Chapter 4

Trigonometric Functions

- 4.2 The Unit Circle

Chapter 4.2

Homework

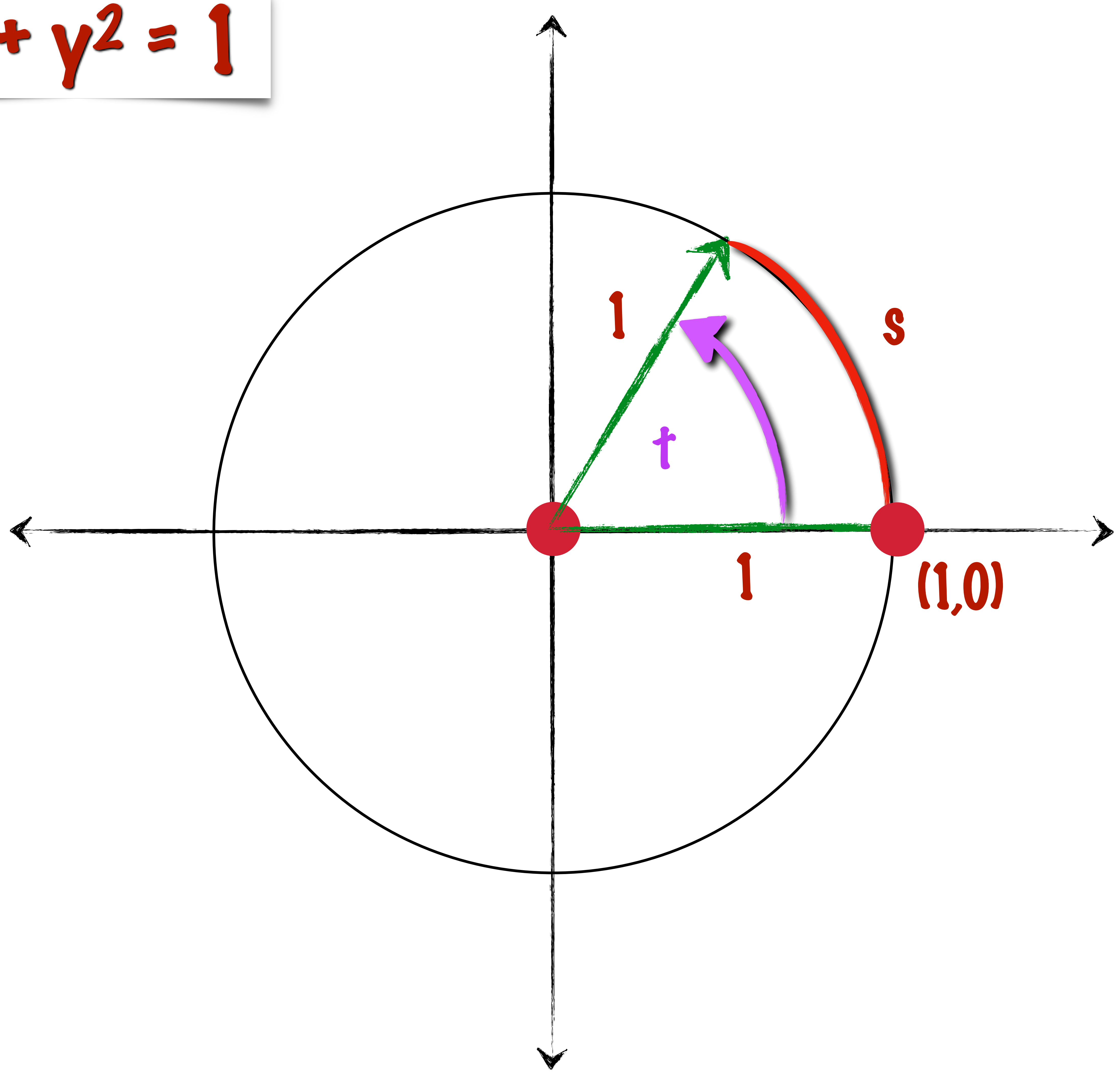
- 4.2 p 299 1-55 odd http://www.mathgraphs.com/mg_pl1e.html

Chapter 4.2

- Use a unit circle to define trigonometric functions of real numbers.
- Recognize the domain and range of sine and cosine functions
- Find exact values of the trigonometric functions at $\frac{\pi}{4}$
- Use even and odd trigonometric functions.
- Recognize and use fundamental identities
- Use periodic properties.
- Evaluate trigonometric functions with a calculator.

The Unit Circle

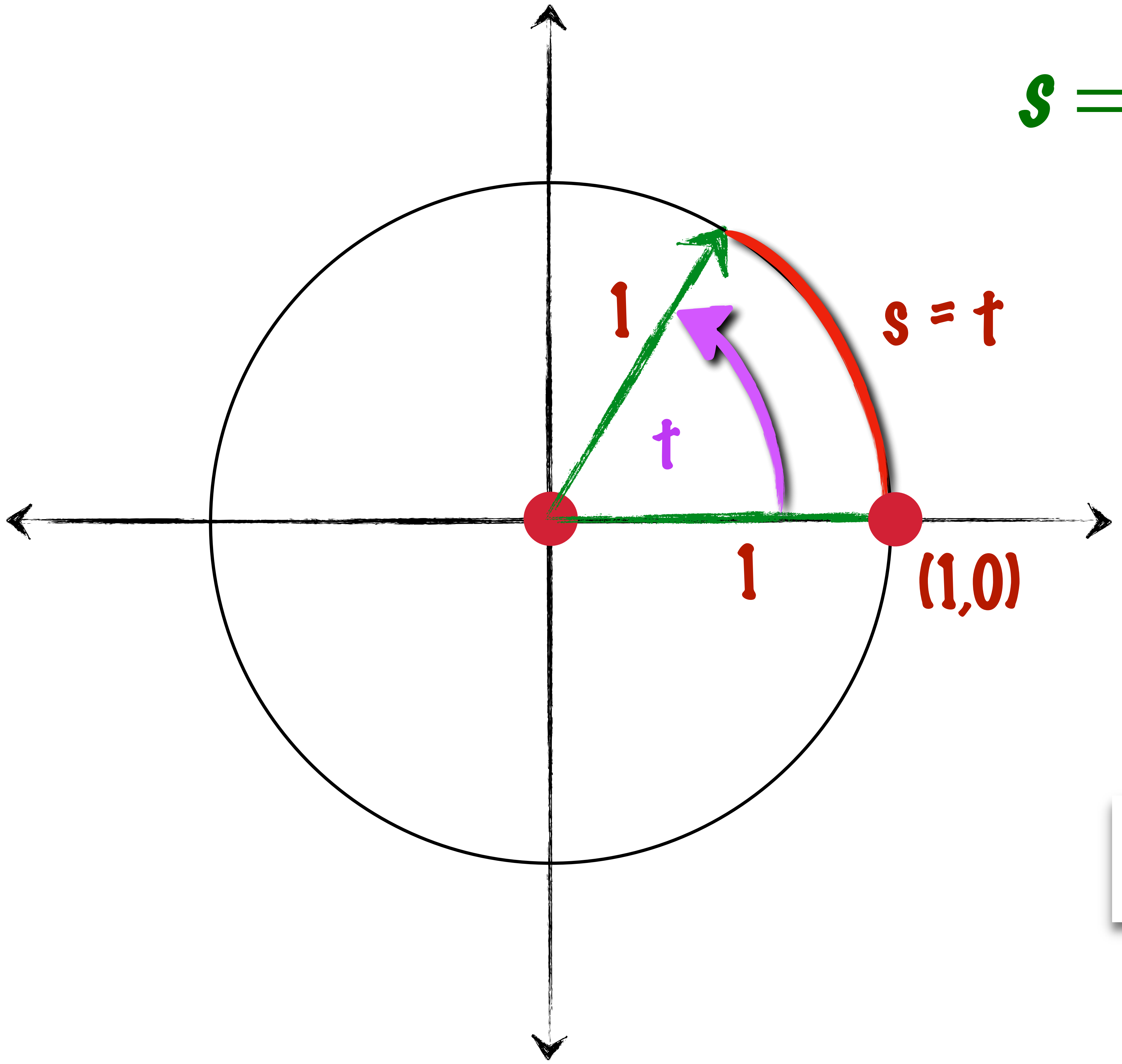
A **unit circle** is a circle of radius 1, with its center at the origin of a rectangular coordinate system. The equation of this circle is $x^2 + y^2 = 1$

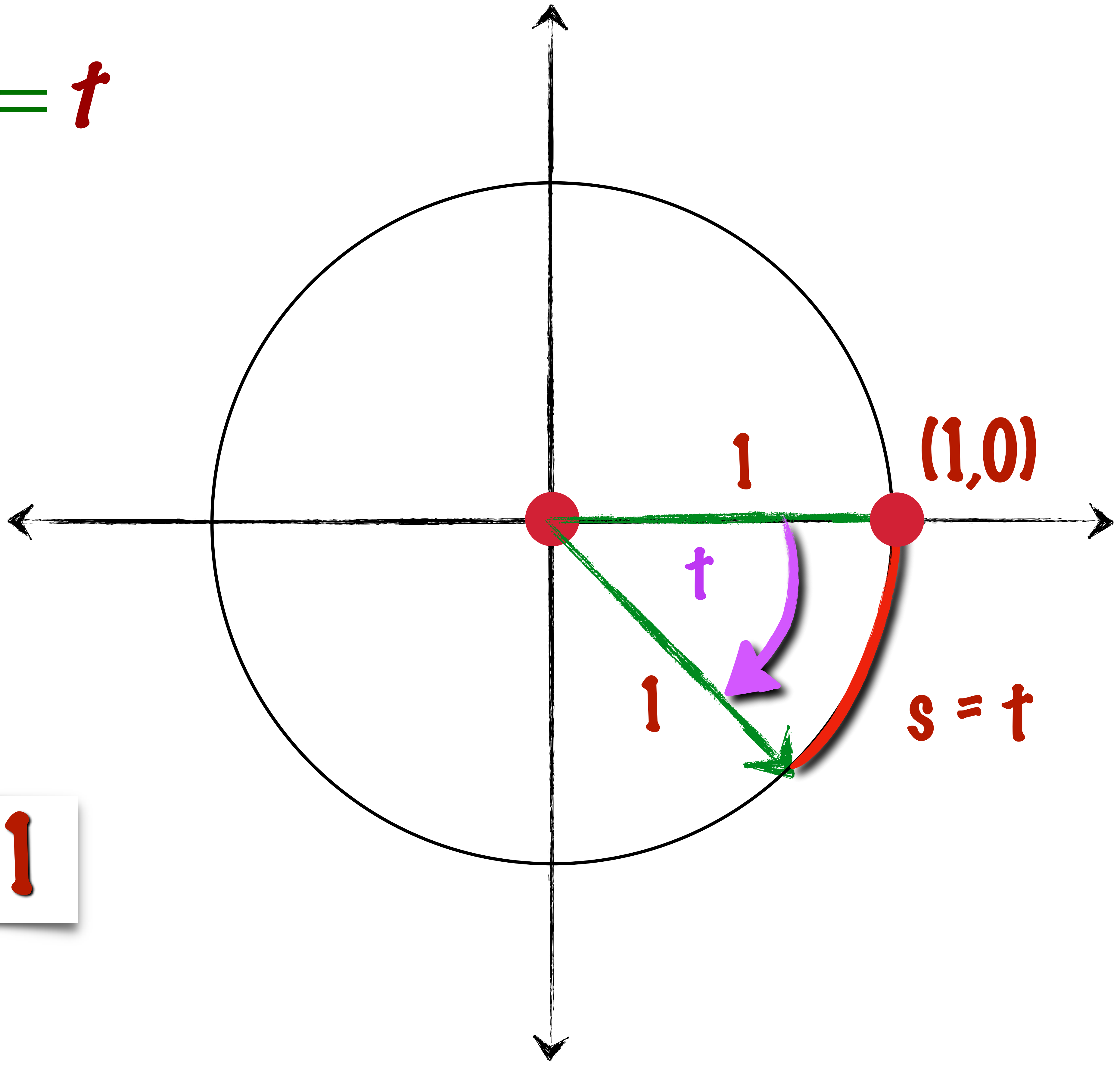


The Unit Circle

In a unit circle, **the radian measure of the central angle is equal to the rotational length of the intercepted arc.** Both are given by the same real number **t**.

$$s = r\theta = 1 \bullet t = t$$





$$x^2 + y^2 = 1$$

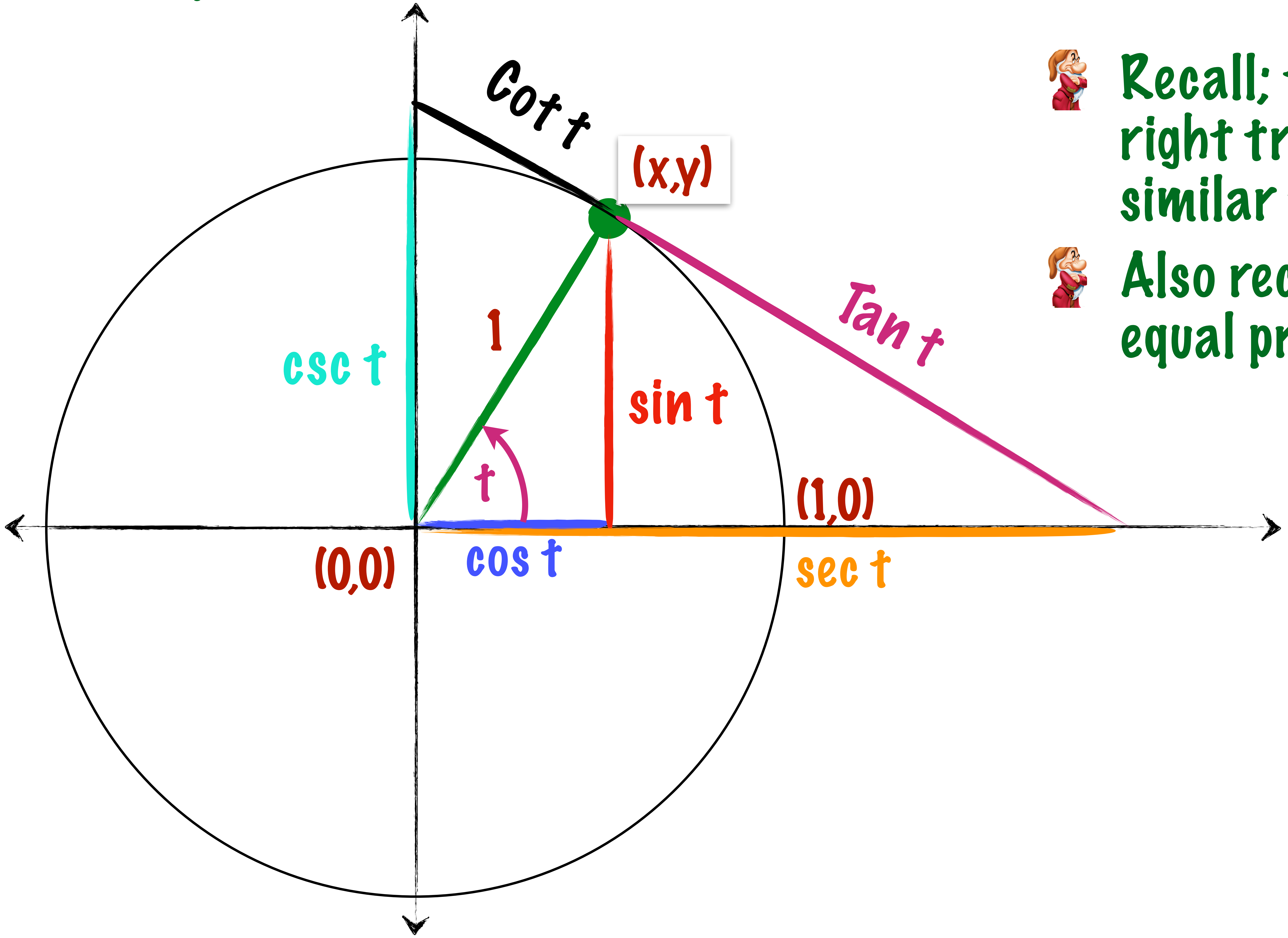
The Six Trigonometric Functions

Six trigonometric Ratios

Name	Abbreviation
Sine	Sin
Cosine	Cos
Tangent	Tan
Cosecant	Csc
Secant	Sec
Cotangent	Cot

