Chapter 4

Trigonometric Functions

- 4.4 Trigonometric Functions of any Angle

Chapter 4.4

Homework

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Chapter 4.4

Objectives

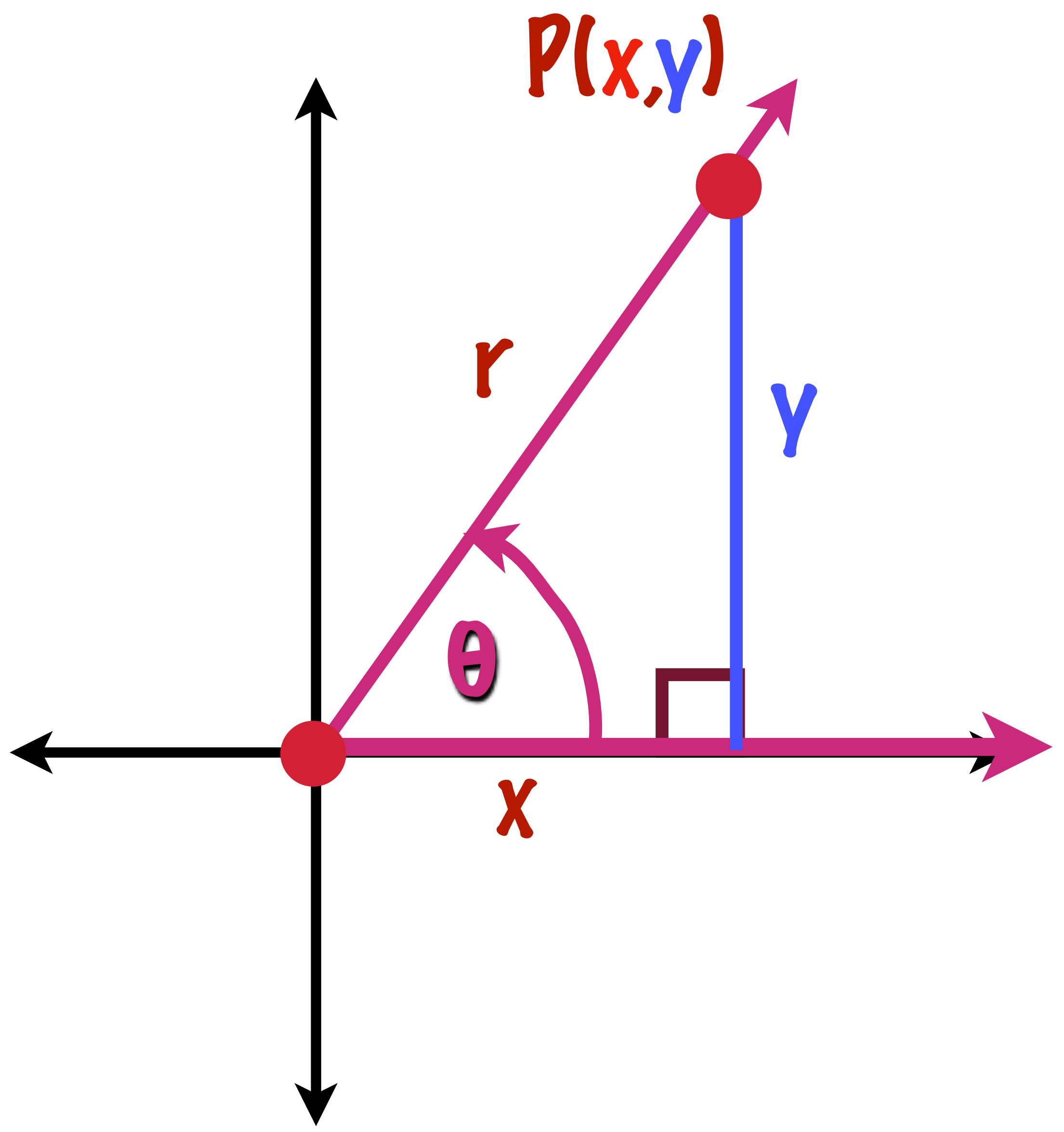
- Use the definitions of trigonometric functions of any angle.
- Use the signs of the trigonometric functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Definitions of Trigonometric Functions of Any Angle

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Let θ be any angle in standard position and let P(x, y) be a point on the terminal side of θ.

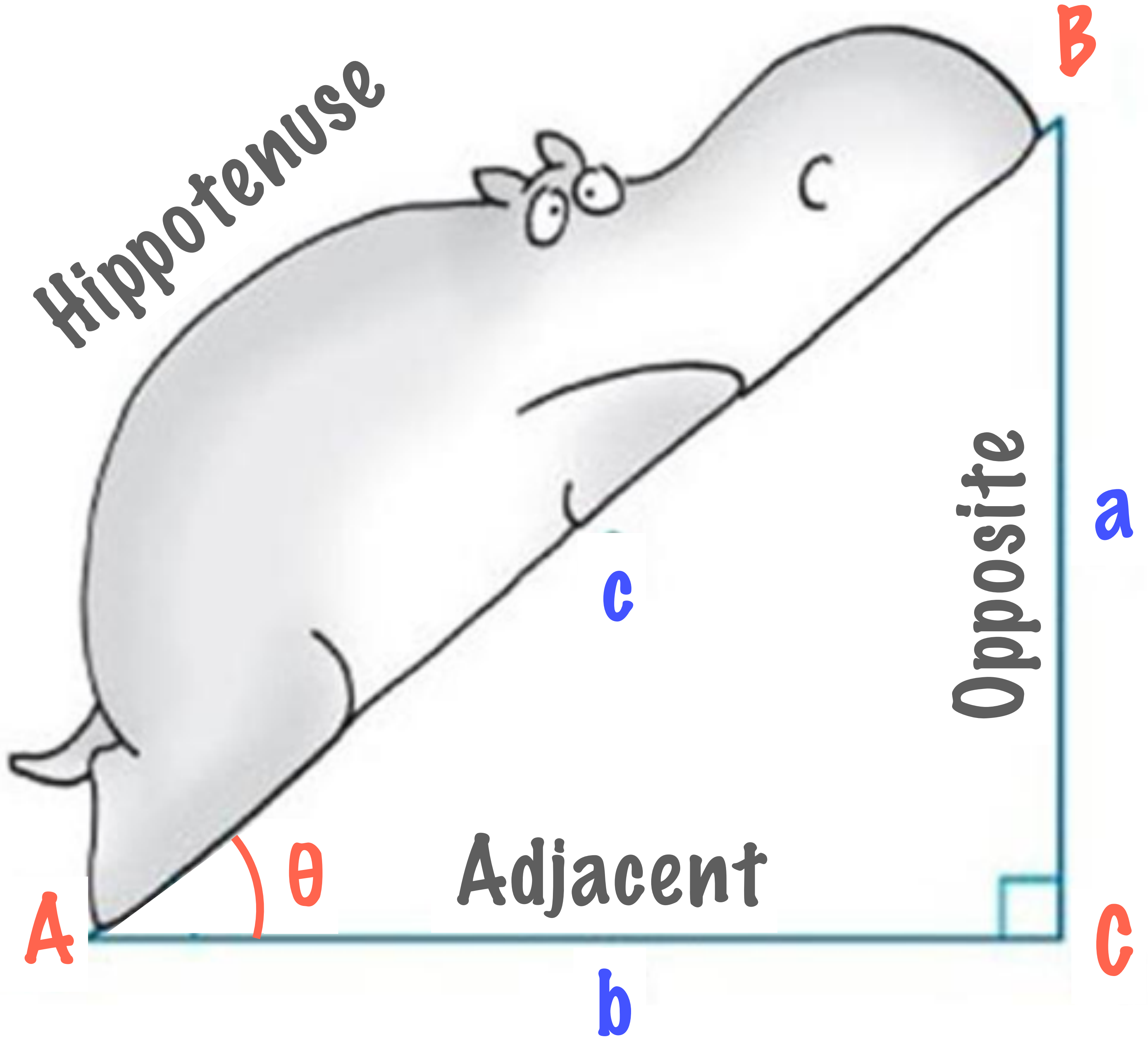
  - If $r^2 = x^2 + y^2$ is the distance from (0, 0) to (x, y) the six trigonometric functions of θ are defined by the following ratios:



