \frac{90m}{1\,radian} \times \frac{.7854\,radians}{1\,\text{second}} \approx 70.6858\,\text{m/sec}$$

That is about 155.5 miles/hour.

Example: Finding Linear Speed

- A typical HDD (hard disk drive in your computer) spins at 7200 revolutions per minute (rpm). In a desktop computer the form factor of an hdd is 3.5 inches. What is the linear speed of a spot 3 inches from the center?

One revolution is 2π radians. Thus 7200 **rpms** is ...

... an angular speed of $7200 \times 2\pi = 14400\pi$ radians per minute.

The **linear speed** at 3 inches is

$$v = r\omega$$

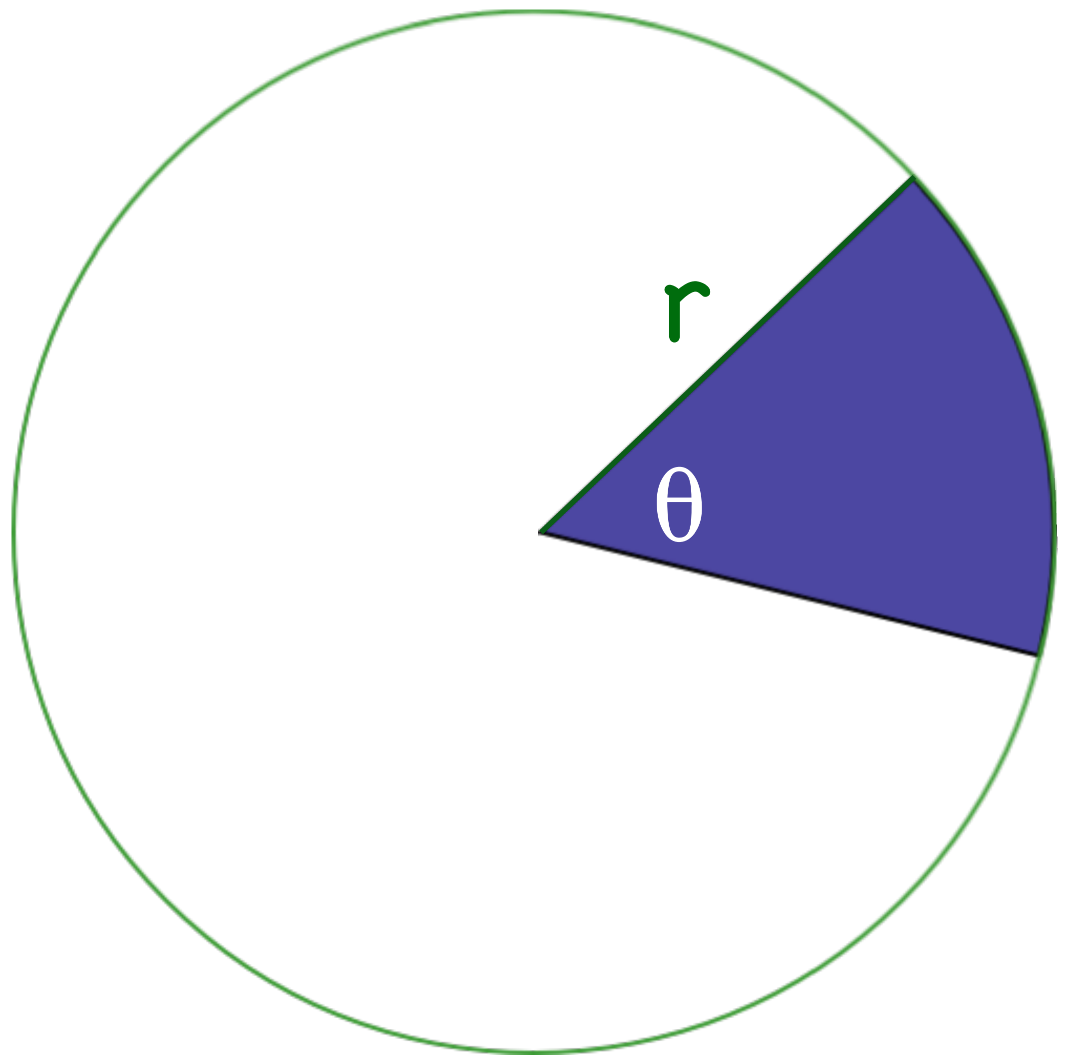
$$v = \frac{3 \text{ in}}{1 \text{ radian}} \times \frac{14400\pi \text{ radians}}{1 \text{ minute}} = \frac{43200\pi \ in}{1 \text{ minute}}$$

$$\approx 135716.8 \text{ in/min}$$

Sectore

- A **sector** of a circle is the region formed by two radii of the circle and the intercepted arc

For a circle with radius r and central angle θ **radians**, the area of a sector is given by **A = (1/2)θr²**.



This is easily confirmed considering the area of circle

$$A = \pi r^2 = \frac{1}{2}(2\pi)r^2$$

Area of a Sector

- A sprinkler on a shameful hotel lawn sprays water over a distance of 60 feet through an angle of 180 degrees. What area of the lawn is watered by the sprinkler?

  - First convert 180^0 to π radians.

$A = (1/2)r^2\theta$

$A = (1/2)60^2$ feet^2 x π radians = 1800π square feet