Trigonometric Ratios

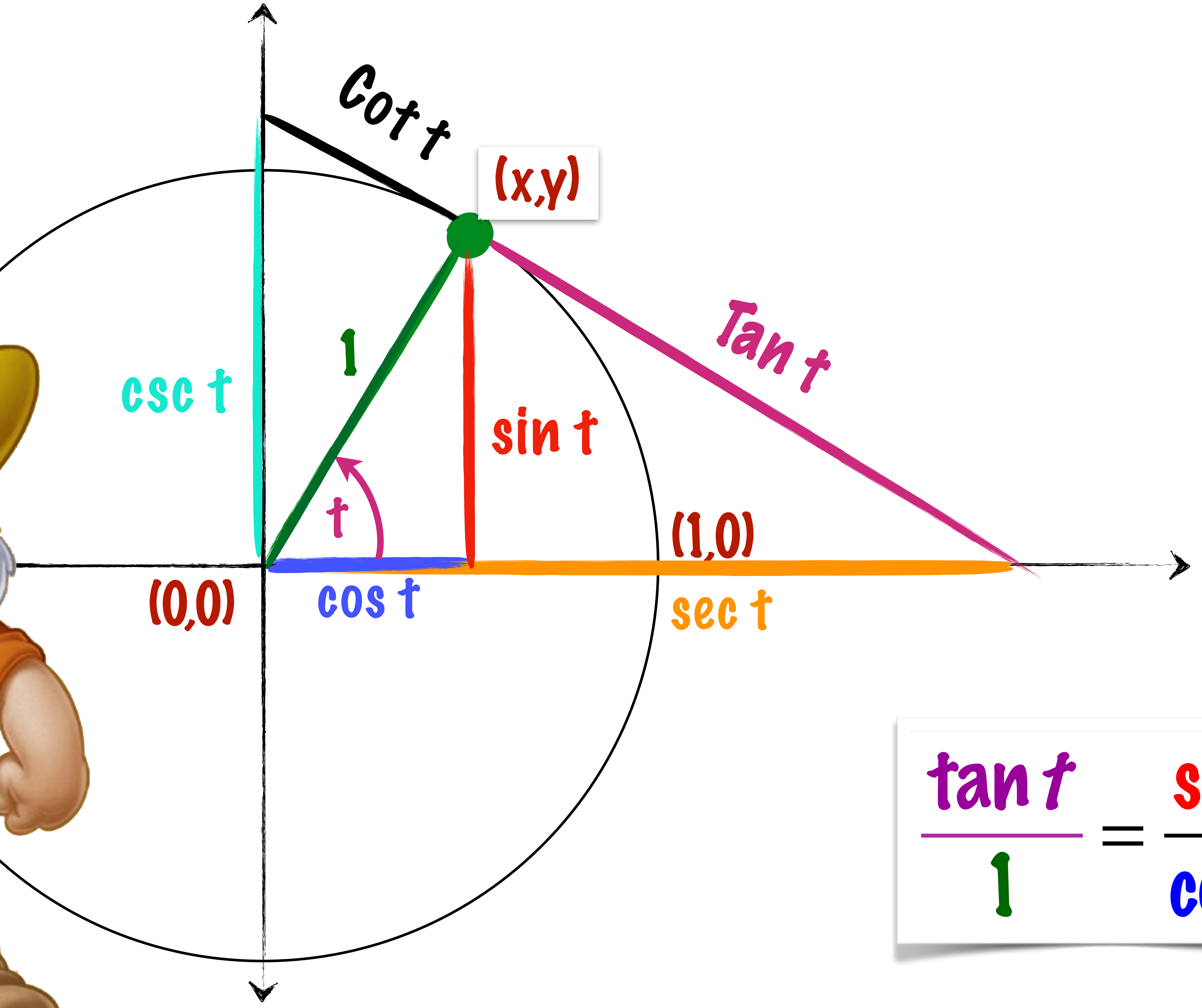
- The six trig ratios can be found geometrically from the tangent to a circle at the point intercepted by the terminal side of an angle in standard position.

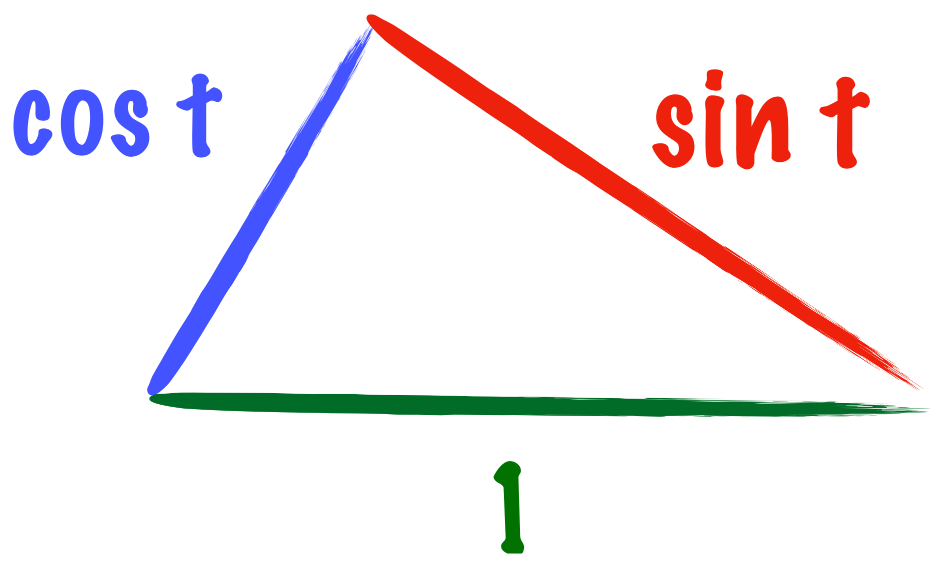


- Recall; the altitude to the hypotenuse of a right triangle divides the triangle into 3 similar triangles.
- Also recall that similar triangles have equal proportions.

Trigonometric Ratios

- By using what we know about **similar** right triangles, we can find the relationships between the trig ratios.