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$



The signs of the Trigonometric Functions

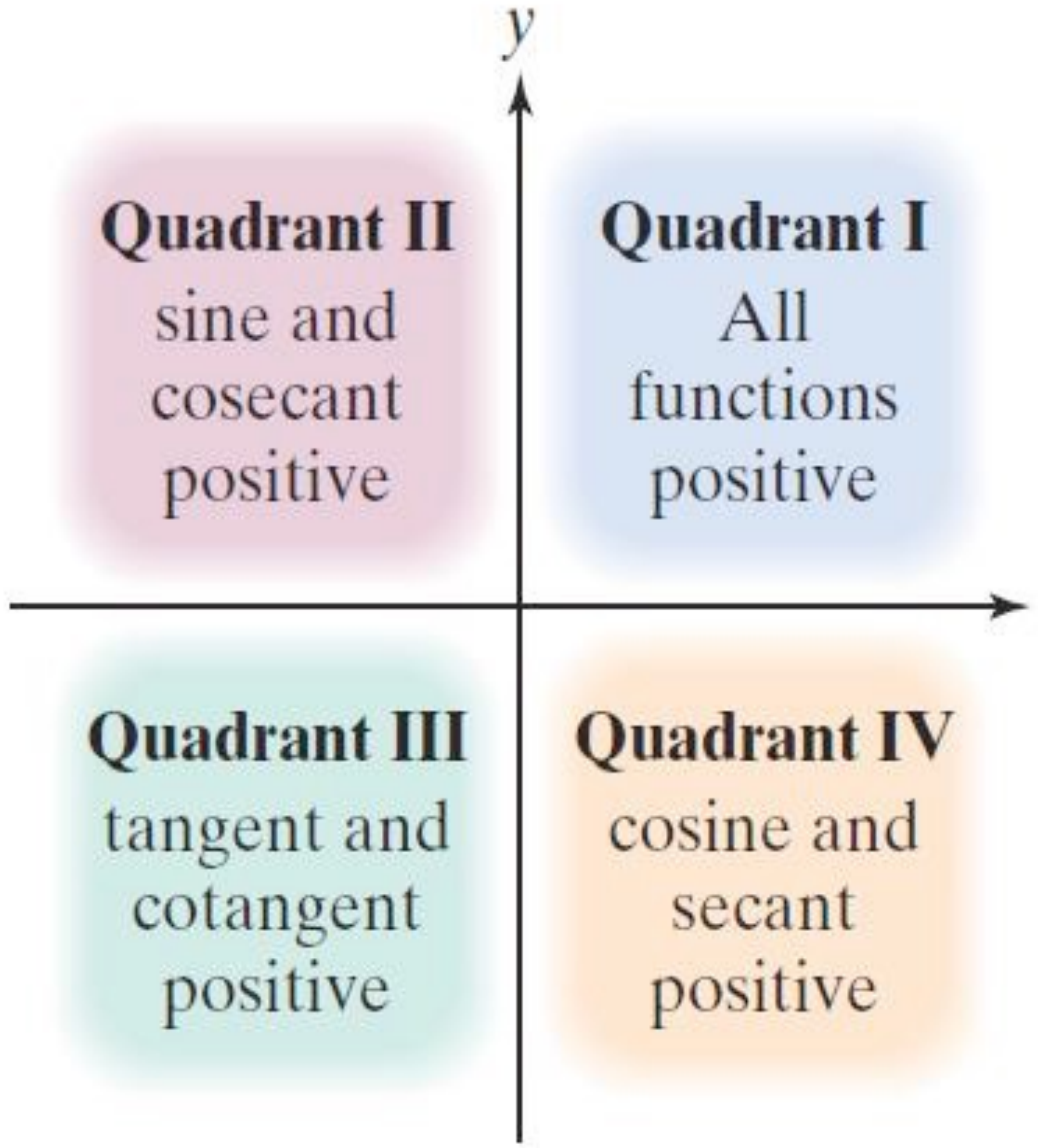
Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- We are now considering angles of any measure. Thus some angles will have the terminal side anywhere on the coordinate plane.
  - The **sign** of the trigonometric ratios are determined by the quadrant in which the terminal side lies.

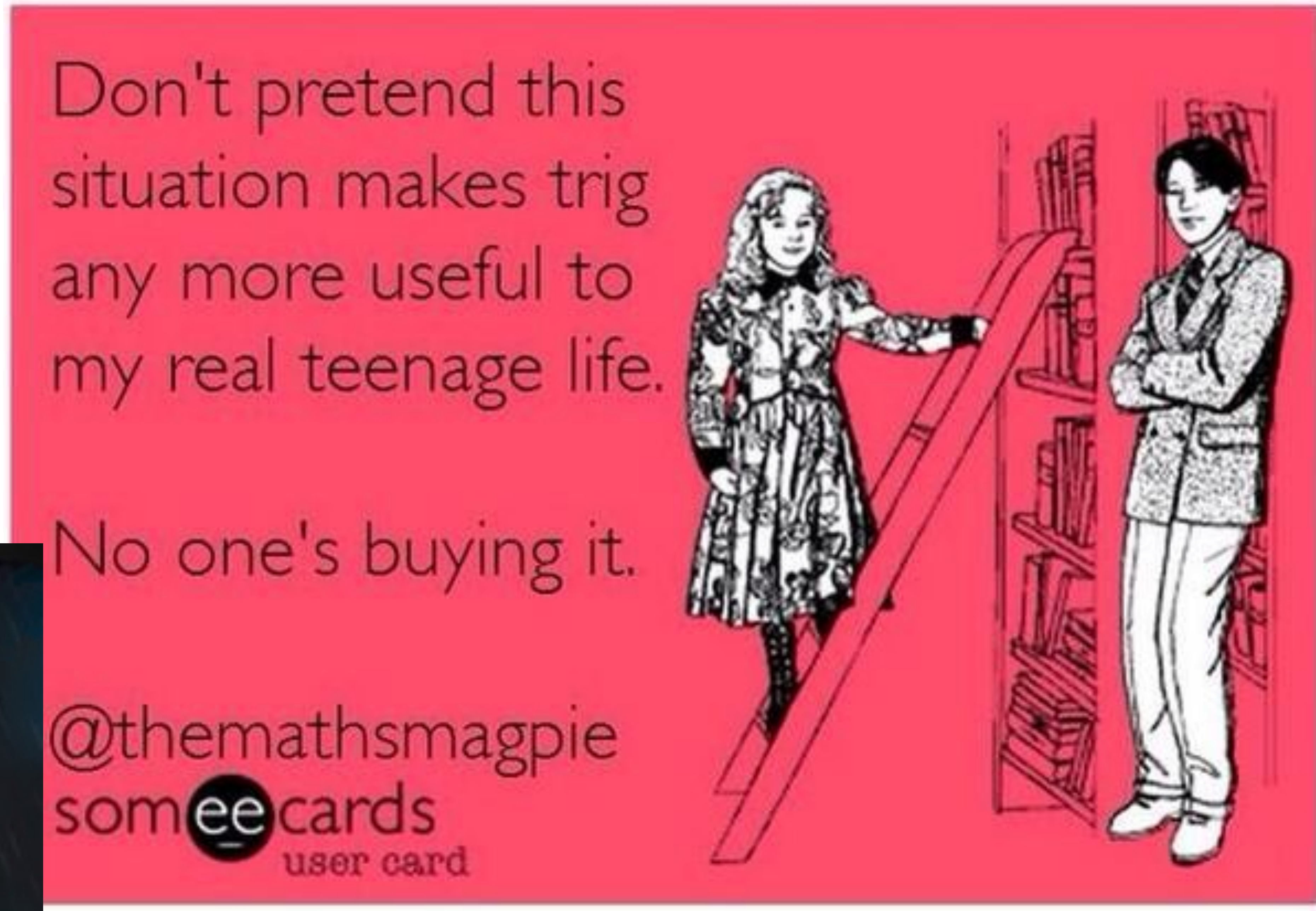
The signs of the Trigonmetric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- If sinθ < 0, and cosθ < 0, name the quadrant in which the angle θ lies.



- Quadrant III



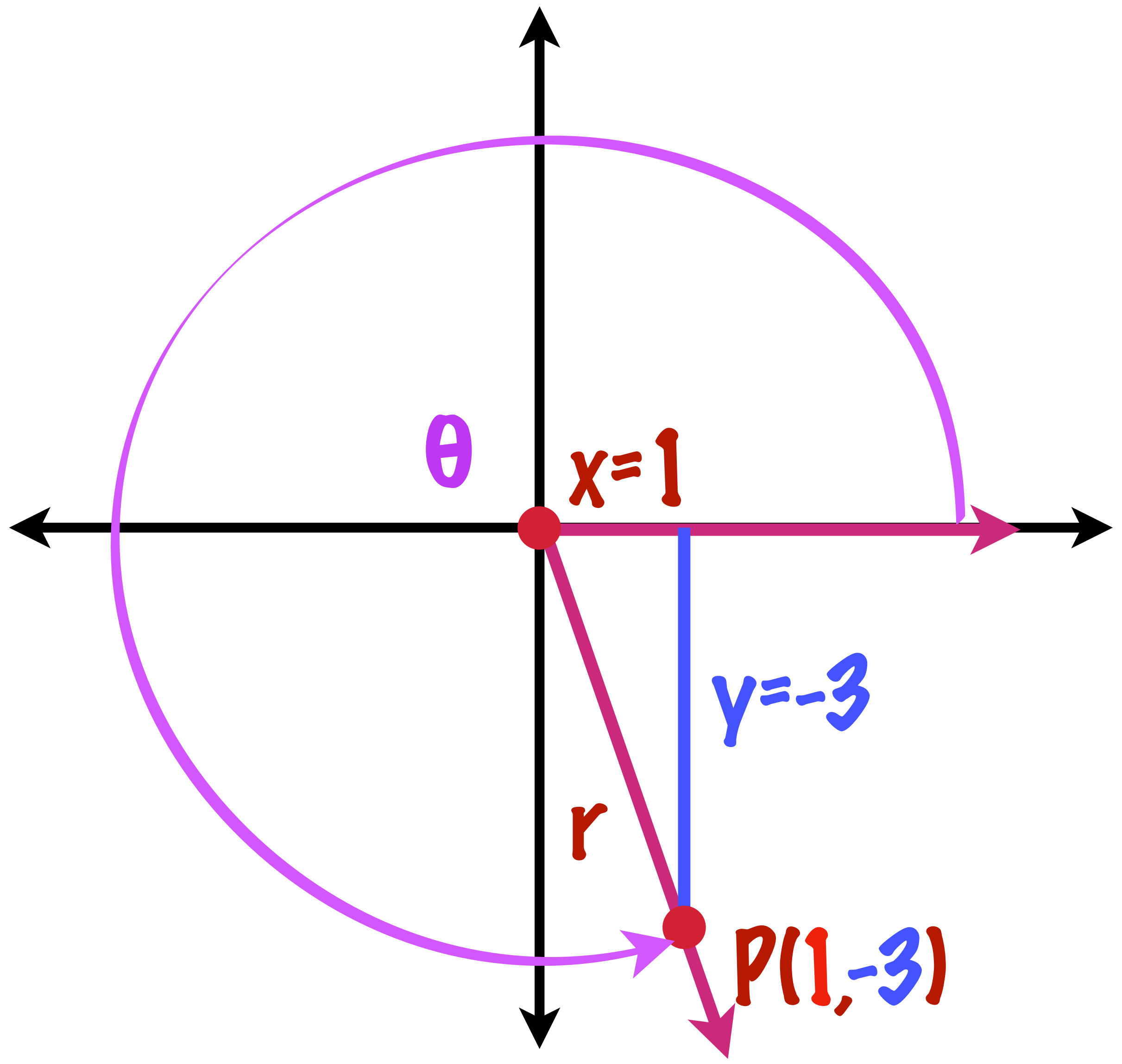
Evaluating Trigonometric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Let P(1, −3) be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

  - P(1, −3) x = 1, y = -3 x>0, y<0 QIV

$$r=\sqrt{x^2+y^2}=\sqrt{1^2+(-3)^2}=\sqrt{10}$$



$$\sin\theta=\frac{y}{r}=\frac{-3}{\sqrt{10}} \qquad \csc\theta=\frac{r}{y}=\frac{\sqrt{10}}{-3}$$

$$\cos\theta=\frac{x}{r}=\frac{1}{\sqrt{10}} \qquad \sec\theta=\frac{r}{x}=\frac{\sqrt{10}}{1}$$

$$\tan\theta=\frac{y}{x}=\frac{-3}{1} \qquad \cot\theta=\frac{x}{y}=\frac{1}{-3}$$

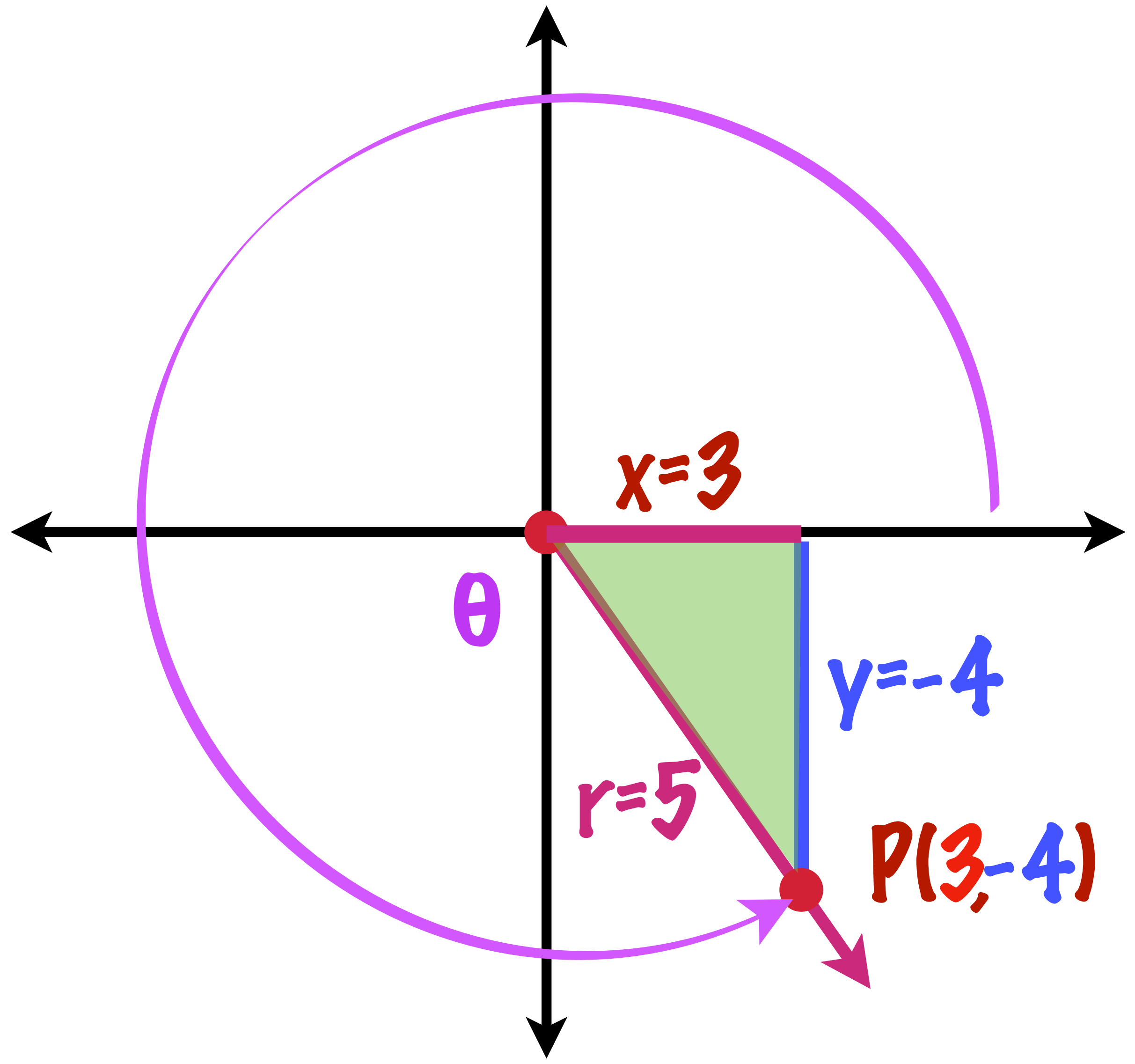
Evaluating Trigonometric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Given that $\cos\theta = 3/5$ and $\tan\theta < 0$. Find each of the six trigonometric functions of θ.