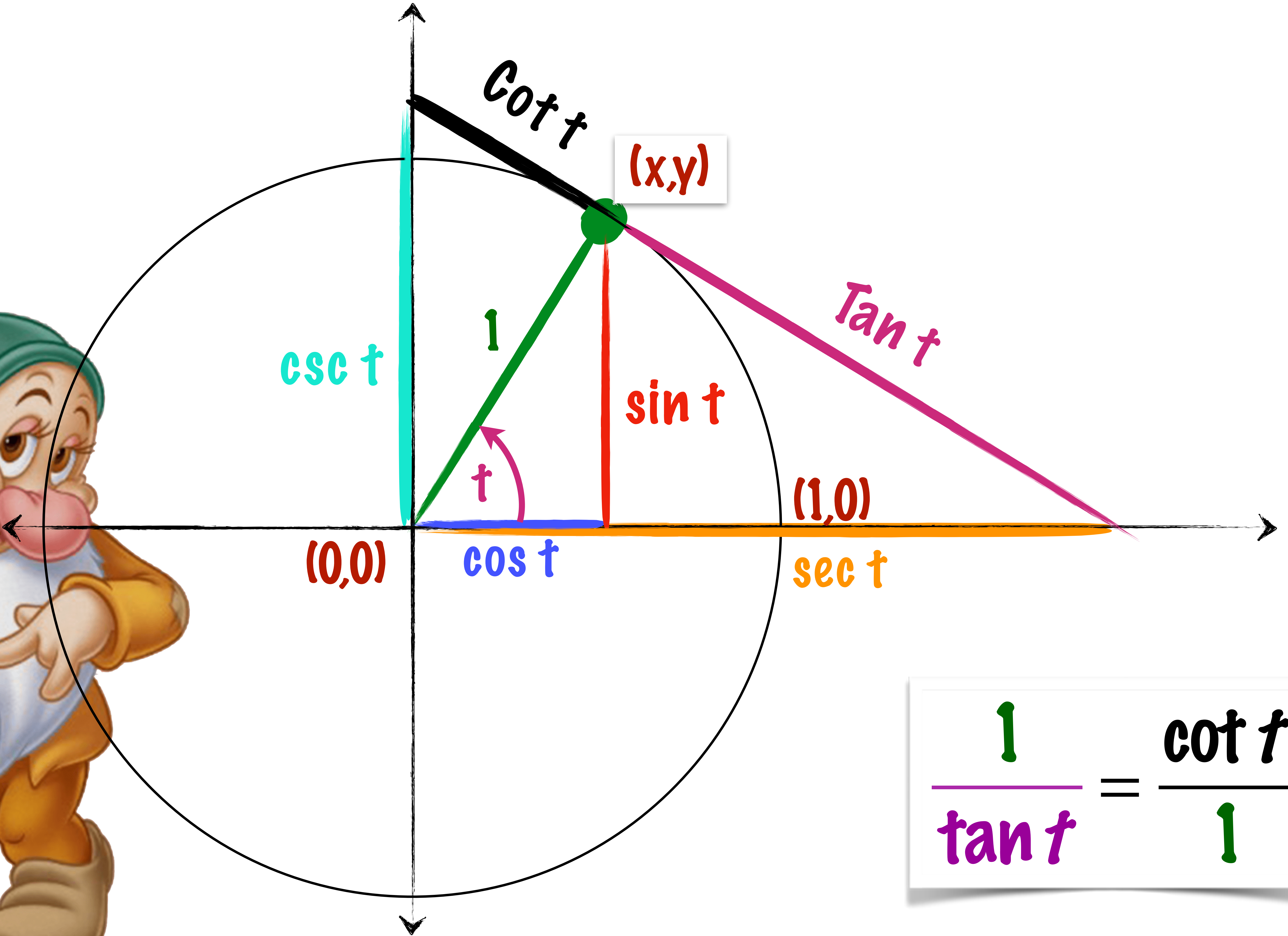




$$\frac{\tan t}{1} = \frac{\sin t}{\cos t}$$

Trigonometric Ratios

Tangent and Cotangent



1

Tan t

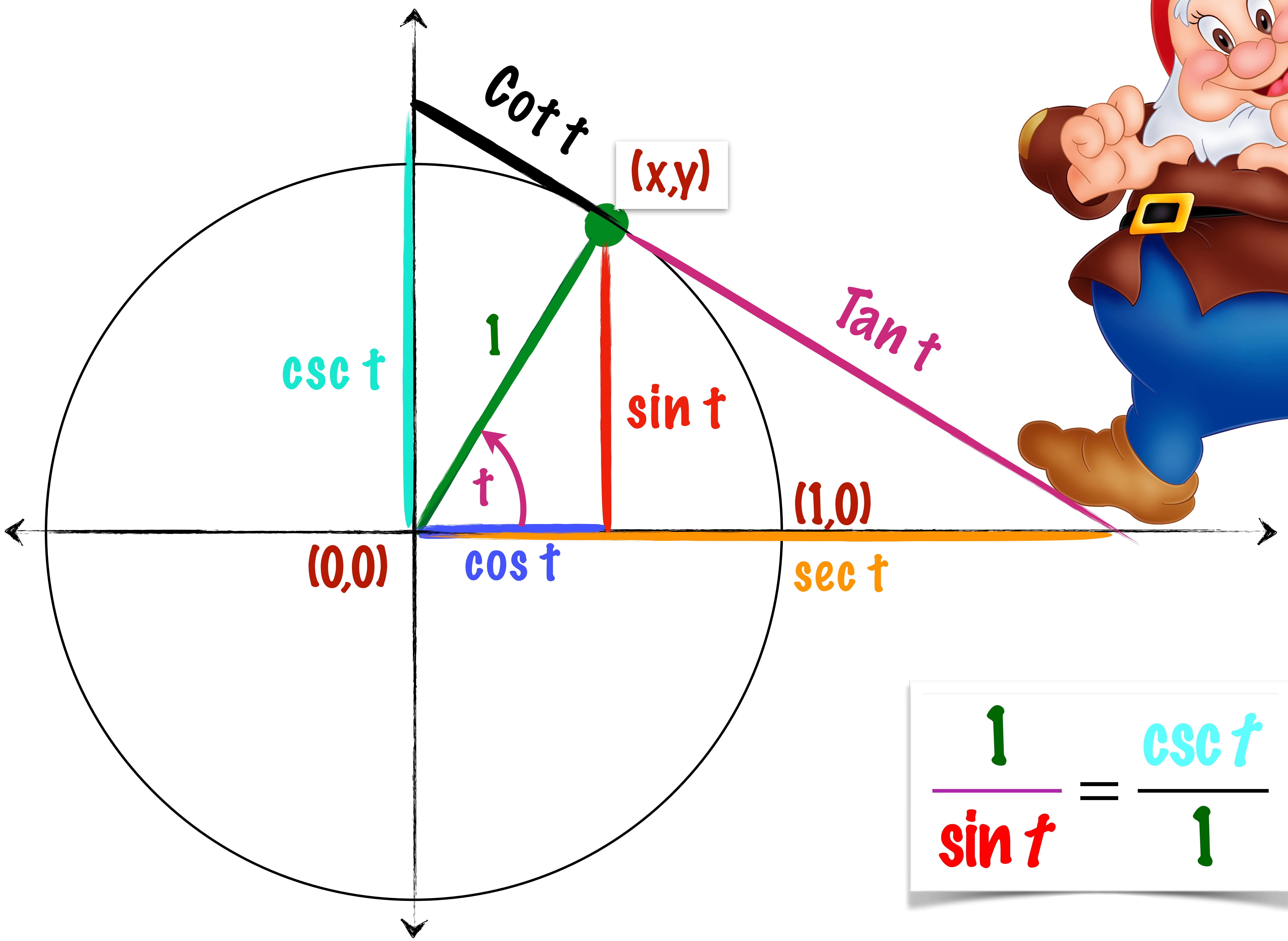
Cot t

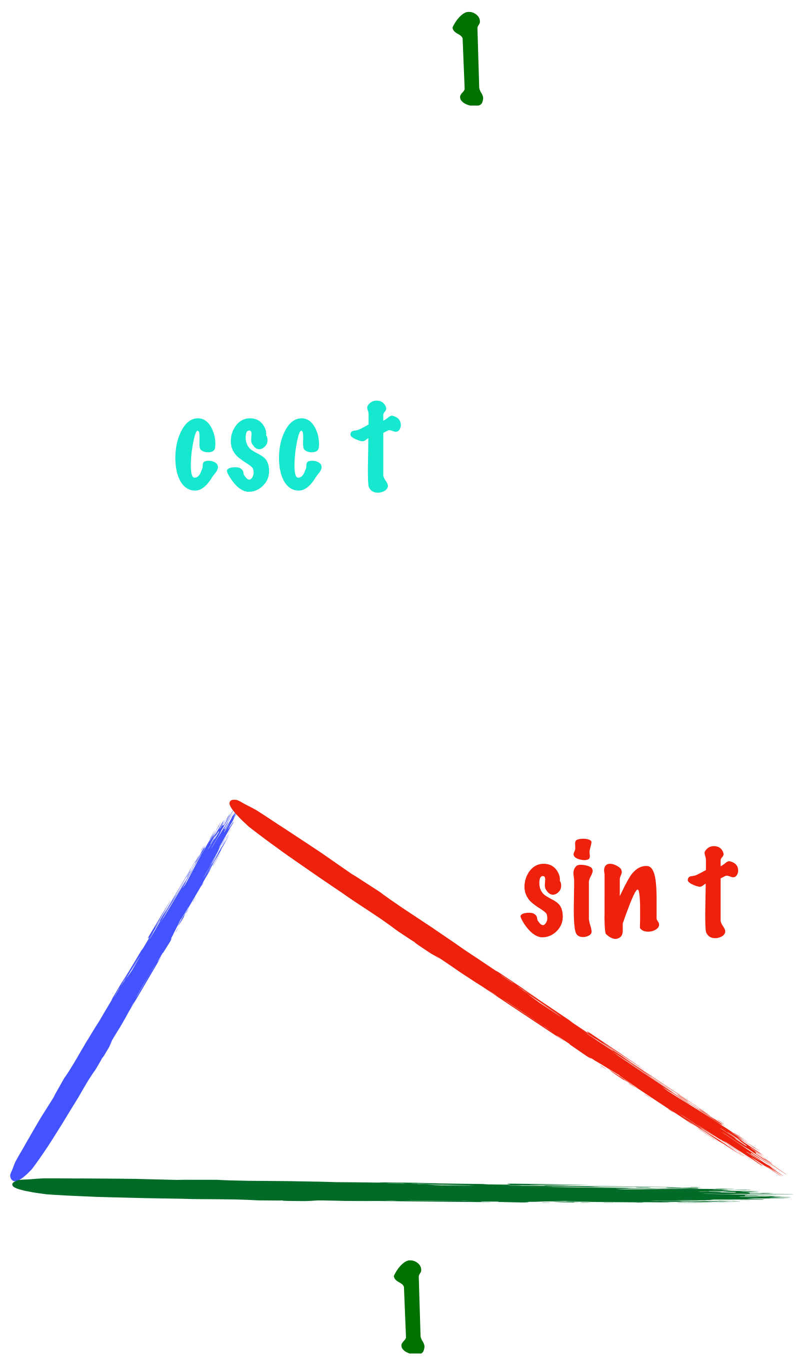
1

$$\frac{1}{\tan t} = \frac{\cot t}{1}$$

Trigonometric Ratios

Sine and Cosecant

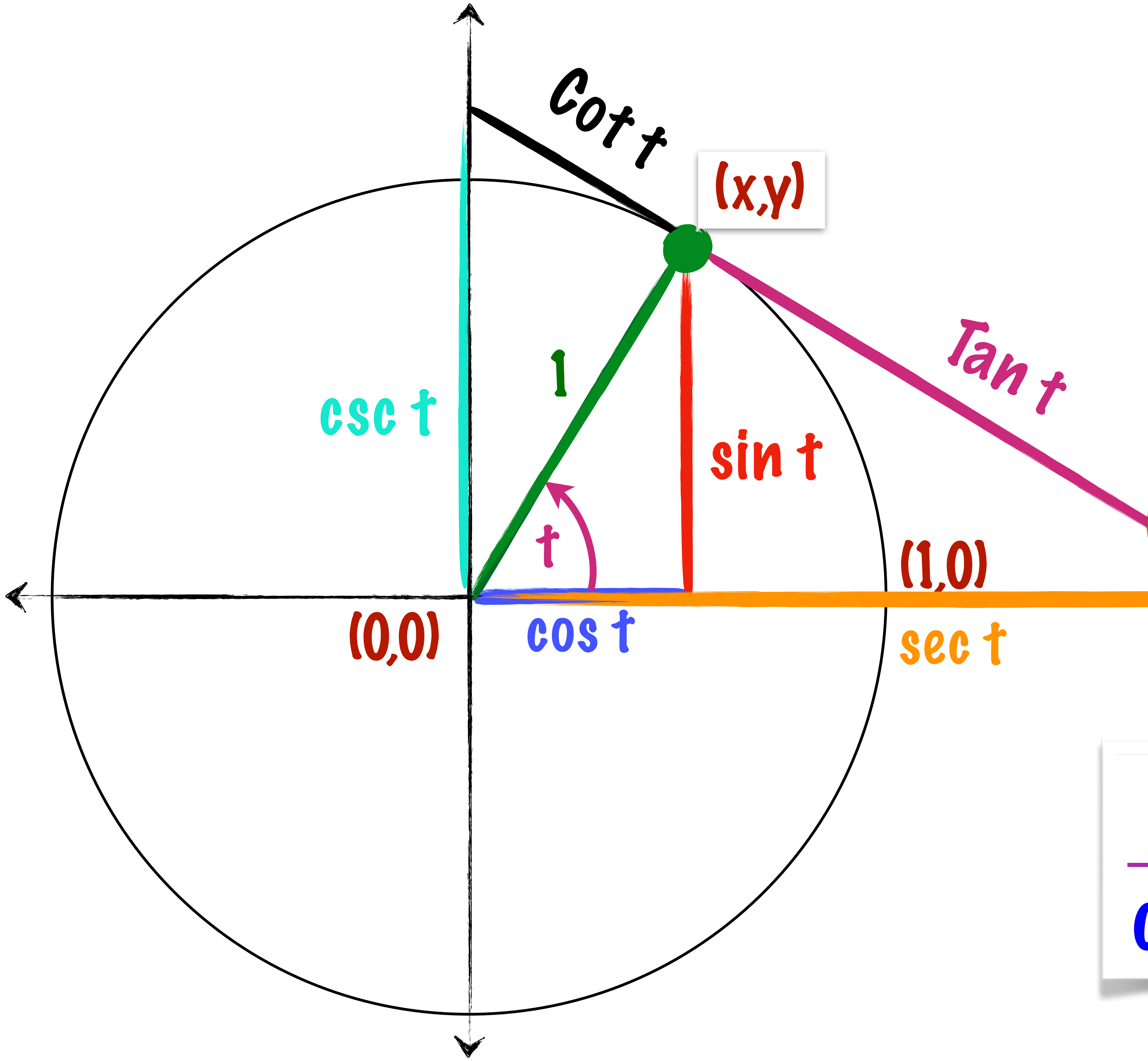




$$\frac{1}{\sin t} = \frac{\csc t}{1}$$

Trigonometric Ratios

Cosine and Secant



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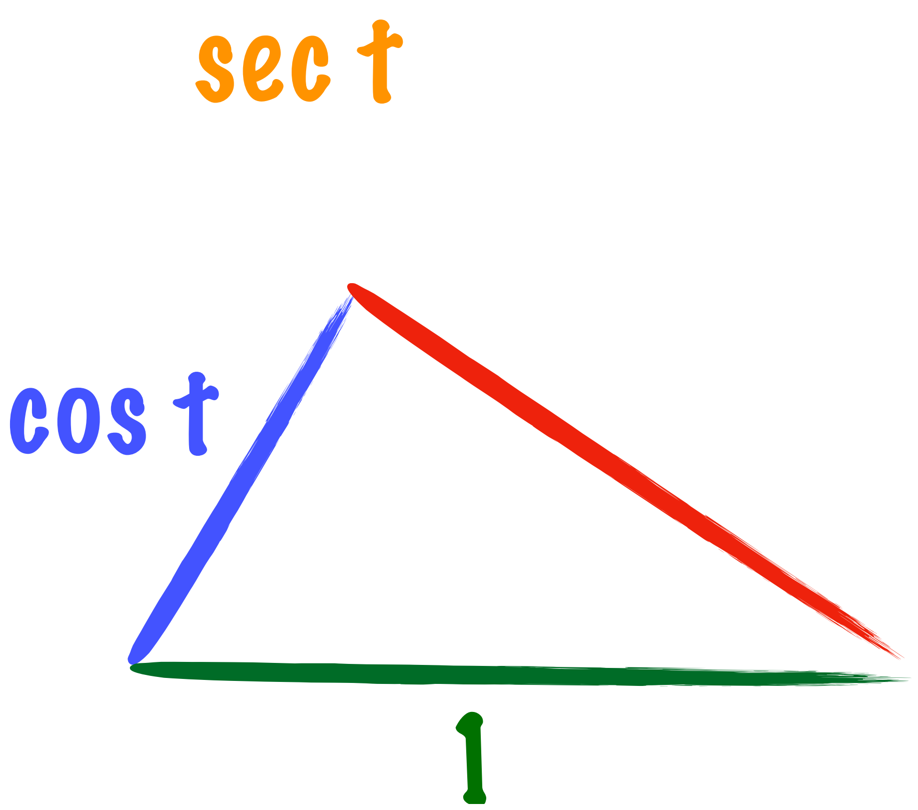


$$\frac{1}{\cos t} = \frac{\sec t}{1}$$

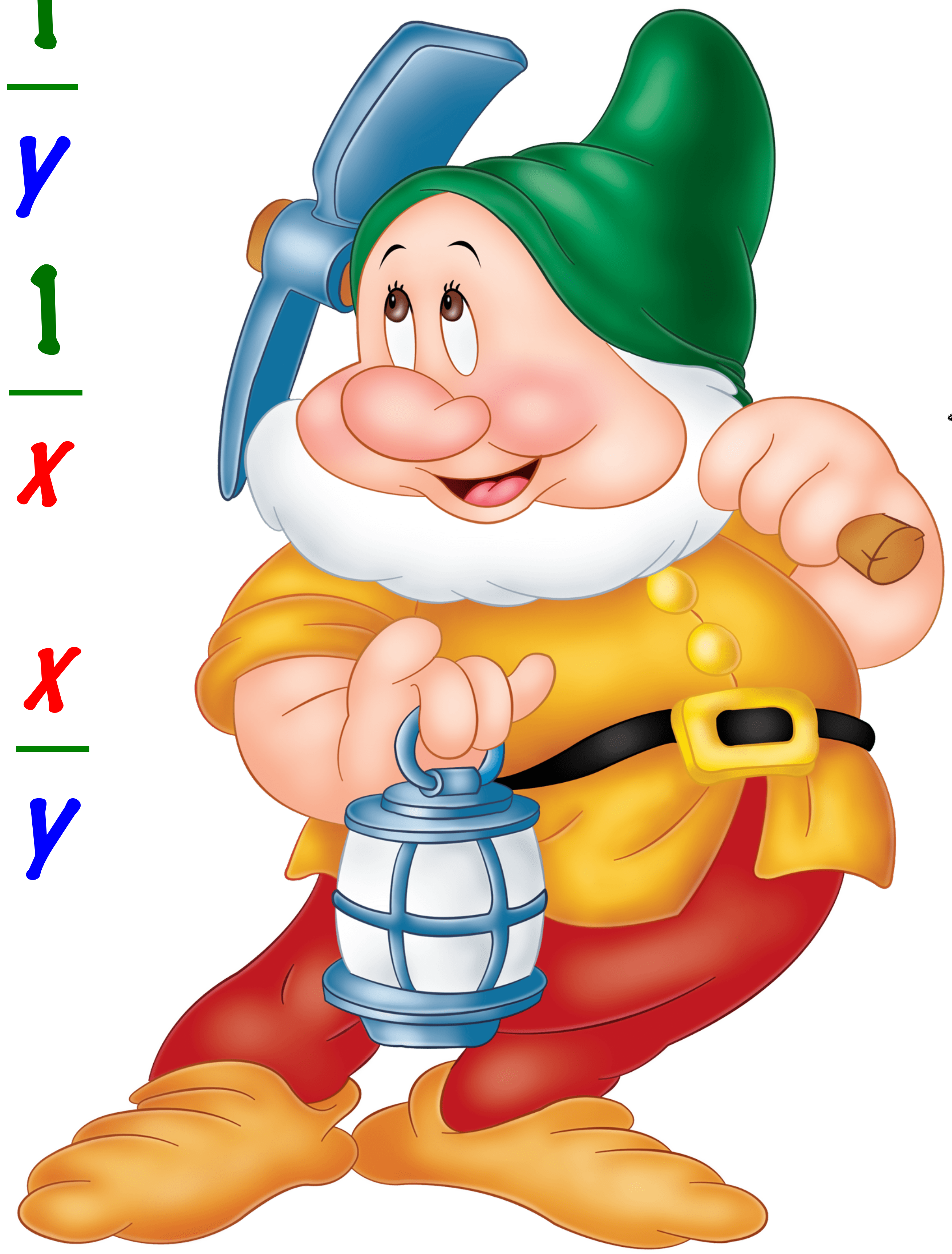
Definitions of the Trigonometric Functions in Terms of a Unit Circle

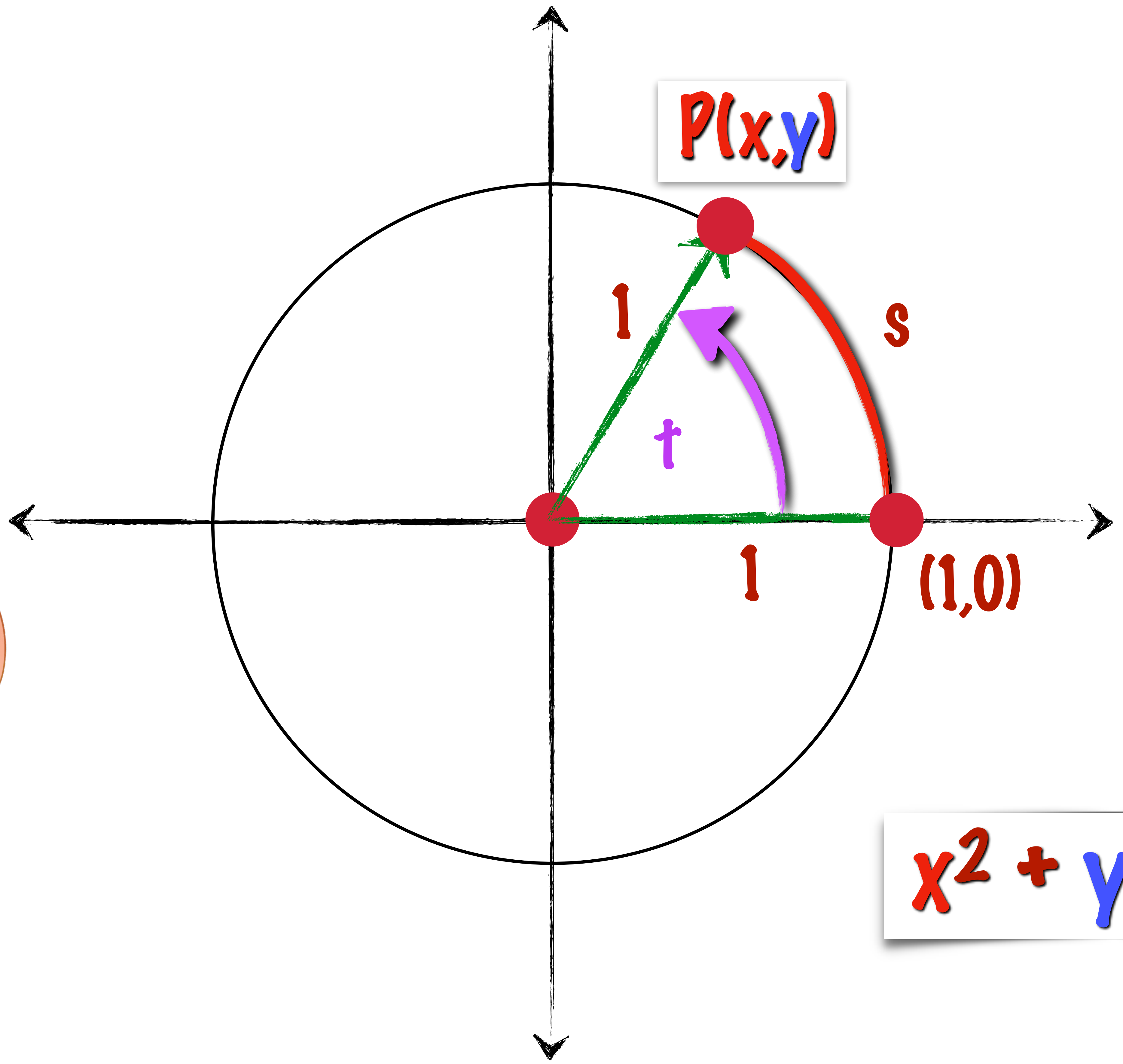
If t is a real number and **P(x, y)** is a point on the **unit circle** that corresponds to **t**, then we can define the trig ratios using the values of the coordinates **x** and **y**.

$$\sin t = y \qquad \csc t = \frac{1}{y}$$

$$\cos t = x \qquad \sec t = \frac{1}{x}$$

$$\tan t = \frac{y}{x} \qquad \cot t = \frac{x}{y}$$

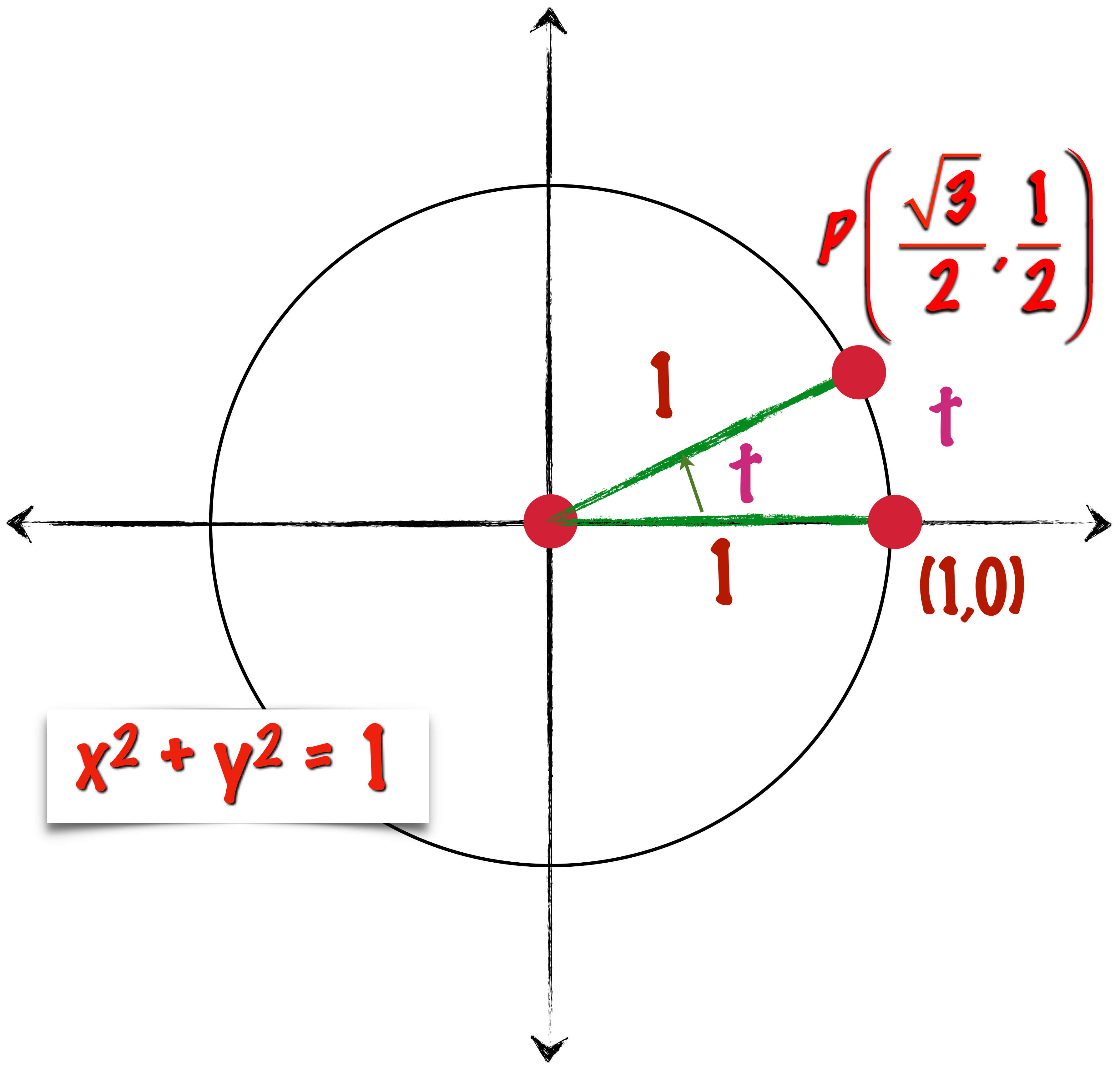




$$x^2 + y^2 = 1$$

Finding Values of the Trigonometric Functions

Use the figure to find the values of the trigonometric functions of t.