  - $\cos\theta = 3/5$ $\quad x = 3,\ r = 5$ $\quad \cos\theta > 0,\ \tan\theta < 0$ so QIV $\quad \sin\theta < 0$

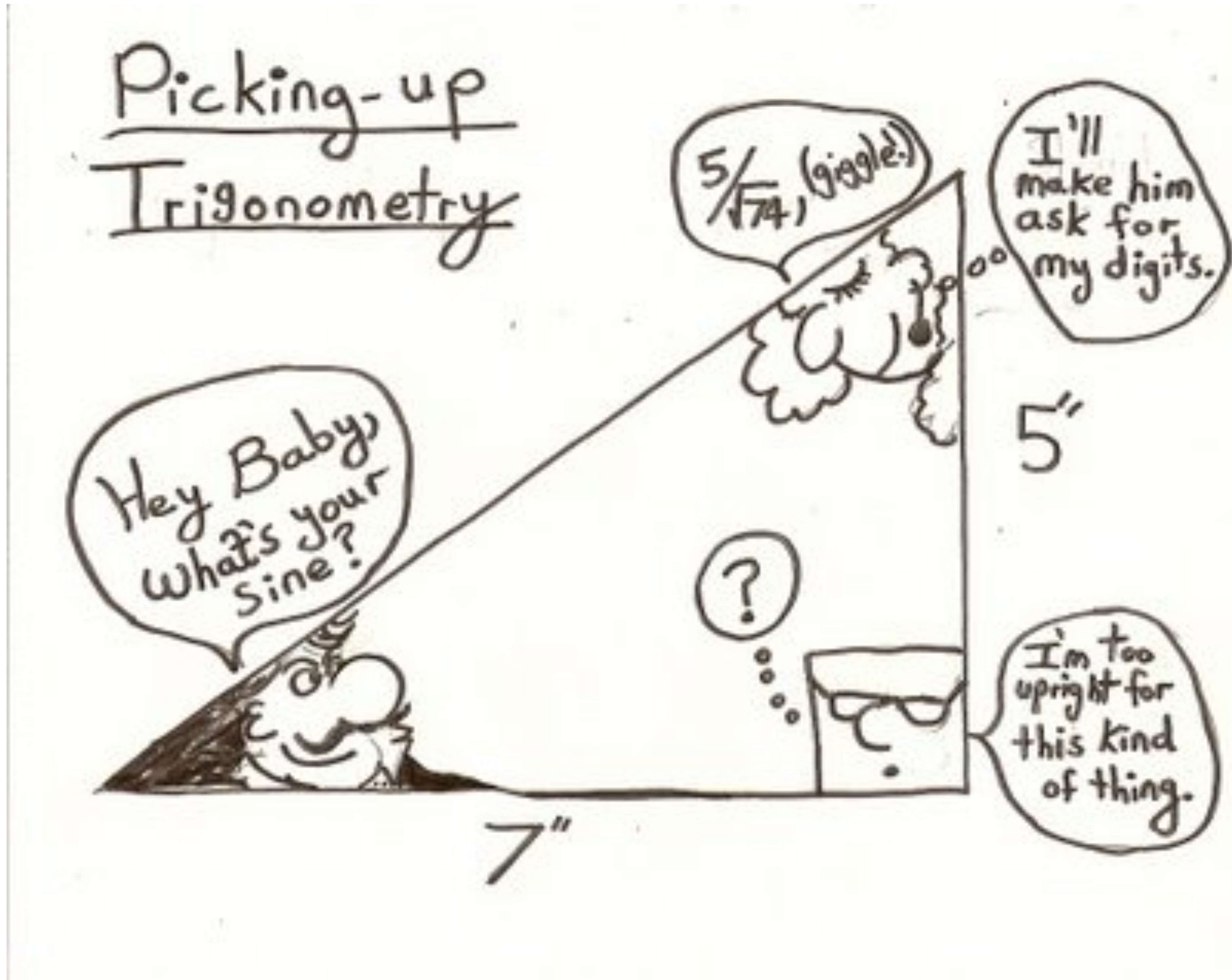
$$y = -\sqrt{r^2 - x^2} = -\sqrt{5^2 + 3^2} = -4$$



$$\sin\theta = \frac{y}{r} = \frac{-4}{5} \qquad \csc\theta = \frac{r}{y} = \frac{5}{-4}$$

$$\cos\theta = \frac{x}{r} = \frac{3}{5} \qquad \sec\theta = \frac{r}{x} = \frac{5}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{-4}{3} \qquad \cot\theta = \frac{x}{y} = \frac{3}{-4}$$



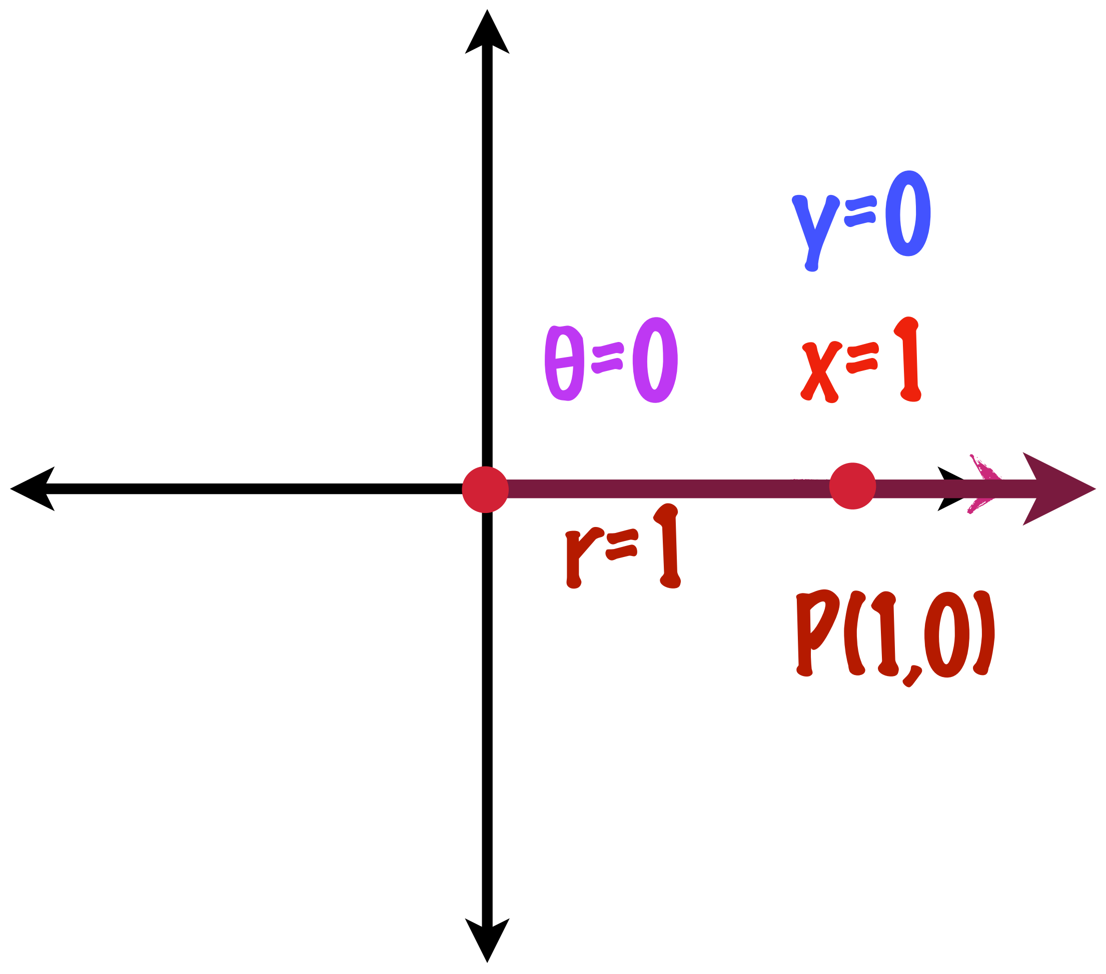
Trigonometric Functions of Quadrantal Angles

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Evaluate, if possible, the trigonometric functions at the following quadrantal angle: $\theta = 0^\circ = 0$ radians.

- If $\theta = 0^\circ = 0$ radians then the terminal side is on the positive x-axis. Select a point on the x-axis (1,0).

- P = (1, 0) with x = 1 and y = 0.



$$\sin\theta = \frac{y}{r} = \frac{0}{1} = 0 \qquad \csc\theta = \frac{r}{y} = \frac{1}{0} = \text{undefined}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{1} = 1 \qquad \sec\theta = \frac{r}{x} = \frac{1}{1} = 1$$

$$\tan\theta = \frac{y}{x} = \frac{0}{1} = 0 \qquad \cot\theta = \frac{x}{y} = \frac{1}{0} = \text{undefined}$$

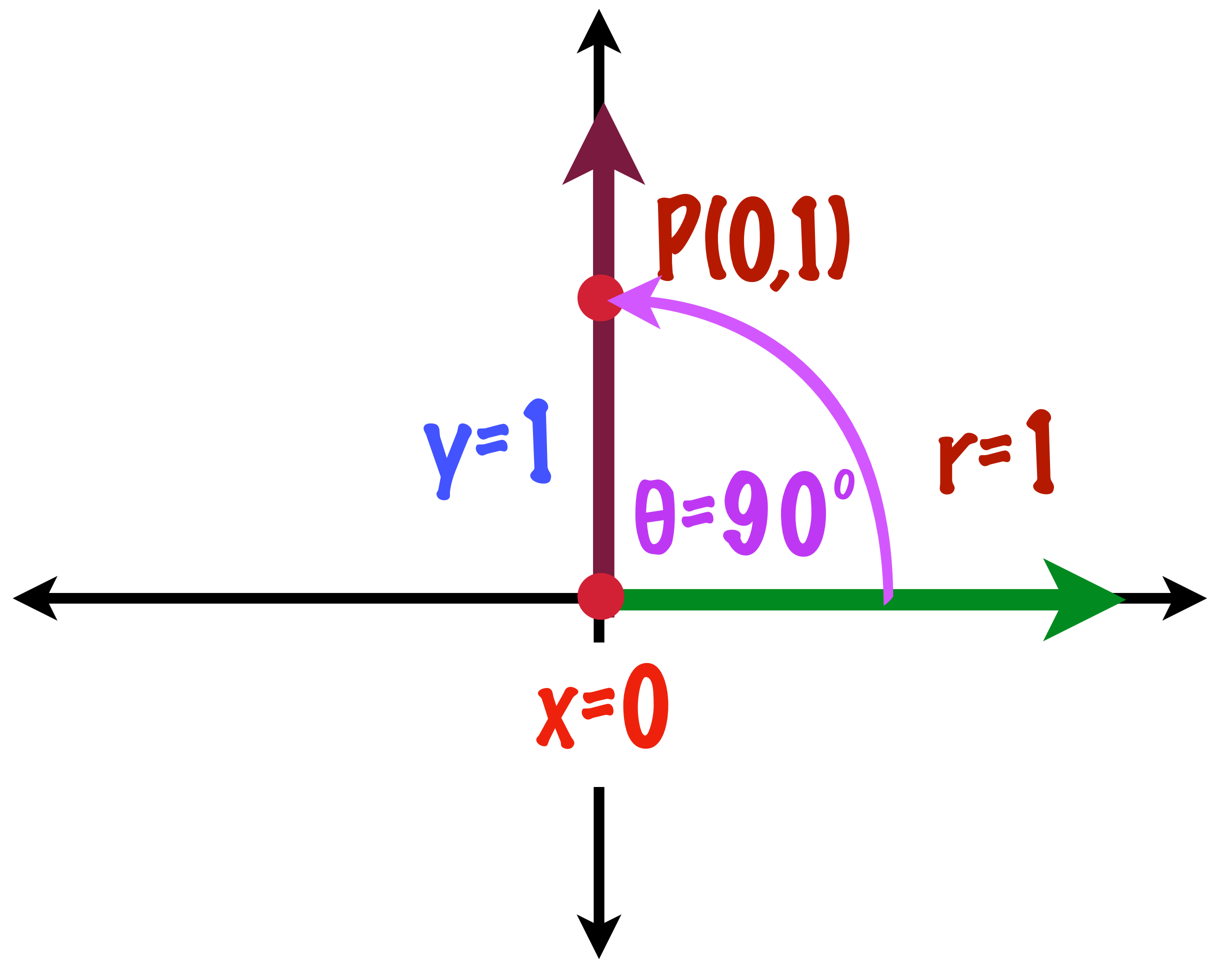
- Now try using another point on the x-axis.

Trigonometric Functions of Quadrantal Angles

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Evaluate, if possible, the trigonometric functions at the following quadrantal angle: $\theta = 90^\circ = \frac{\pi}{2}$ radians.

  - If $\theta = 90^\circ = \frac{\pi}{2}$ radians then the terminal side of the angle is on the positive y-axis.
  - Choose P(0, 1); $x = 0$ and $y = 1$.



$$\sin\theta = \frac{y}{r} = \frac{1}{1} = 1 \qquad \csc\theta = \frac{r}{y} = \frac{1}{1} = 1$$

$$\cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \qquad \sec\theta = \frac{r}{x} = \frac{1}{0} = \textit{undefined}$$

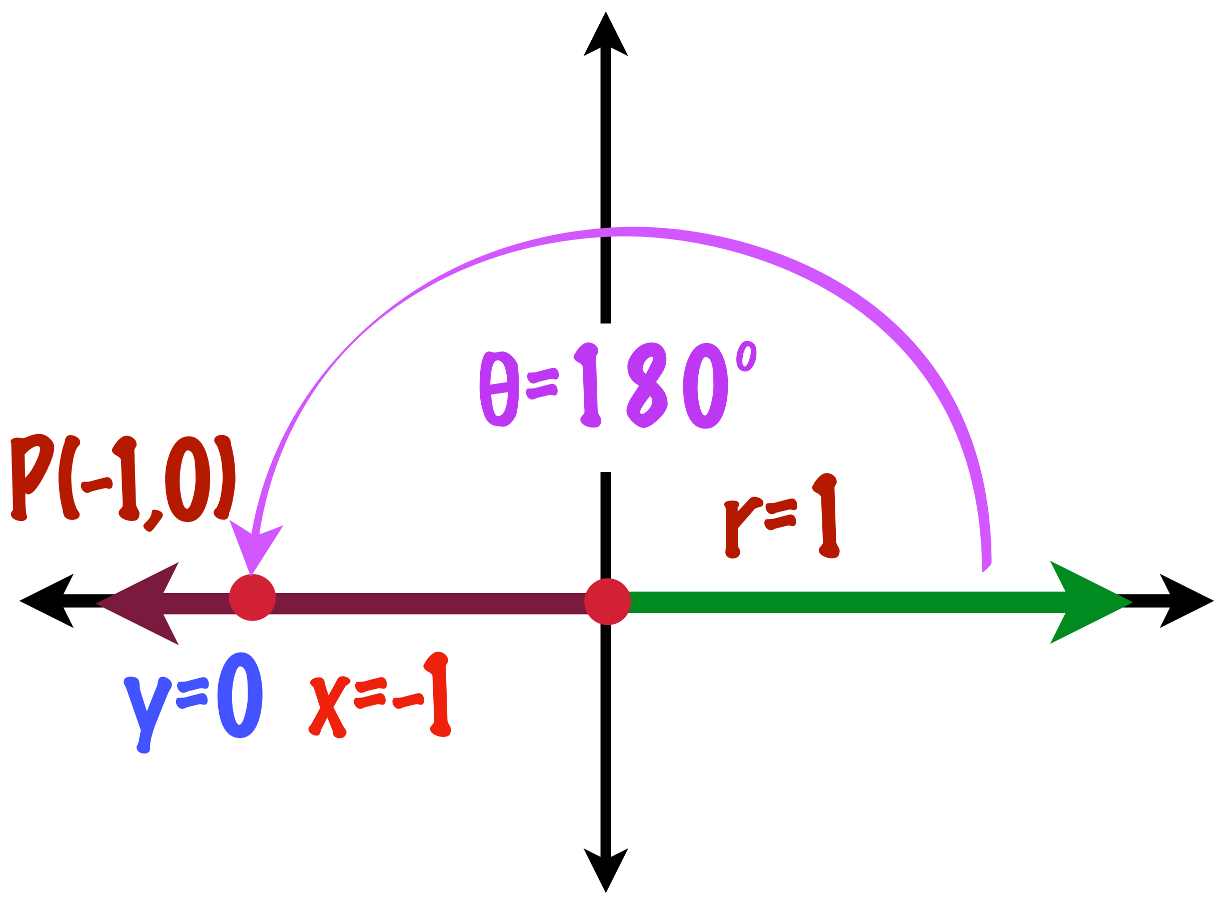
$$\tan\theta = \frac{y}{x} = \frac{1}{0} = \textit{undefined} \qquad \cot\theta = \frac{x}{y} = \frac{0}{1} = 0$$

- Now try using another point on the y-axis.