$$\sin t = \frac{1}{2}$$

$$\csc t = \frac{1}{\frac{1}{2}} = 2$$

$$\cos t = \frac{\sqrt{3}}{2}$$

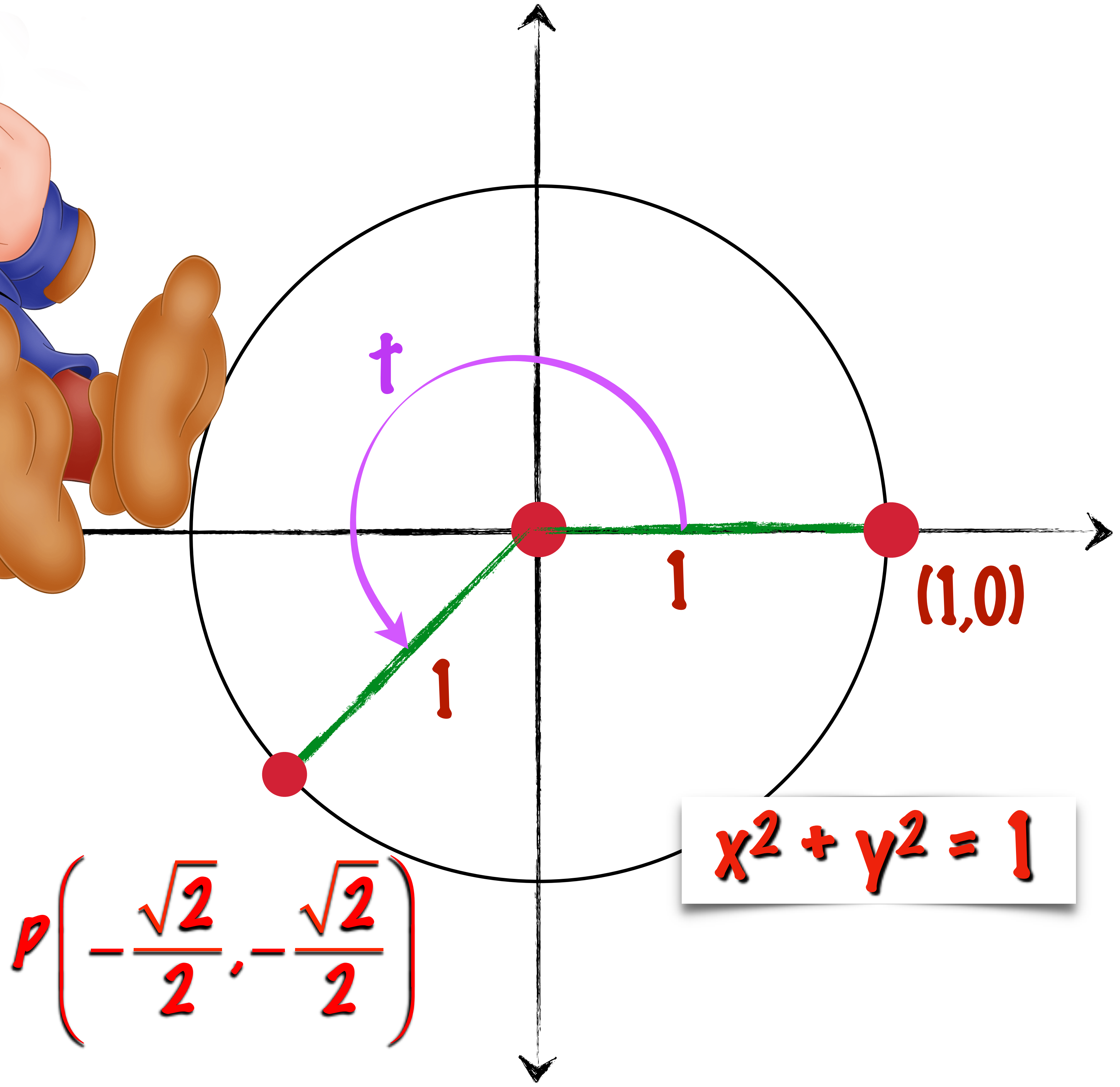
$$\sec t = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\tan t = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot t = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Finding Values of the Trigonometric Functions

Use the figure to find the values of the trigonometric functions at t.



$$\sin t = -\frac{\sqrt{2}}{2}$$

$$\csc t = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$$

$$\cos t = -\frac{\sqrt{2}}{2}$$

$$\sec t = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$$

$$\tan t = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

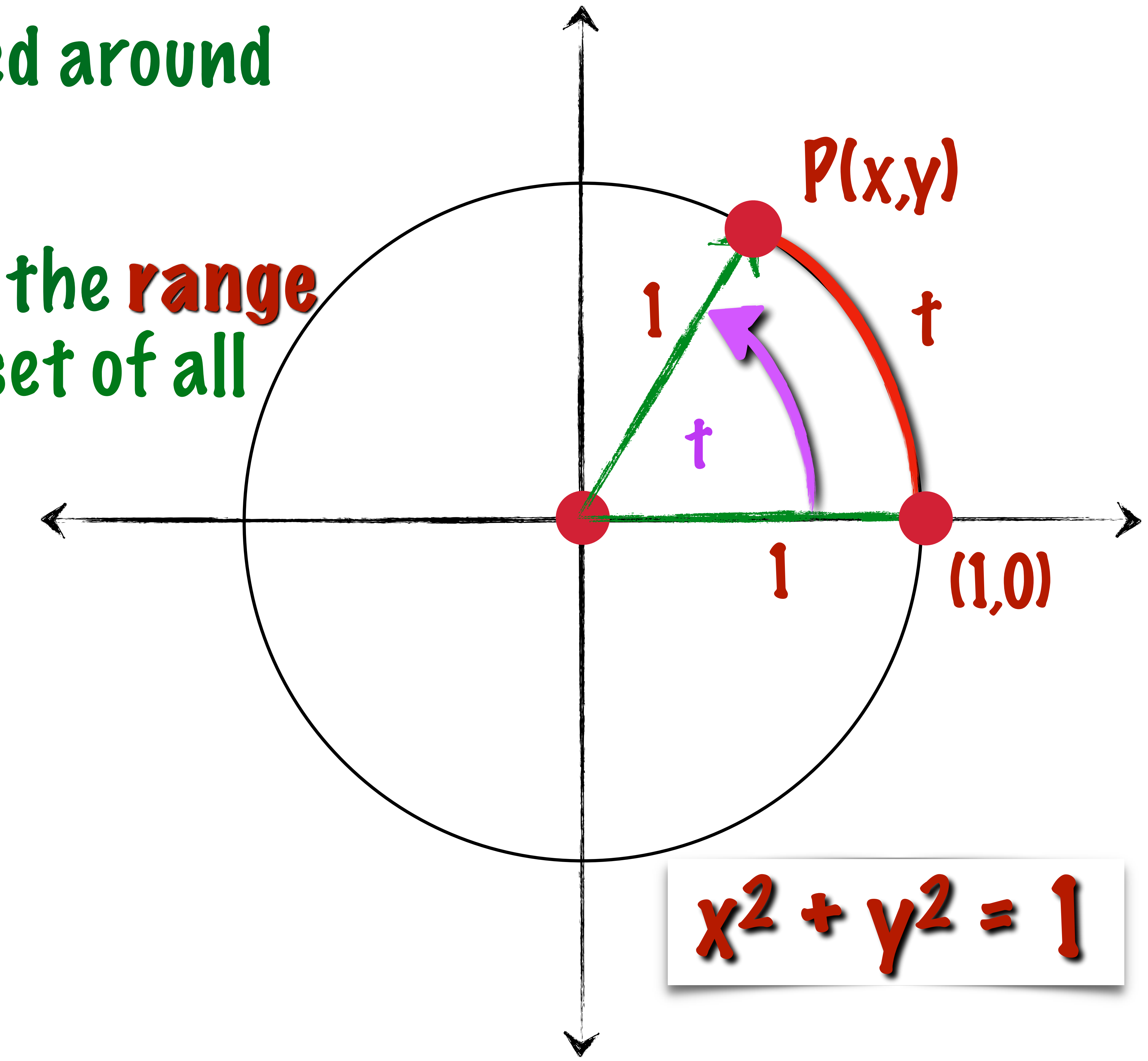
$$\cot t = 1$$

The Domain and Range of the Sine and Cosine Functions

Because **t** can have any value, the **domain** of the sine function and the cosine function is **(-∞, ∞)**, the set of all real numbers.

In other words, the circle is a number line for **t** wrapped around onto itself infinitely many times.

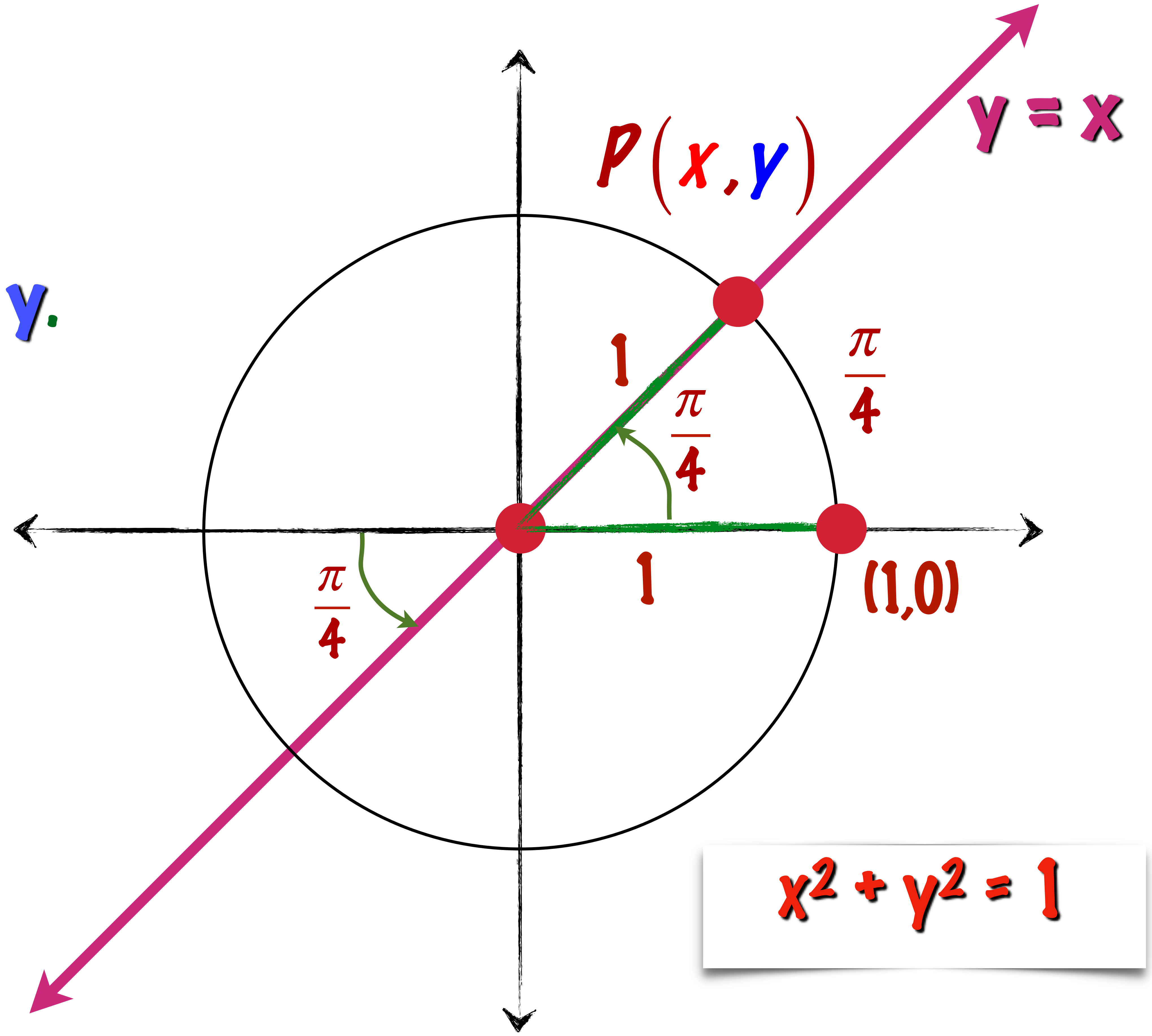
Since the values of x and y are restricted to the circle, the **range** of the sine function and cosine function is **[-1, 1]**, the set of all real numbers from -1 to 1, inclusive.



Exact Values of the Trigonometric Functions at $\frac{\pi}{4}$

Trigonometric functions at $t = \frac{\pi}{4}$ occur frequently. We can use the unit circle to find values of the trigonometric functions at $t = \frac{\pi}{4}$

The point P(x,y) lies on the line y = x. Thus, point P has equal x- and y-coordinates: x = y.



Exact Values of the Trigonometric Functions at $\frac{\pi}{4}$

- Given that $x^2 + y^2 = 1$, and $x = y$, then:

The point $P(x,y)$ lies on the line $y = x$. Thus, point P has equal x- and y-coordinates: $x = y$.

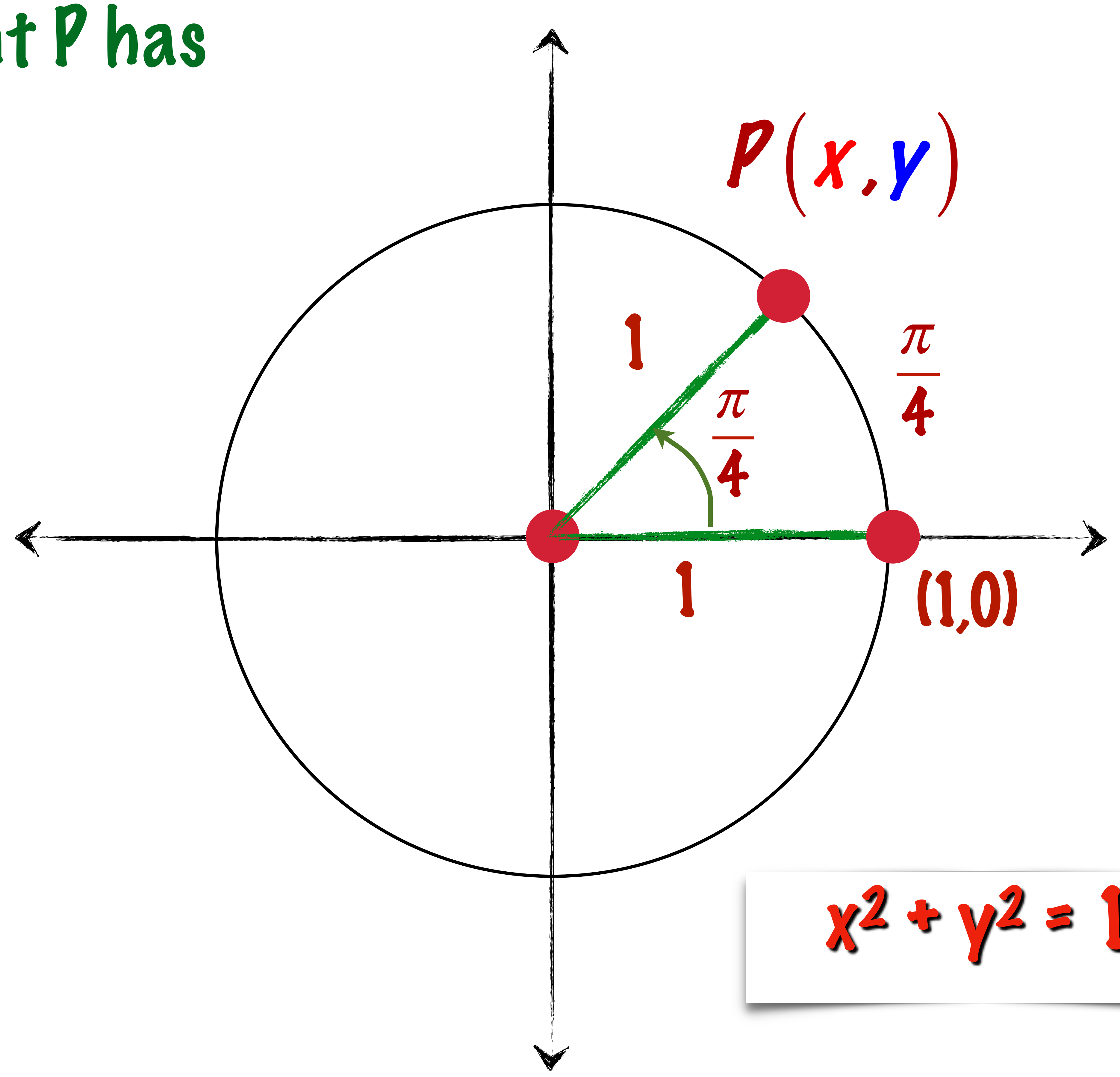
$\measuredangle\ x^2 + y^2 = x^2 + x^2 = 2x^2 = 1$

$$x^2 = \frac{1}{2}$$

$$x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$



$x^2 + y^2 = 1$

Exact Values of the Trigonometric Functions at $\frac{\pi}{4}$

- We can also use the triangle formed by the point P(x,y) and (0,0).