Trigonometric Functions of Quadrantal Angles

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Evaluate, if possible, the trigonometric functions at the following quadrantal angle: $\theta = 180^\circ = \pi$ radians.
  - If $\theta = 180^\circ = \pi$ radians then the terminal side is on the negative x-axis. Select a point on the x-axis (-1,0).
    - P = (-1, 0) with x = -1 and y = 0.



$$\sin\theta = \frac{y}{r} = \frac{0}{1} = 0 \qquad \csc\theta = \frac{r}{y} = \frac{1}{0} = \textit{undefined}$$

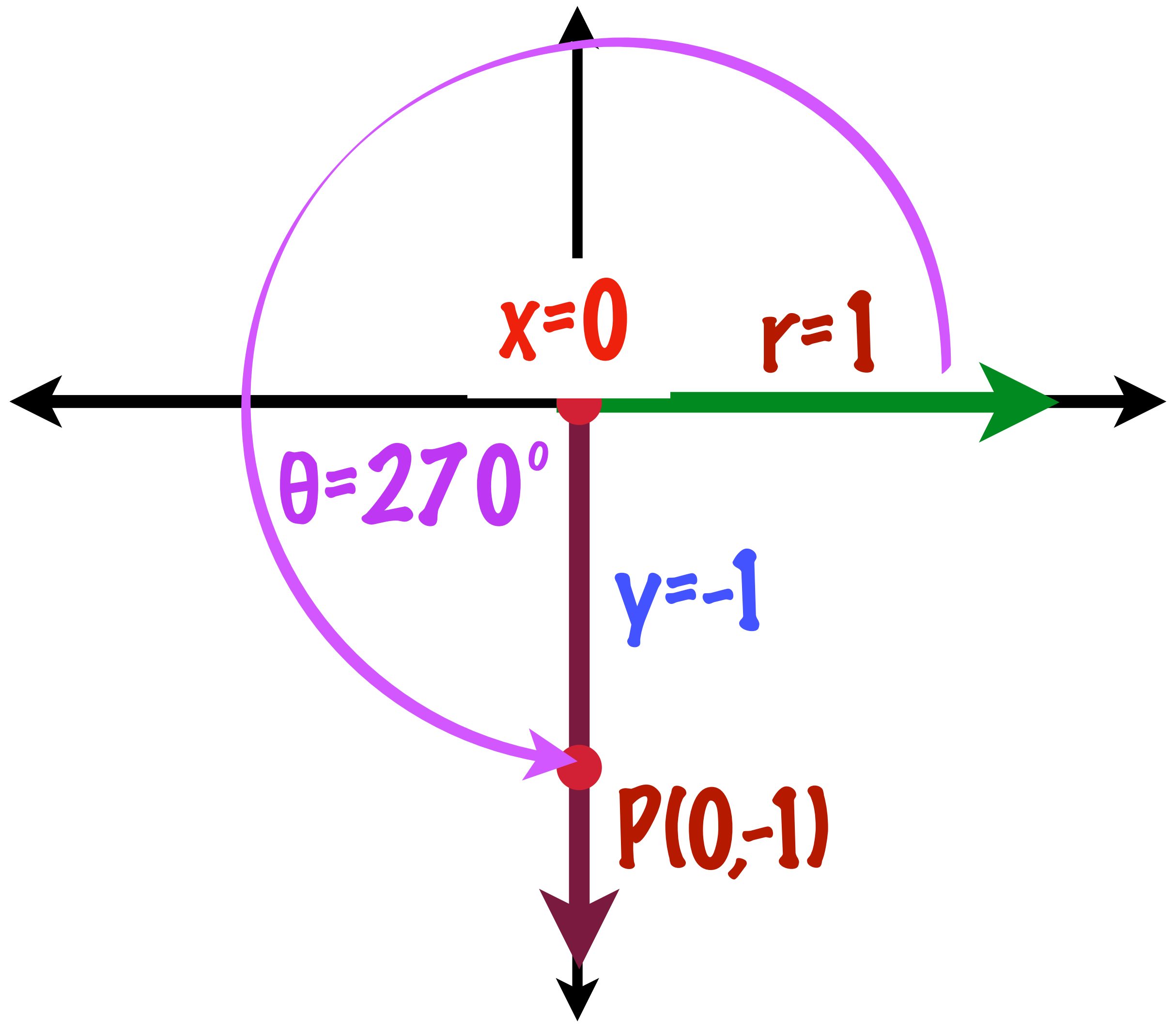
$$\cos\theta = \frac{x}{r} = \frac{-1}{1} = -1 \qquad \sec\theta = \frac{r}{x} = \frac{1}{-1} = -1$$

$$\tan\theta = \frac{y}{x} = \frac{0}{-1} = 0 \qquad \cot\theta = \frac{x}{y} = \frac{-1}{0} = \textit{undefined}$$

Trigonometric Functions of Quadrantal Angles

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Evaluate, if possible, the trigonometric functions at the following quadrantal angle: $\theta = 270^\circ = \frac{3\pi}{2}$ radians.
  - If $\theta = 270^\circ = \frac{3\pi}{2}$ radians then the terminal side of the angle is on the negative y-axis.
    - Choose P(0, -1); $x = 0$ and $y = -1$.



$$\sin\theta = \frac{y}{r} = \frac{-1}{1} = -1 \qquad \csc\theta = \frac{r}{y} = \frac{1}{-1} = -1$$

$$\cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \qquad \sec\theta = \frac{r}{x} = \frac{1}{0} = \text{undefined}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{0} = \text{undefined} \qquad \cot\theta = \frac{x}{y} = \frac{0}{-1} = 0$$

Trig Functions for Quadrantal Angles

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

	0, 2π	90, π/2	180, π	270, 3π/2
Sinθ	0	1	0	-1
Cosθ	1	0	-1	0
Tanθ	0	undefined	0	undefined

Trig Functions for Special Angles

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

	30°, π/6	60°, π/3	45, π/4
Sinθ	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$
Cosθ	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
Tanθ	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	1

- You will save yourself a ton of time, heartache, and anxiety if you memorize these values.

Trig Functions for Special Angles

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

STUDY TIP

Learning the table of values at the right is worth the effort because doing so will increase both your efficiency and your confidence. Here is a pattern for the sine function that may help you remember the values.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

Reverse the order to get cosine values of the same angles.

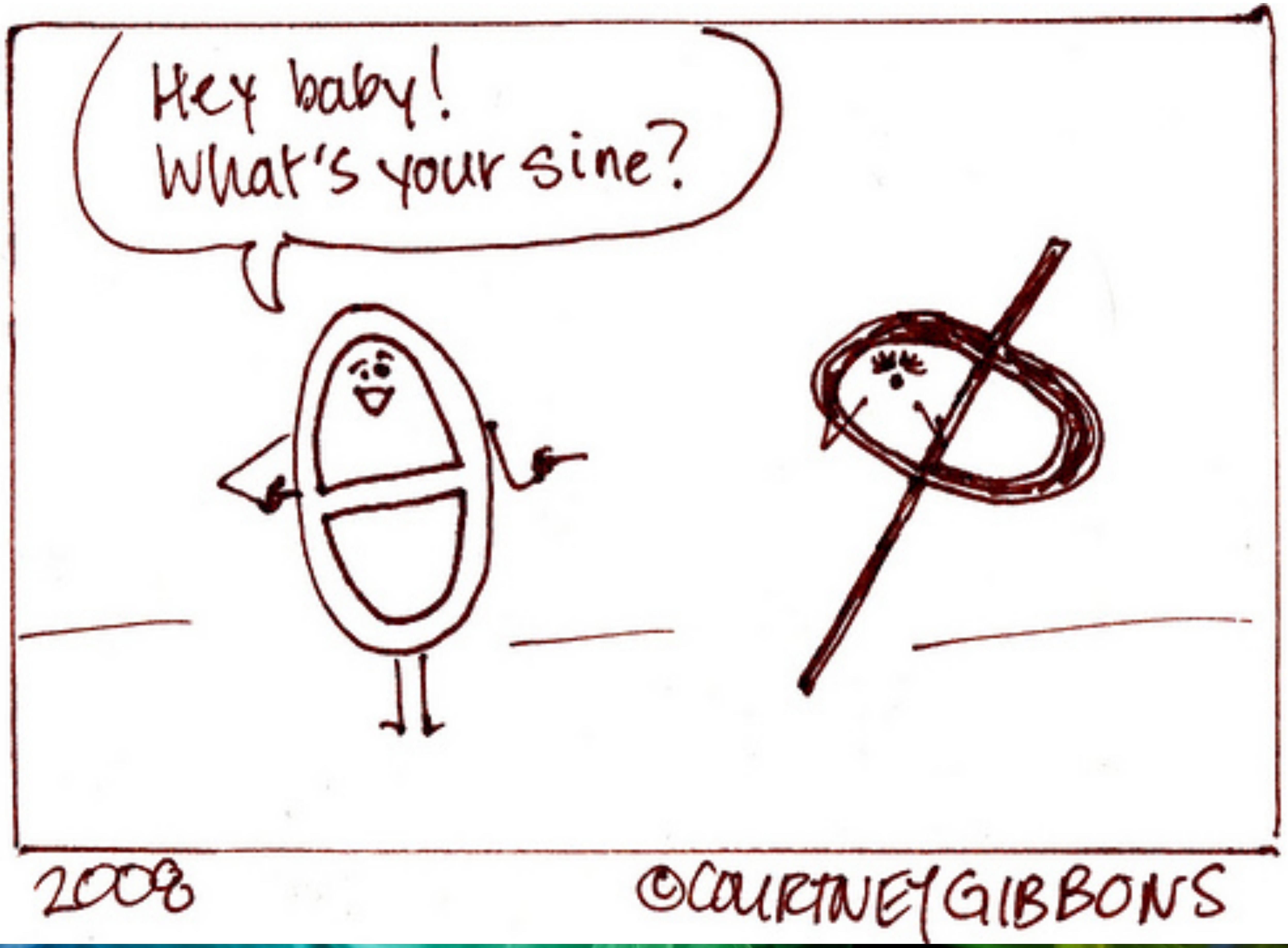
Trigonometric Values of Common Angles

θ (degrees)	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.