$\measuredangle \quad x^2 + y^2 = x^2 + x^2 = 2x^2 = 1$

$x^2 = \frac{1}{2}$

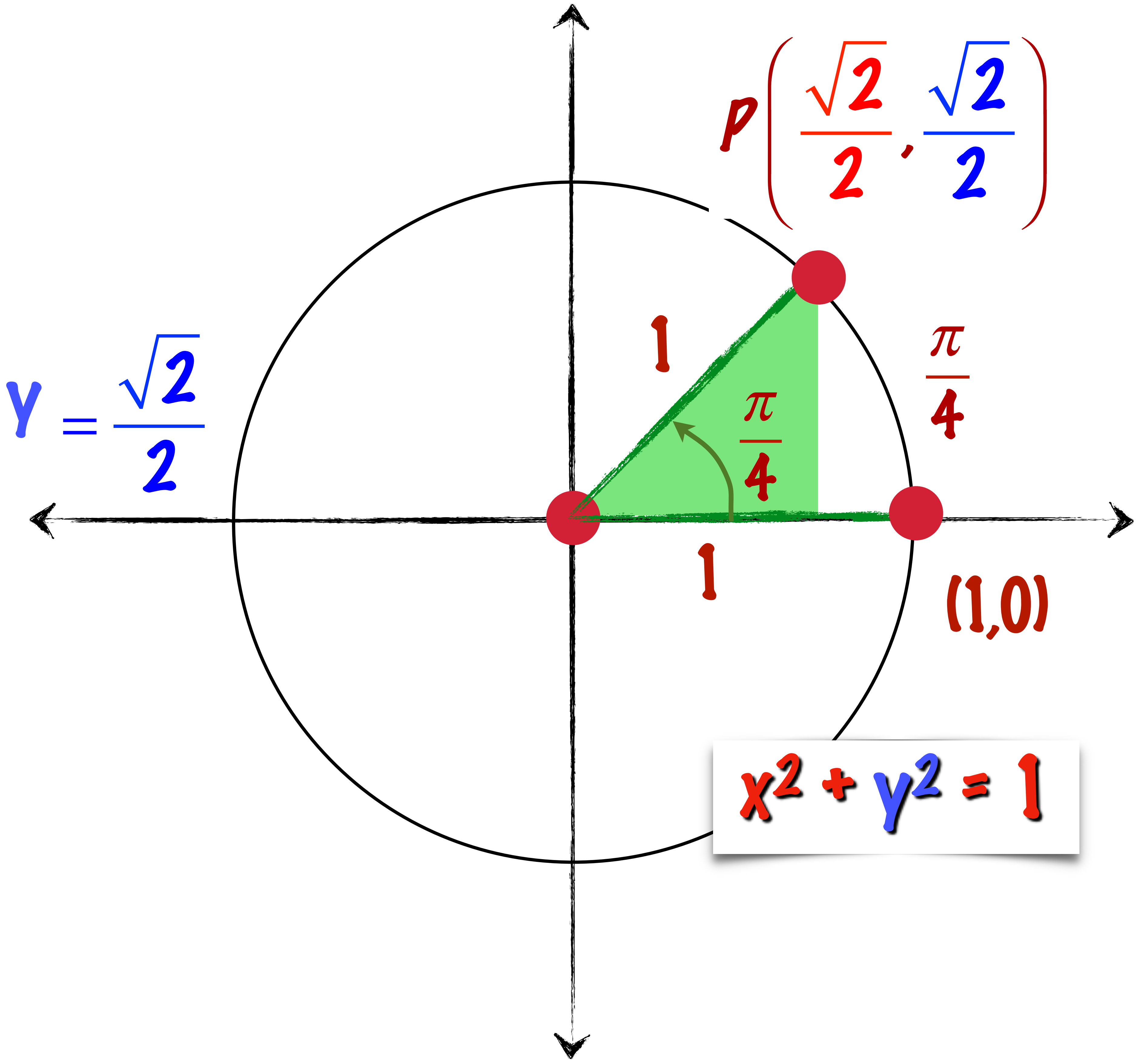
1

$x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$x = \frac{\sqrt{2}}{2}$

$y = \frac{\sqrt{2}}{2}$



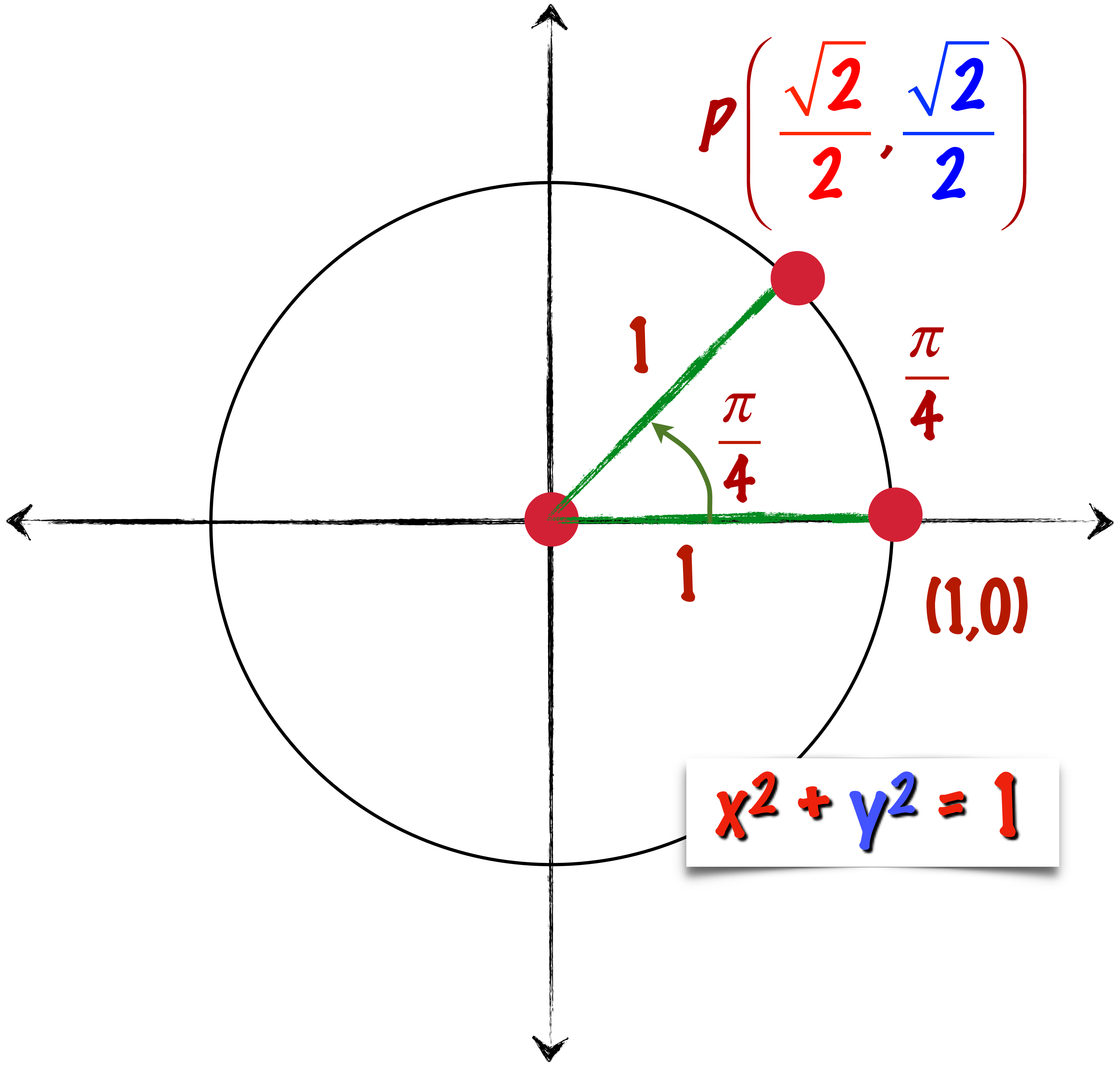
Exact Values of the Trigonometric Functions at $\frac{\pi}{4}$

We have used the unit circle to find the coordinates of point P(a, b) that correspond to $t = \frac{\pi}{4}$, now find the rest of the trig functions for $\frac{\pi}{4}$

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\csc\frac{\pi}{4} = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2} \qquad \sec\frac{\pi}{4} = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$$

$$\tan\frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 \qquad \cot\frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$



Trigonometric Functions at $\frac{\pi}{4}$

Trigonometric Ratios for θ = π/4

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\csc\frac{\pi}{4} = \sqrt{2} \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \cot\frac{\pi}{4} = 1$$

TI-84

- Parametric Equations are relationships between two variables defined through a third variable called the parameter. That third variable is most often time so your calculator uses t to represent the parameter, which is nice since we have been using t for our angle measure in radians.

- We will discuss parametric equations later, but to graph a circle on the calculator we need to use parametric equations.

- Set your calculator to radian and parameter modes.

MODE ∨ **RADIAN** DEGREE ∨ FUNCTION **PARAMETRIC** POLAR SEQ

Graphing Circle on TI-84

Now enter the equations to define our circle. X and y are defined as functions of a central angle, t (parameter), we will enter those functions.

Y=

X_{1T}= COS X,T,θ,n)

Y_{1T}= SIN X,T,θ,n)

To make certain our circle looks like a circle, to graph ZOOM 5

To see points on the circle, TRACE

To make the unit circle, 2nd WINDOW TBLSET

TblStart= 0

ΔTbl= 2nd π ^ ÷ 12

Indpnt: Auto Ask

Depend: Auto Ask

To see the values, 2nd GRAPH TABLE

Even and Odd Trigonometric Functions

- Remember that a function is **even** if f(-x) = f(x), and a function is **odd** if f(-x) = -f(x).

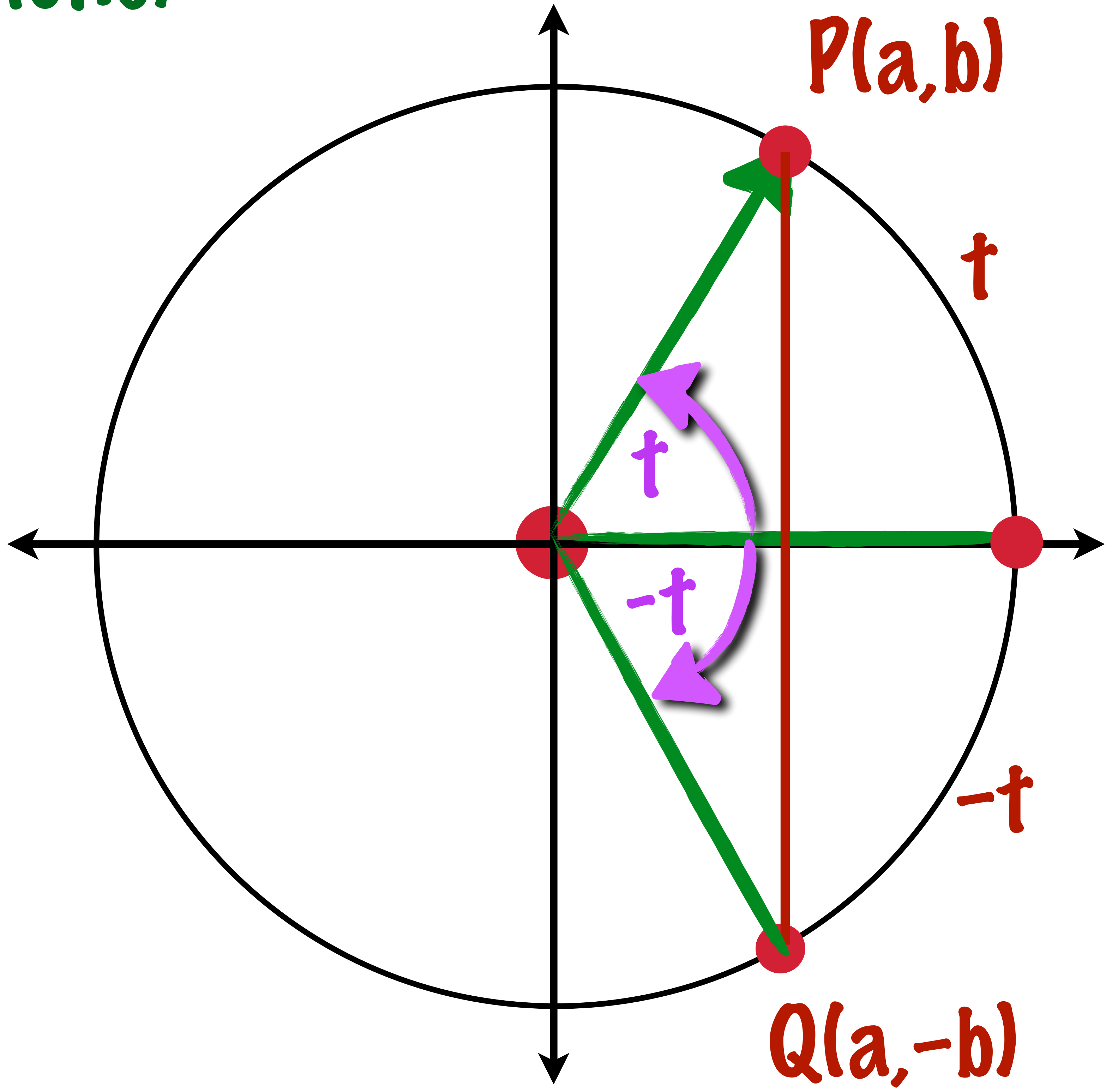
  - The cosine and secant functions are **even** functions.

$$\cos t = a \qquad \cos(-t) = a$$

$$\cos t = \cos(-t) = a$$

$$\sec t = \frac{1}{a} \qquad \sec(-t) = \frac{1}{a}$$

$$\sec t = \sec(-t) = \frac{1}{a}$$



Even and Odd Trigonometric Functions

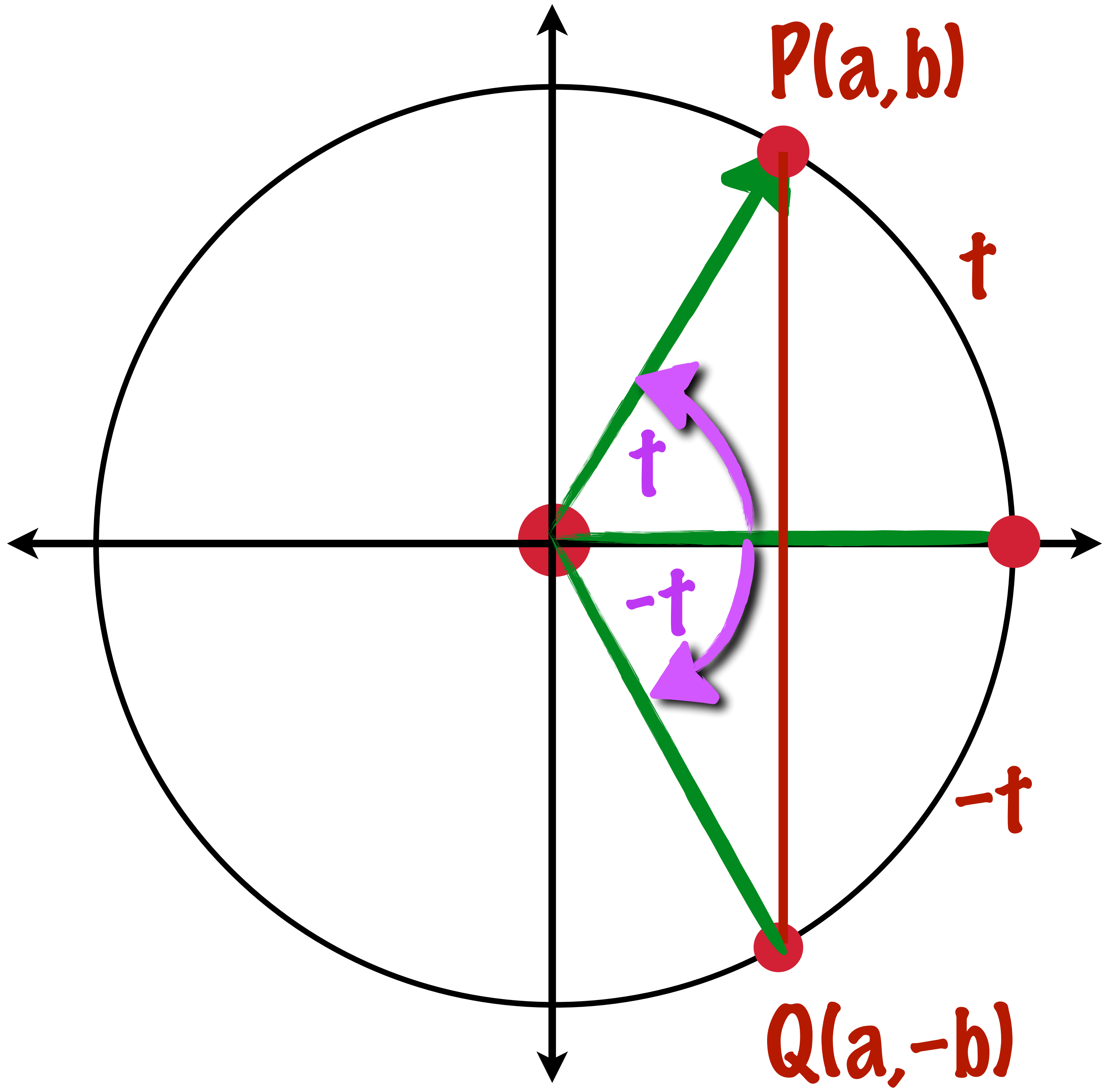
- The sine, cosecant, tangent, and cotangent functions are odd.

$\sin t = b \qquad \sin(-t) = -b \qquad \sin(-t) = -\sin t = -b$

$\csc t = \frac{1}{b} \qquad \csc(-t) = \frac{1}{-b} \qquad \csc(-t) = -\csc t = -\frac{1}{b}$

$\tan t = \frac{b}{a} \qquad \tan(-t) = \frac{-b}{a} \qquad \tan(-t) = -\tan t = -\frac{b}{a}$

$\cot t = \frac{a}{-b} \qquad \cot(-t) = \frac{a}{-b} \qquad \cot(-t) = -\cot t = -\frac{a}{b}$



Using Even and Odd Functions to Find Values of Trigonometric Functions

Find the value of each trigonometric function:

$$\sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\sec\left(-\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$$

Fundamental Trigonometric Identities

Reciprocal Identities

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t}$$

Quotient Identities

$$\tan t = \frac{\sin t}{\cos t}$$

$$\cot t = \frac{\cos t}{\sin t}$$

Pythagorean Identities

$$\sin^2 t + \cos^2 t = 1$$

$$\cot^2 t + 1 = \csc^2 t$$

$$1 + \tan^2 t = \sec^2 t$$

Using Quotient and Reciprocal Identities

Given $\sin t = \frac{2}{3}$ and $\cos t = \frac{\sqrt{5}}{3}$ find the value of each of the four remaining trigonometric functions.

$$\csc t = \frac{1}{\sin t} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{2}{3}}{\frac{\sqrt{5}}{3}} = \frac{2}{\sqrt{5}}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{\sqrt{5}}{3}} = \frac{3}{\sqrt{5}}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{\frac{2}{\sqrt{5}}} = \frac{\sqrt{5}}{2}$$

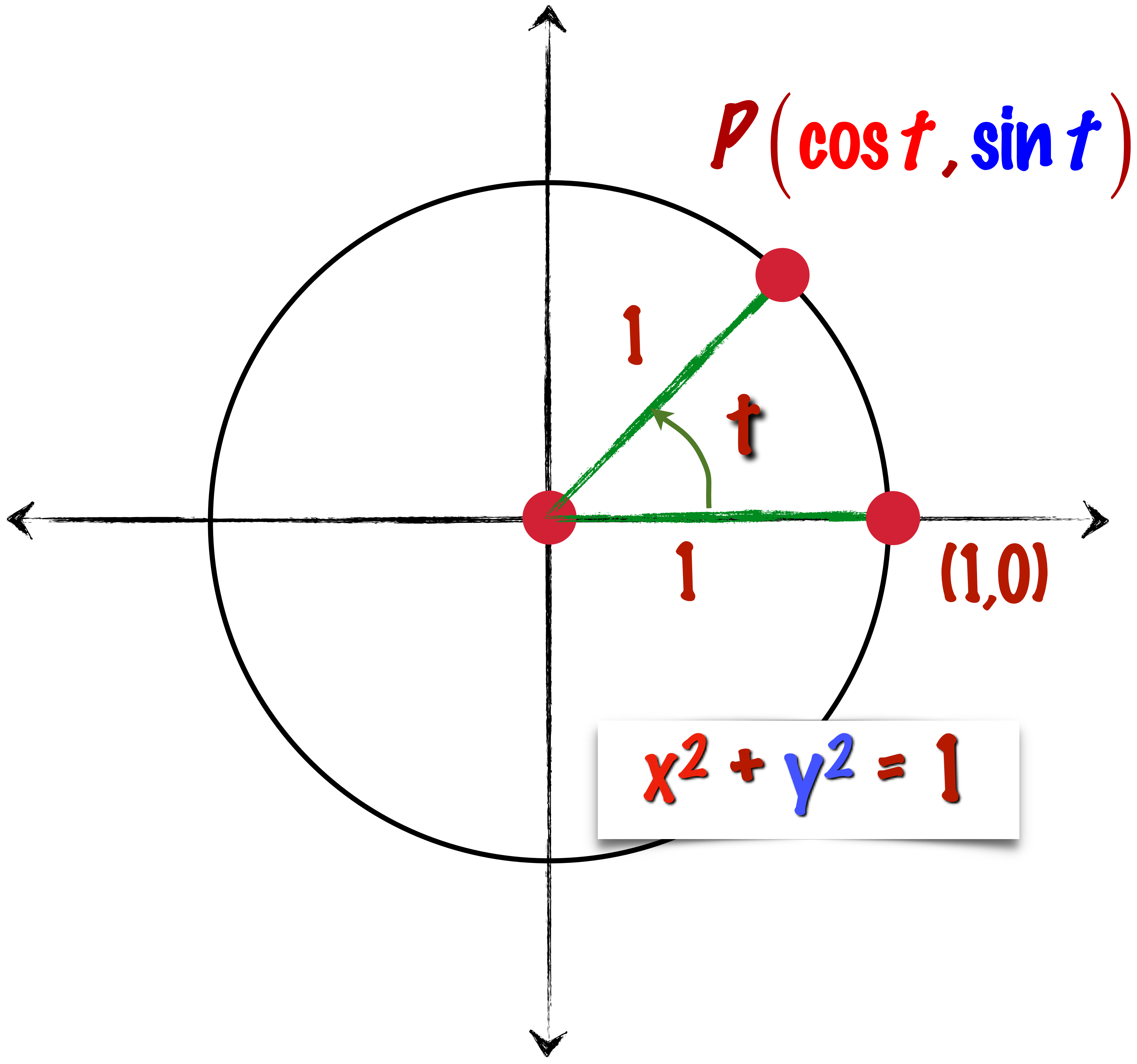
The Pythagorean Identities

Here is the biggie

$$x^2 + y^2 = 1$$

$\sin t = y \quad \cos t = x$

$$\sin^2 t + \cos^2 t = 1$$



The Pythagorean Identities

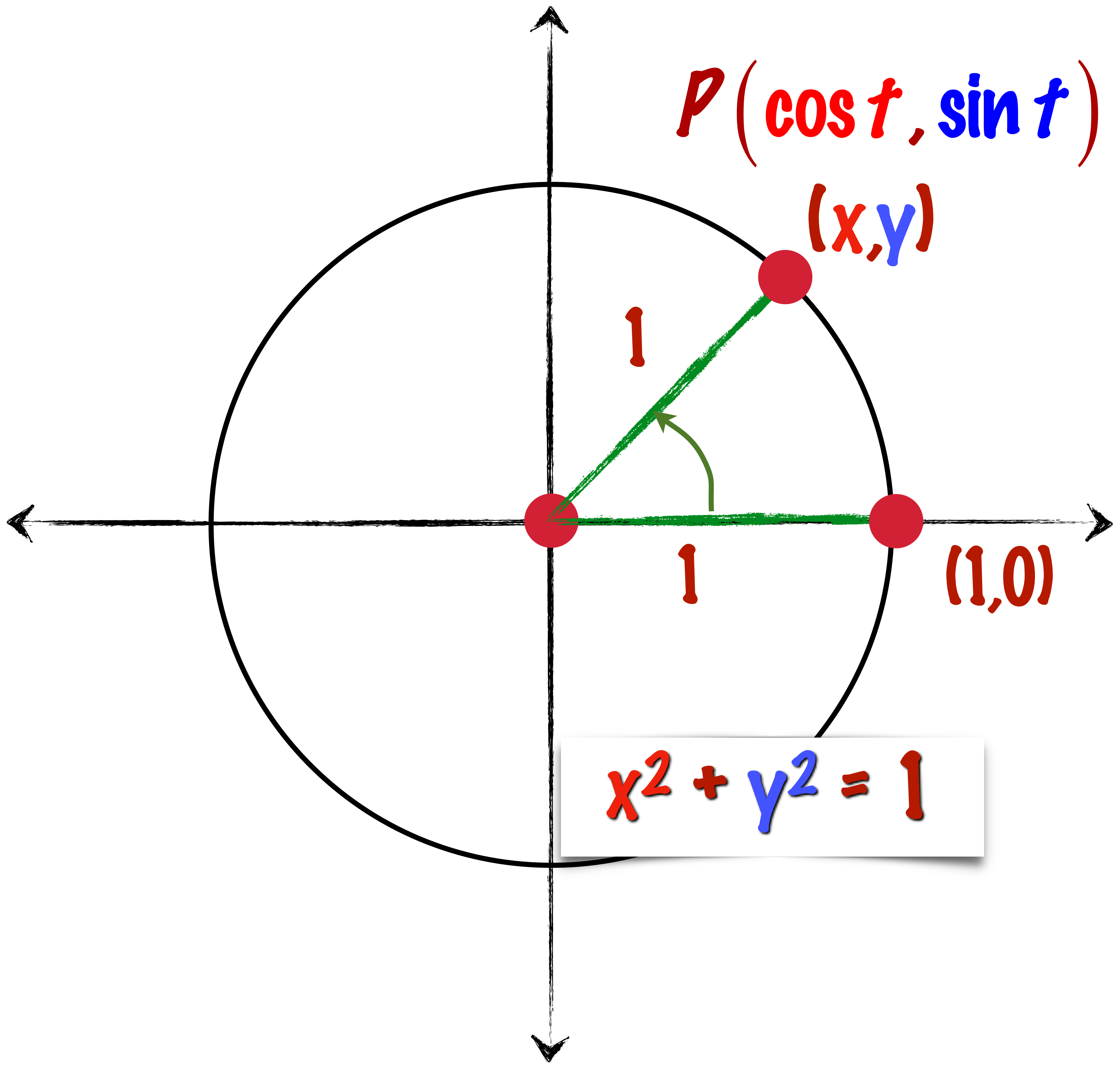
From $x^2 + y^2 = 1$ we can also derive two additional identities.

$$\frac{x^2}{y^2} + 1 = \frac{1}{y^2} \qquad \left(\frac{x}{y}\right)^2 + 1 = \left(\frac{1}{y}\right)^2$$

$$\cot^2 t + 1 = \csc^2 t$$

$$1 + \frac{y^2}{x^2} = \frac{1}{x^2} \qquad 1 + \left(\frac{y}{x}\right)^2 + 1 = \left(\frac{1}{x}\right)^2$$

$$1 + \tan^2 t = \sec^2 t$$



Example: Using a Pythagorean Identity

Given that $\sin t = \frac{1}{2}$ and $0 \le t < \frac{\pi}{2}$ find the value of cos t.

- Why did we restrict t?

$\sin^2 t + \cos^2 t = 1$

$$\left(\frac{1}{2}\right)^2 + \cos^2 t = 1$$

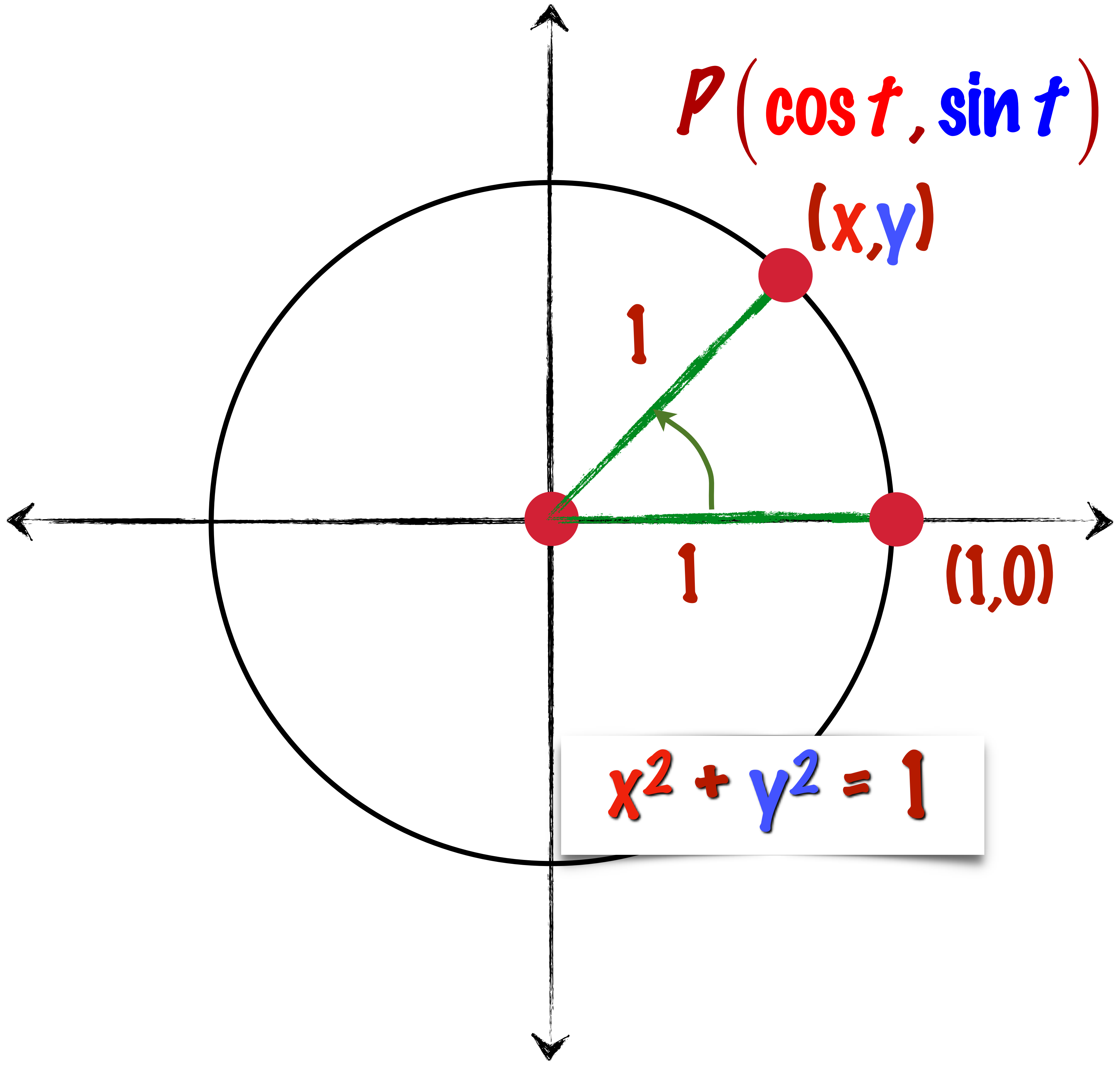
$$\cos^2 t = 1 - \frac{1}{4}$$

$$\cos^2 t = \frac{3}{4}$$

$$\cos t = \pm\frac{\sqrt{3}}{2}$$

$$0 \le t < \frac{\pi}{2}$$

$$\cos t = +\frac{\sqrt{3}}{2}$$



Definition of a Periodic Function

- A function f is a **periodic function** if there exists a positive number p such that **f(t + p) = f(t)** for all t in the domain of f. The smallest possible value of p is the **period** of f.
  - Note: This suggests f(t + p) = f(t) and f(t + p + p) = f(t + p) = f(t), and this continues without end. Notice also that p need not be a positive number.
    - Sin(t + 2π) = sin t, a periodic function with period 2π.
    - Cos(t + 2π) = cos t, a periodic function with period 2π.
    - Sec(t + 2π) = sec t, a periodic function with period 2π.
    - Csc(t + 2π) = csc t, a periodic function with period 2π.
- For example: $\sin\frac{\pi}{3} = \sin\frac{7\pi}{3} = \sin\frac{13\pi}{3} = \sin\frac{19\pi}{3} \ldots$