The Signs of the Trigonometric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- The **sign** of the trigonometric ratios are determined by the quadrant in which the terminal side lies.



Example: Finding the Quadrant in Which an Angle Lies

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

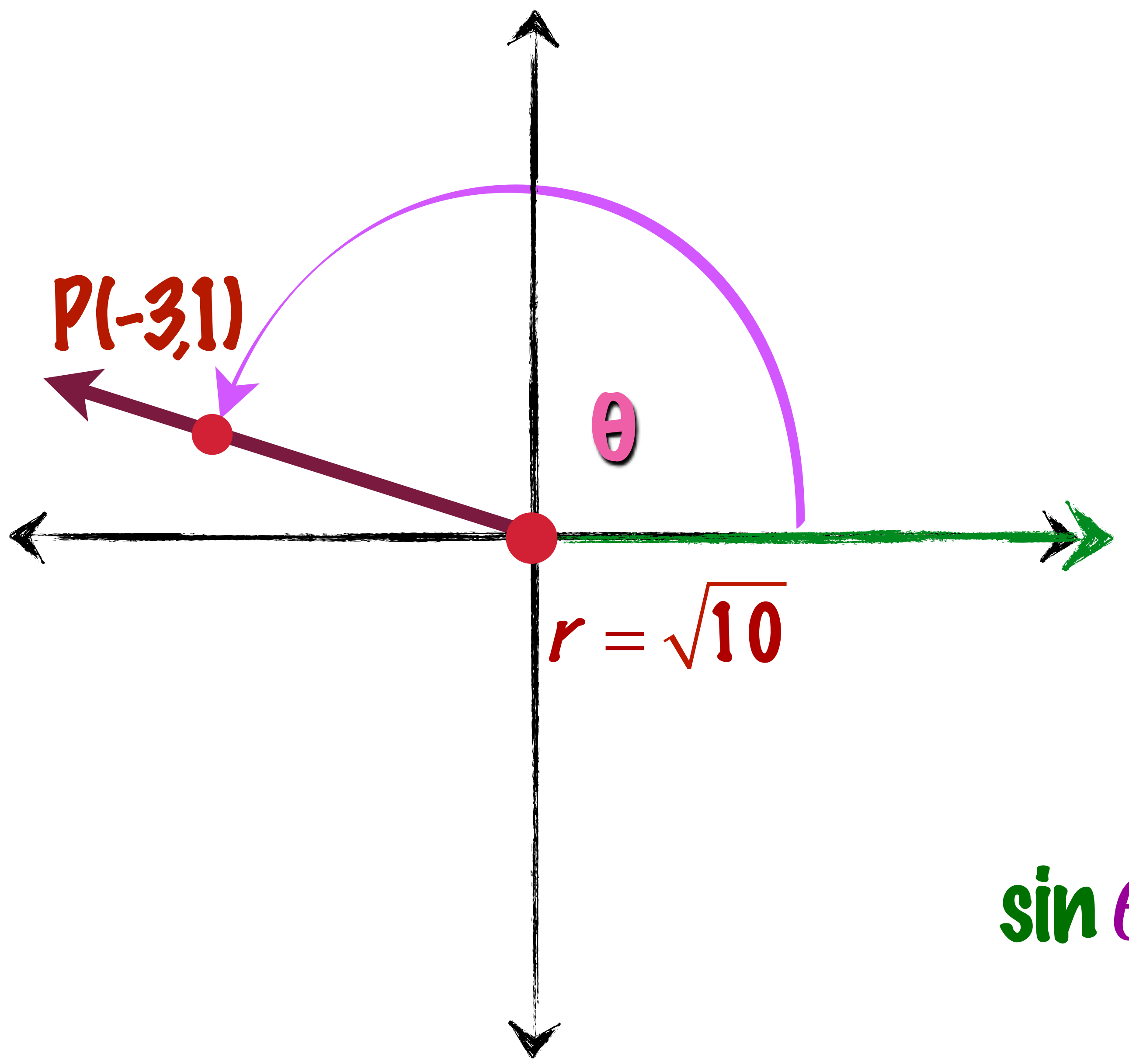
- If $\sin\theta < 0$, and $\cos\theta < 0$, name the quadrant in which the angle θ lies.

- Quadrant III

Evaluating Trigonometric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Given $\tan\theta = -\frac{1}{3}$ and $\cos\theta < 0$, find $\sin\theta$ and $\sec\theta$.
  - Because both the tangent and the cosine are negative, θ lies in Quadrant II.



$$\tan\theta = \frac{y}{x} = -\frac{1}{3} \qquad x = -3,\ y = 1$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$$

$$\sin\theta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10} \qquad \sec\theta = \frac{\sqrt{10}}{-3} = -\frac{\sqrt{10}}{3}$$

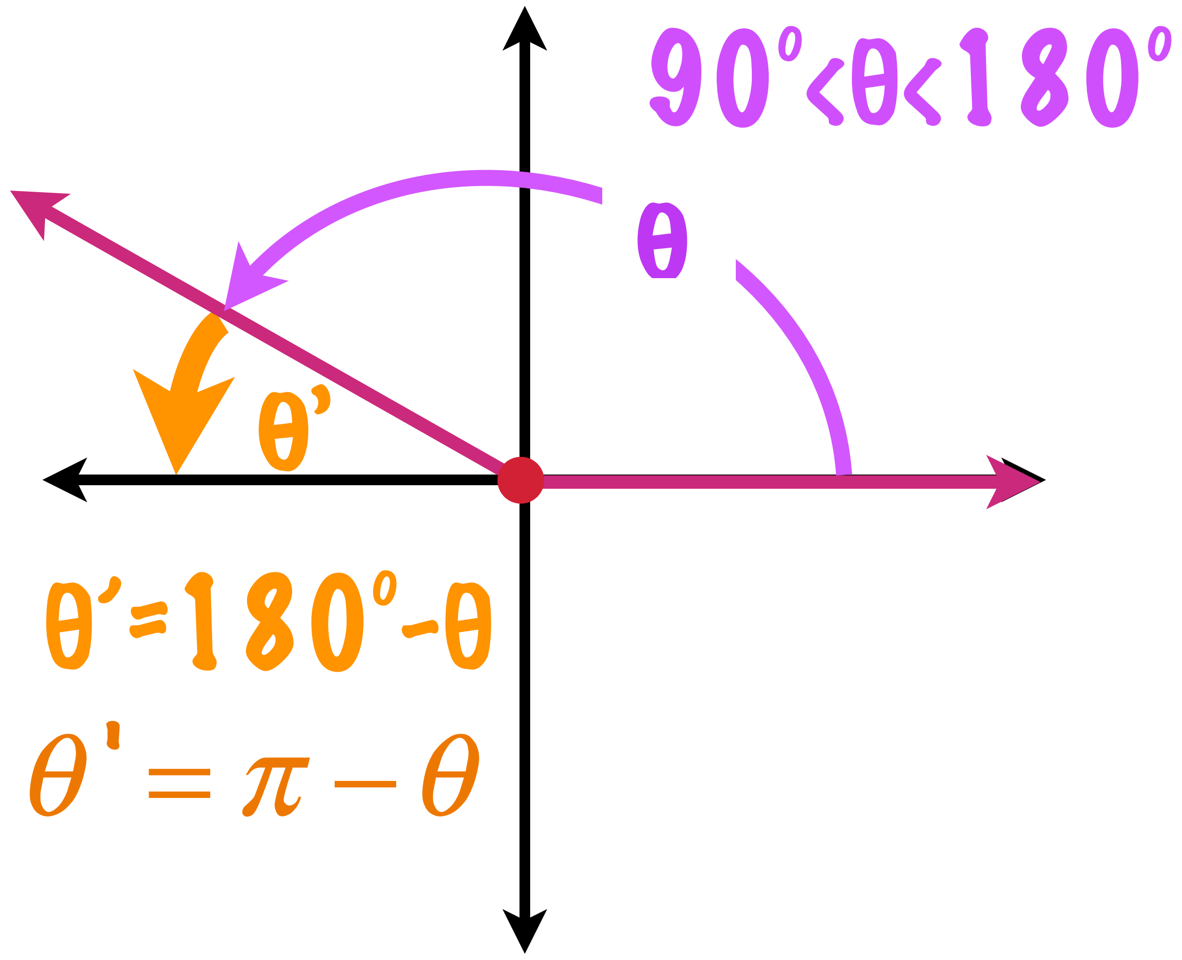
Definition of a Reference Angle

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Let θ be a non-acute angle (θ ≥ 90°) in standard position that lies in a quadrant. Its **reference angle** is the **positive** acute angle θ' formed by the terminal side of θ and the x-axis.