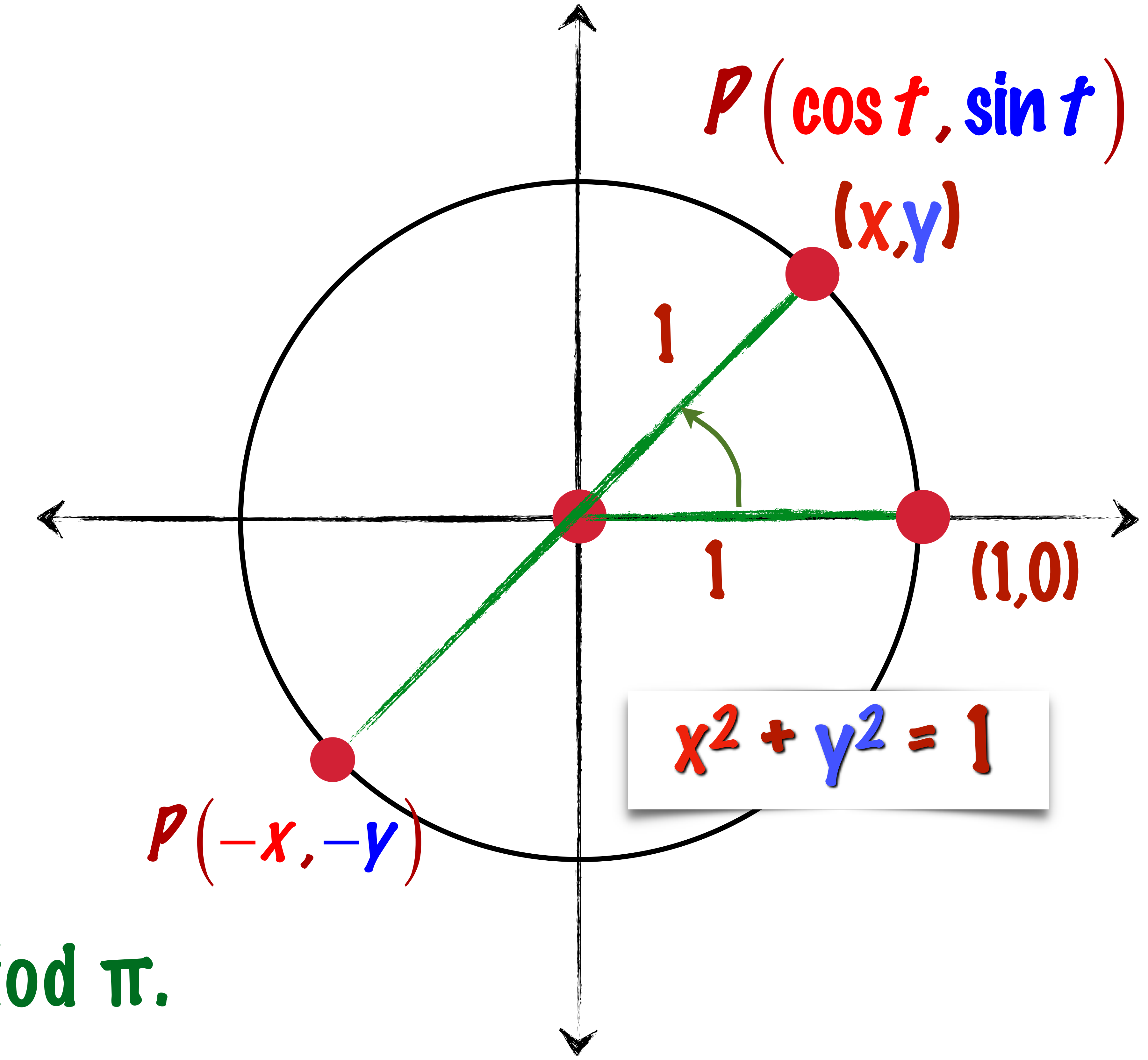
Definition of a Periodic Function

- Tangent and Cotangent are also periodic functions with period π (not 2π).

$$\tan t = \frac{y}{x} \qquad \tan(t+\pi) = \frac{-y}{-x}$$

$$\frac{y}{x} = \frac{-y}{-x}$$

$$\tan t = \tan(t+\pi)$$



- Similarly cotangent is also periodic functions with period π.

- For example: $\tan\frac{\pi}{3} = \tan\frac{4\pi}{3} = \tan\frac{7\pi}{3} = \tan\frac{10\pi}{3} \ldots$

Using Periodic Properties

Find the value of each trigonometric function:

$$\cot\left(\frac{5\pi}{4}\right) = \cot\left(\frac{\pi}{4}+\frac{4\pi}{4}\right) = \cot\left(\frac{\pi}{4}+\pi\right) = \cot\frac{\pi}{4} = 1$$

$$\cos\left(-\frac{9\pi}{4}\right) = \cos\left(-\frac{\pi}{4}+\frac{8\pi}{4}\right) = \cos\left(-\frac{\pi}{4}+2\pi\right)$$

$$= \cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

Repetitive Behavior of the Sine, Cosine, and Tangent Functions

Periodic functions repeat indefinitely, thus for any integer n and real number t,

$$\sin(t + 2\pi n) = \sin t \qquad \csc(t + 2\pi n) = \csc t$$

$$\cos(t + 2\pi n) = \cos t \qquad \sec(t + 2\pi n) = \sec t$$

$$\tan(t + \pi n) = \tan t \qquad \cot(t + \pi n) = \cot t$$

45°, $\frac{\pi}{4}$ radians

- We can also use the triangle formed by the point P(x,y) and (0,0).

$\measuredangle \; x^2 + y^2 = x^2 + x^2 = 2x^2 = 1$

$$x^2 = \frac{1}{2}$$

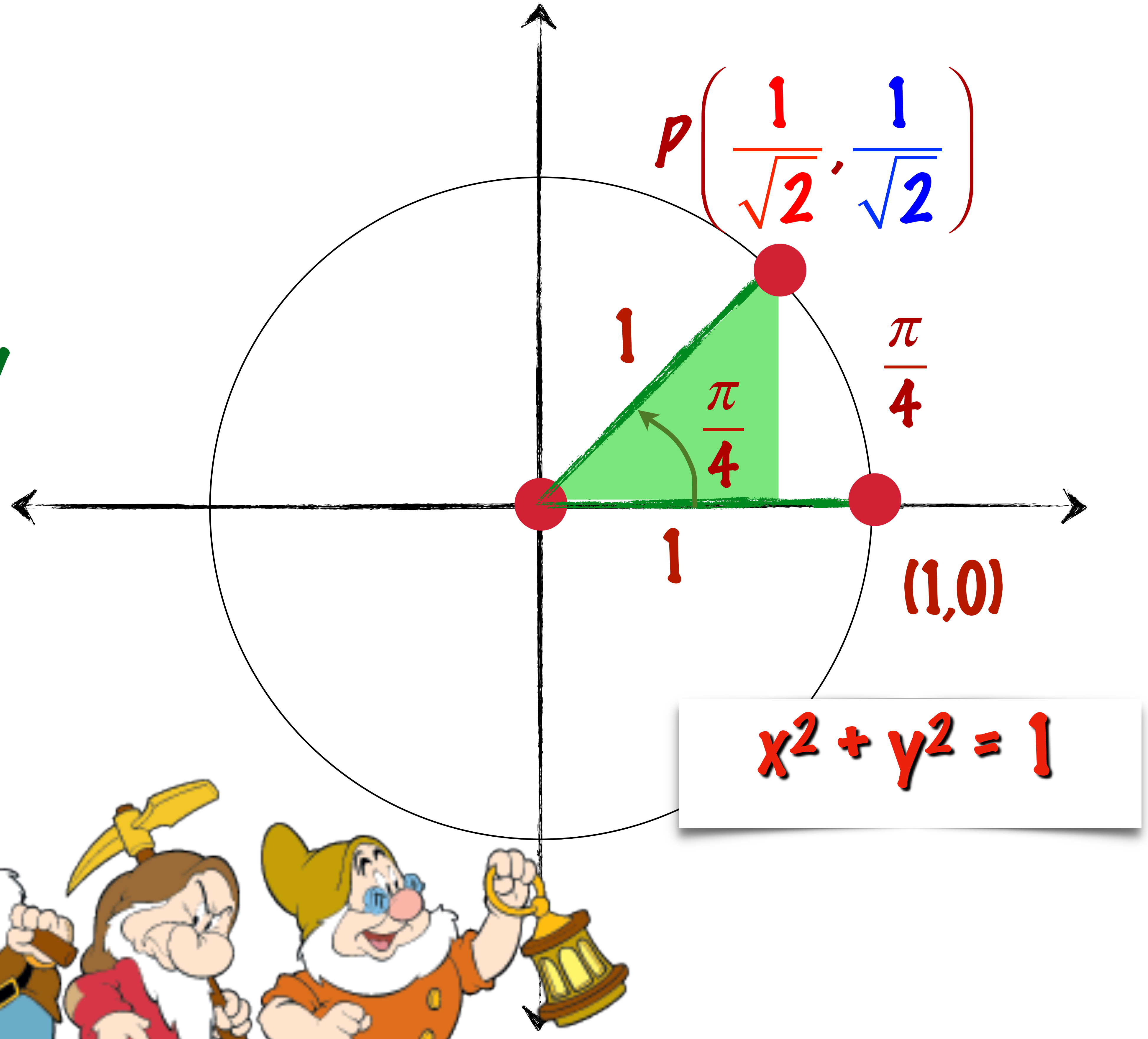
$$x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

1

y

x



30°, $\frac{\pi}{6}$ radians

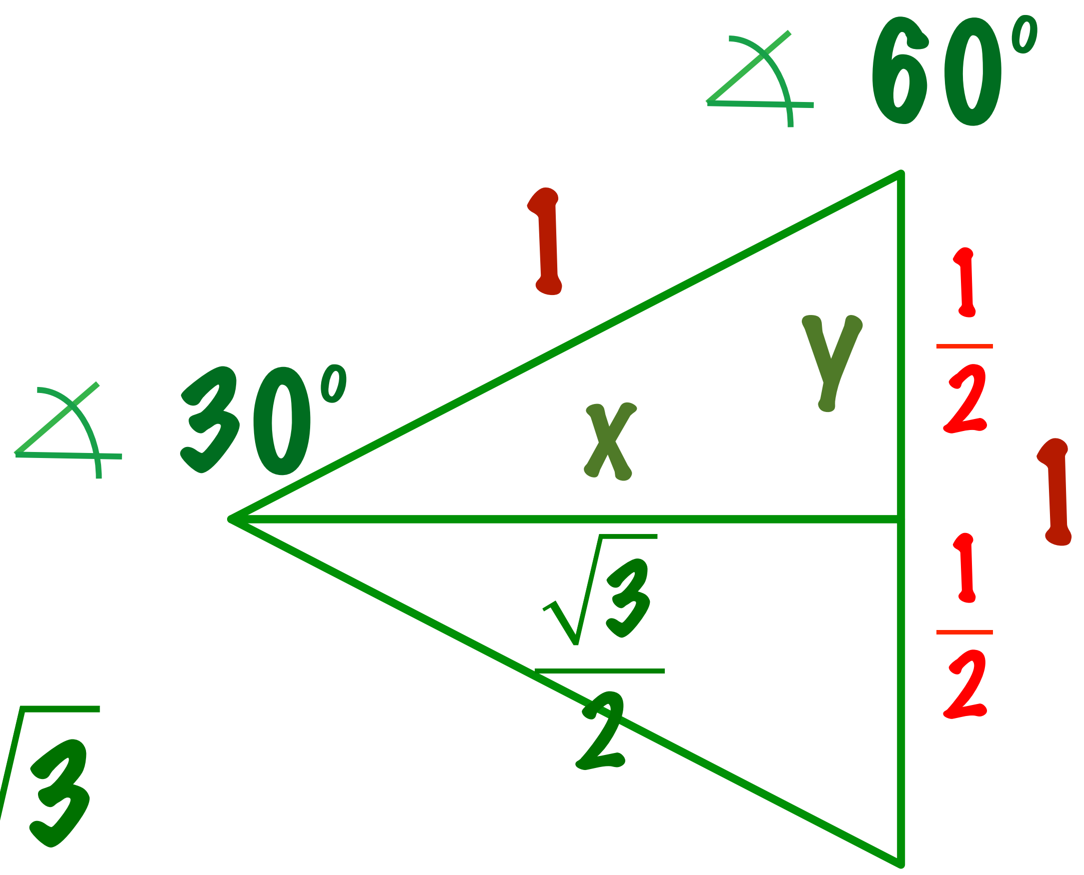
We can also find the 30-60-90° triangle formed by the point P(x,y) and (0,0).

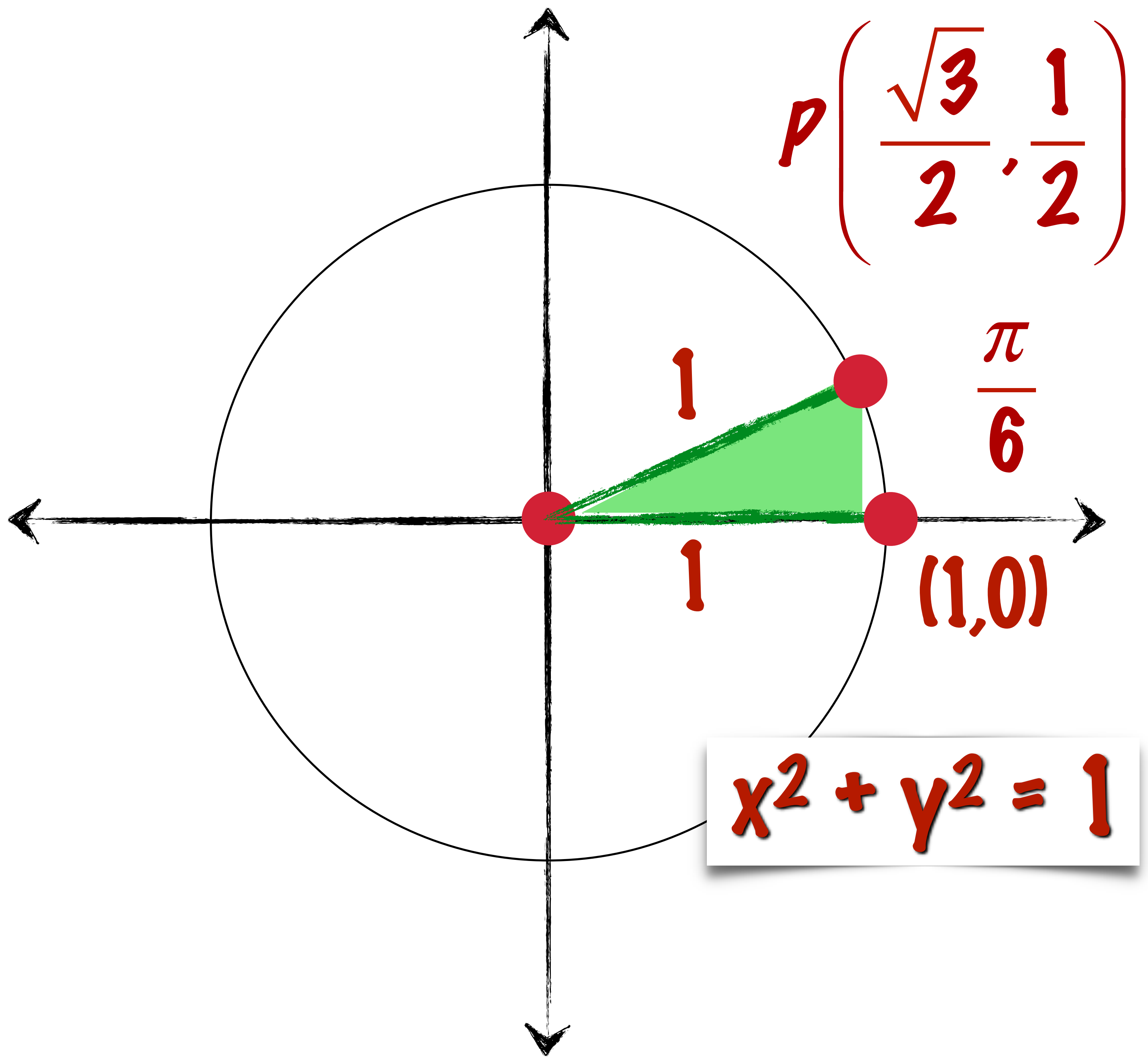
$x^2 + y^2 = 1$

$$x^2 + \left(\frac{1}{2}\right)^2 = 1$$

$$x^2 = \frac{3}{4}$$

$$x = \pm\frac{\sqrt{3}}{2} \qquad x = \frac{\sqrt{3}}{2}$$





$60°, \frac{\pi}{3}$ radians

- We can also find the 30-60-90° triangle formed by the point P(x,y) and (0,0).

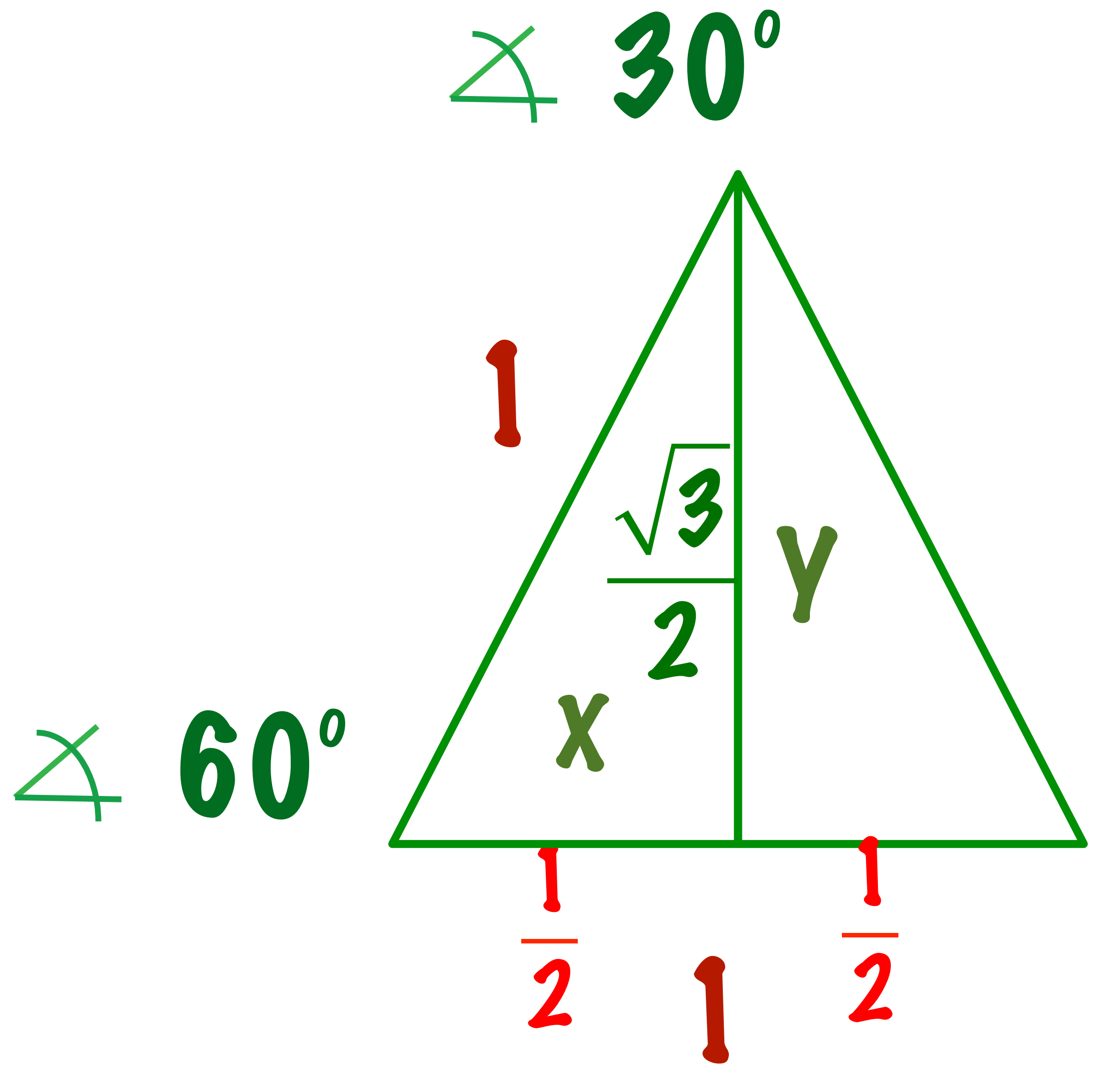
$x^2 + y^2 = 1$

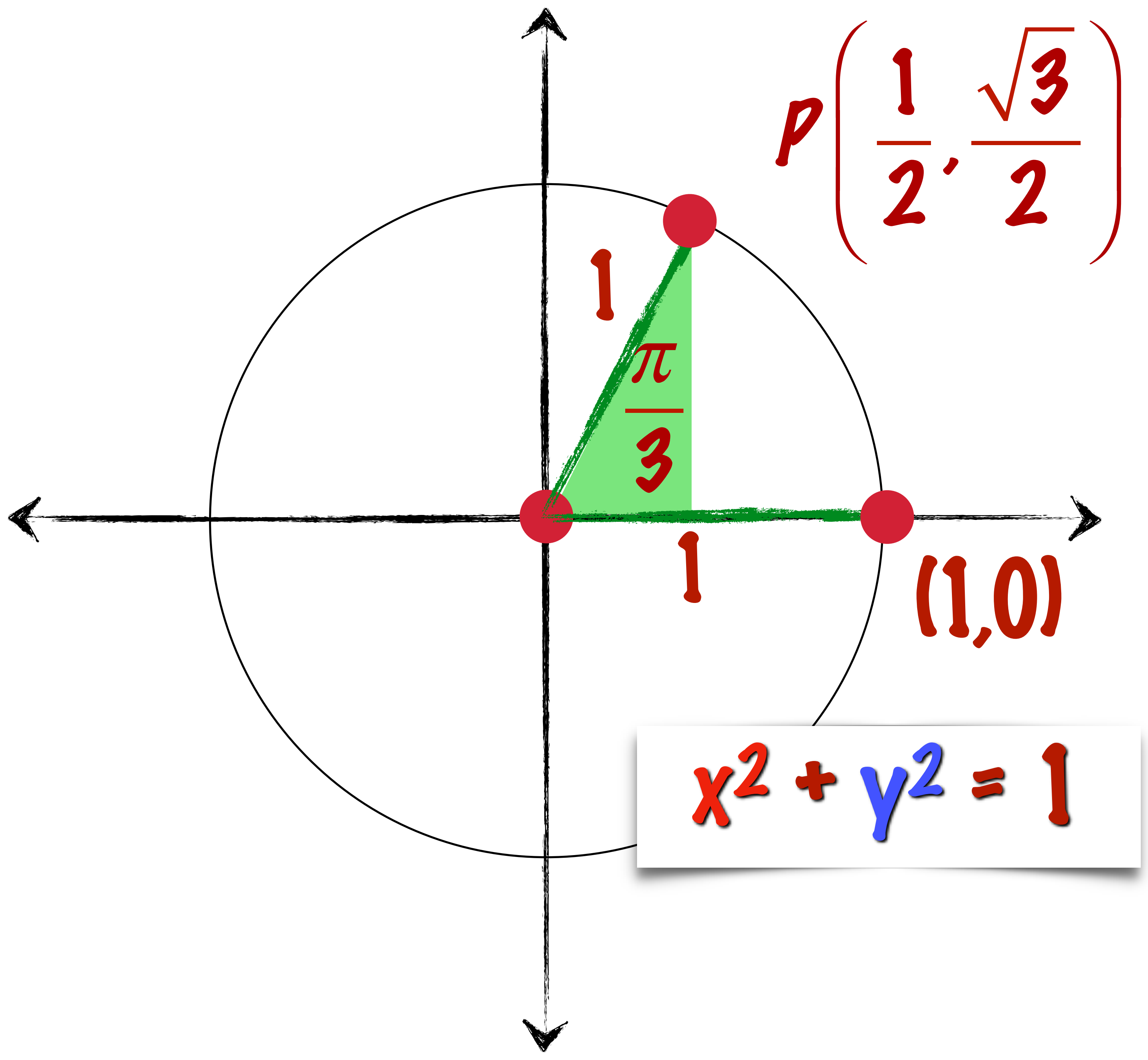
$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

$$y^2 = \frac{3}{4}$$

$$y = \pm\frac{\sqrt{3}}{2}$$

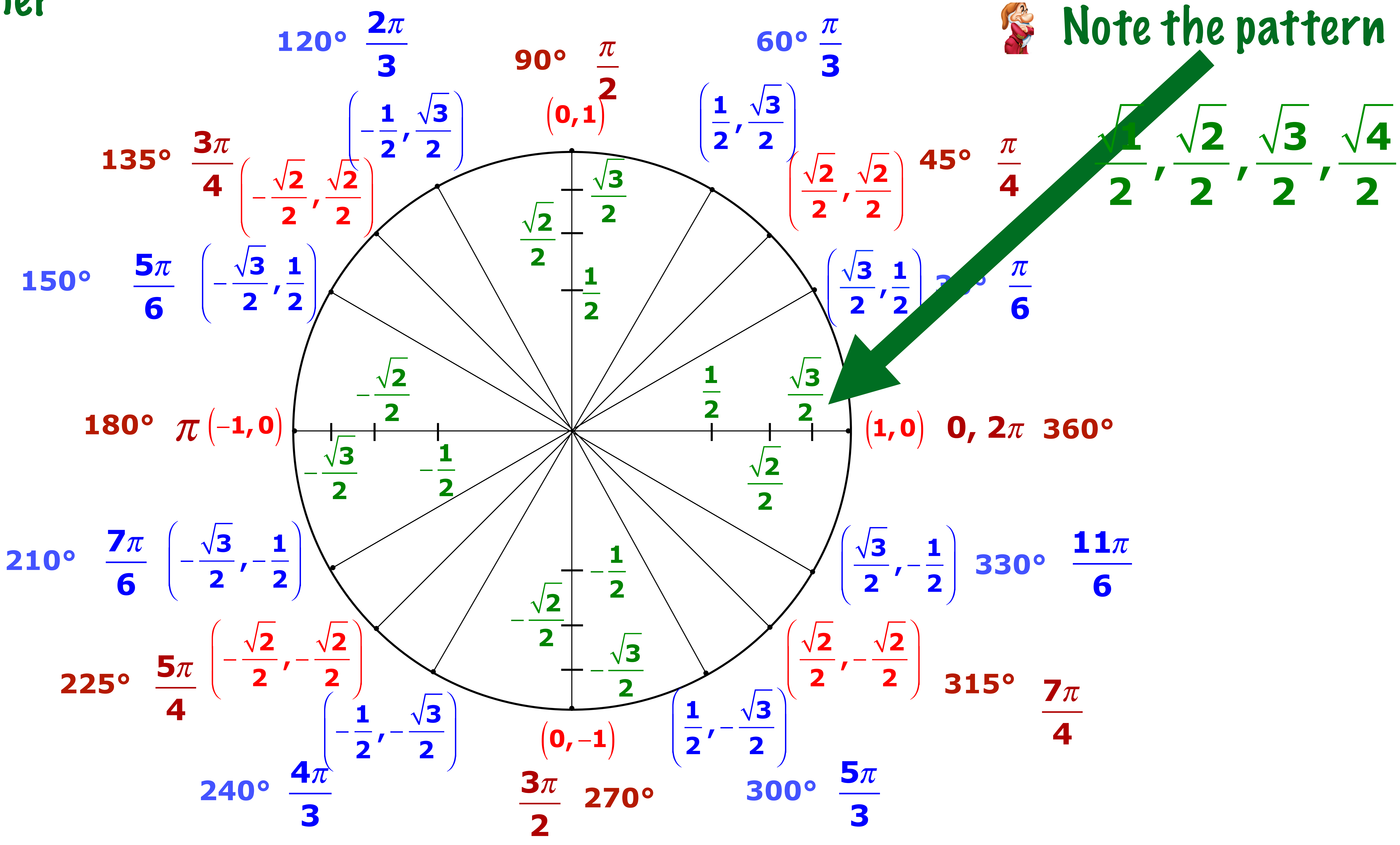
$$y = \frac{\sqrt{3}}{2}$$





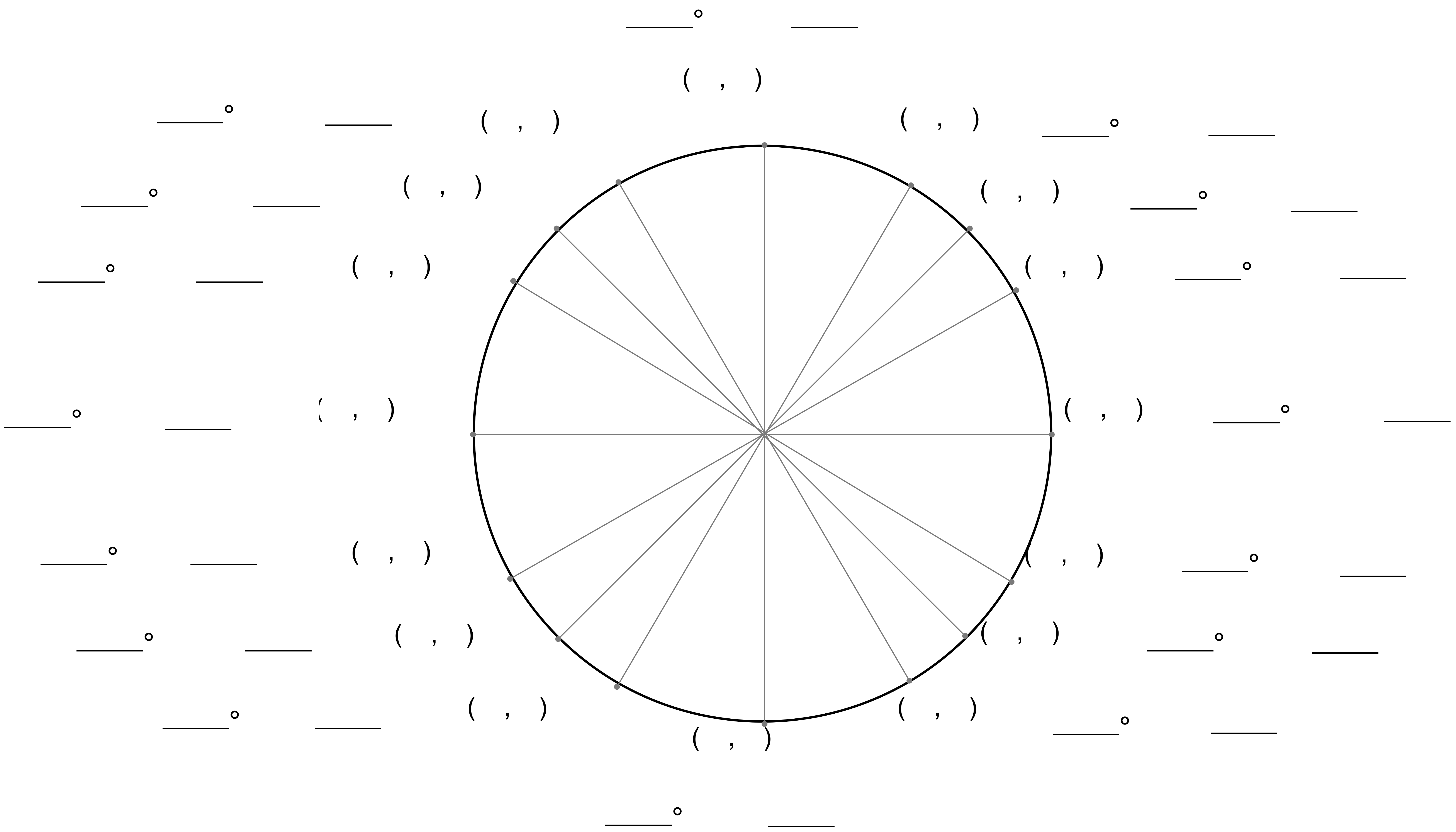
The Unit Circle

- Put it all together
- Note the pattern



Fill in the Blanks

- Fill in the blank unit circle



Using a Calculator to Evaluate Trigonometric Functions

To evaluate trigonometric functions, we will use the keys on a calculator that are marked SIN, COS, and TAN. **Be sure to set the mode to degrees or radians**, depending on the function that you are evaluating. You may consult the manual for your calculator for specific directions for evaluating trigonometric functions.

Get in the habit of closing the parentheses. COS (π + 6) is very different from COS (π) + 6

- On the TI-84 the MODE button sets degrees and radians.
- The SIN COS and TAN buttons are obvious.
- There are no CSC, SEC, and COT buttons.
  - The calculator figures you can handle those on your own.

Evaluating Trigonometric Functions with a Calculator

Use a calculator to find the value to four decimal places:

$\sin\left(\frac{\pi}{4}\right) \approx .7071$

$\cos 47^\circ \approx .6820$

$\csc 1.5 \approx 1.0025$

$\sec 138^\circ \approx -1.3456$

$\tan(3) \approx -0.1425$

$\cot 283^\circ \approx -.2309$