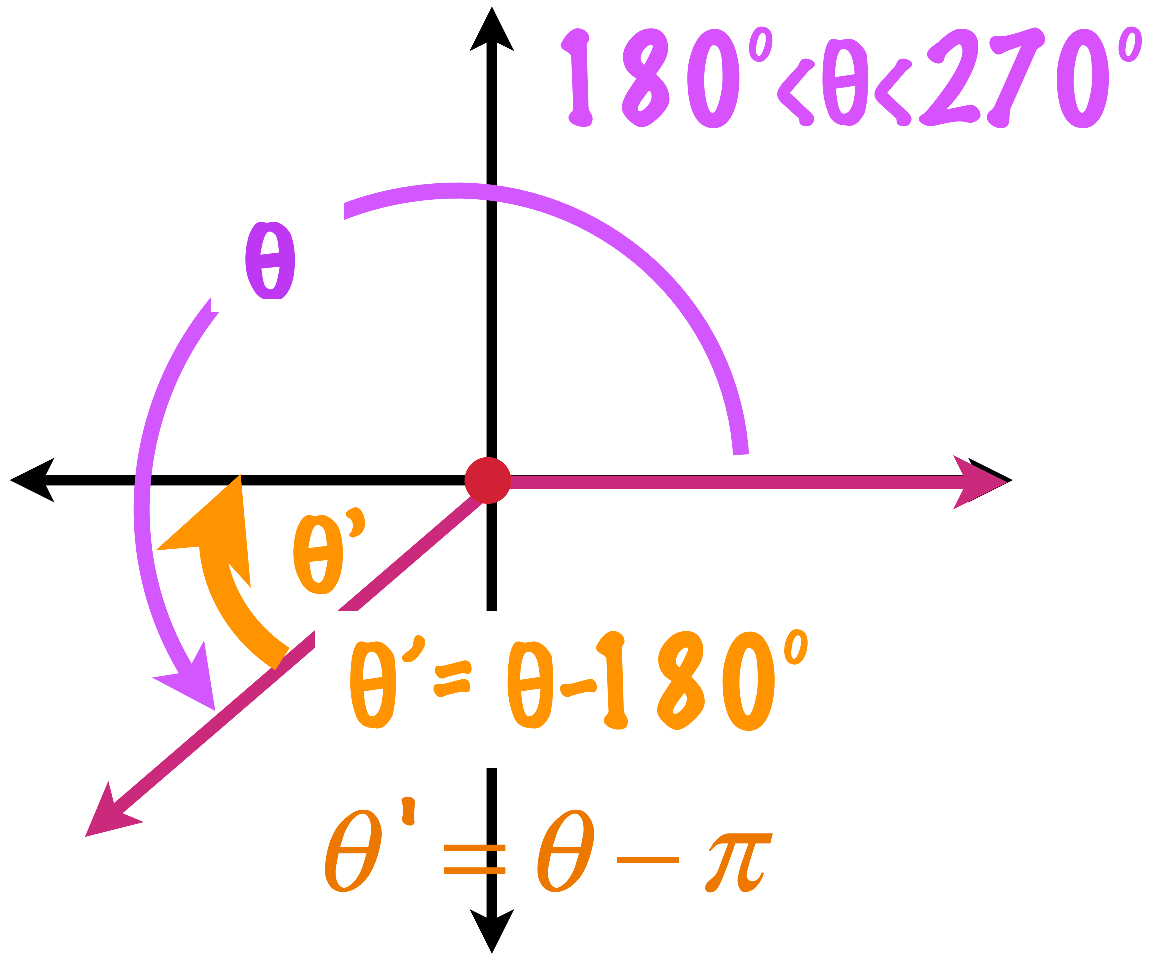
- Quadrant II

$$\frac{\pi}{2} < \theta < \pi$$



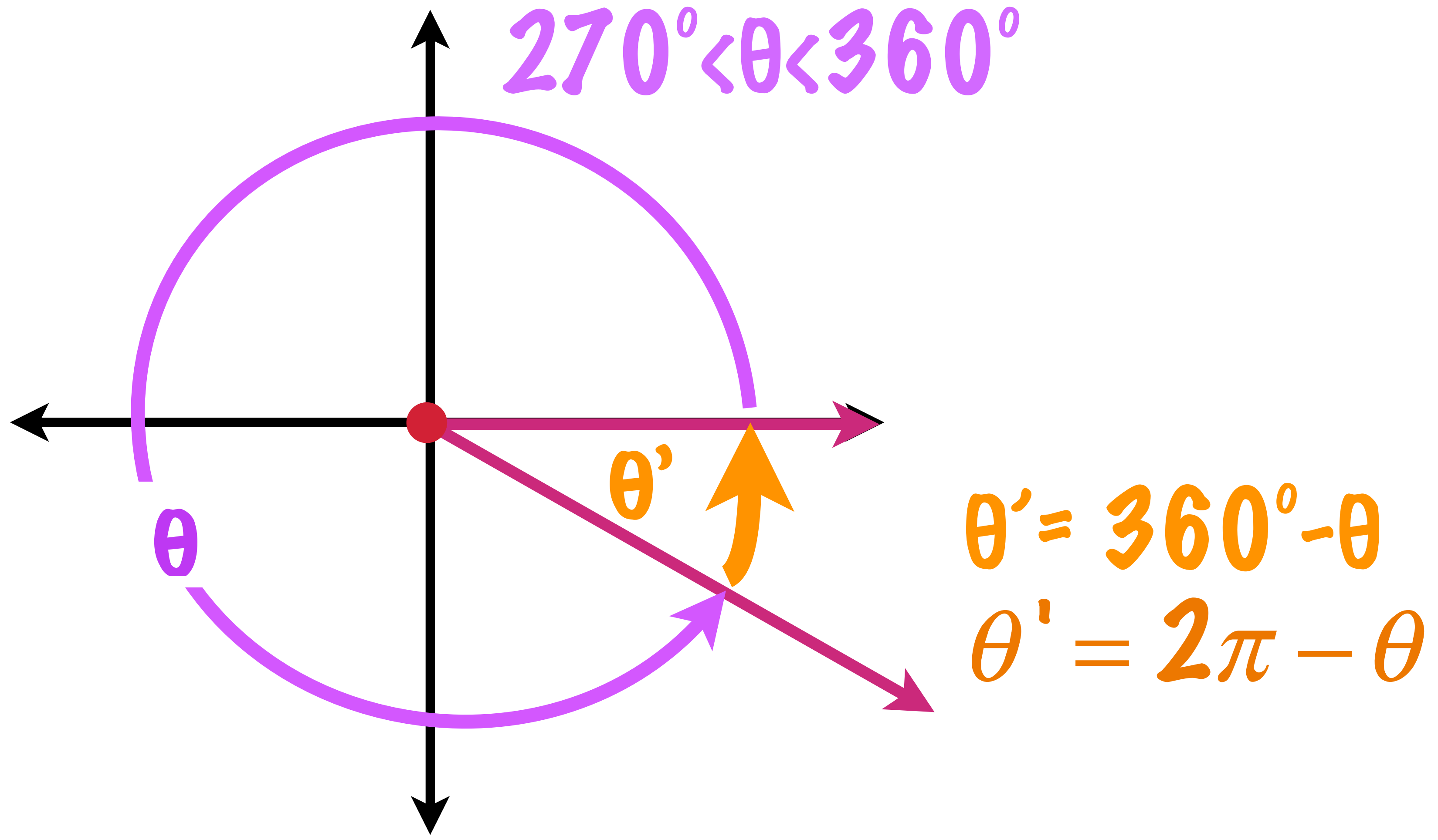
- Quadrant III

$$\pi < \theta < \frac{3\pi}{2}$$



- Quadrant IV

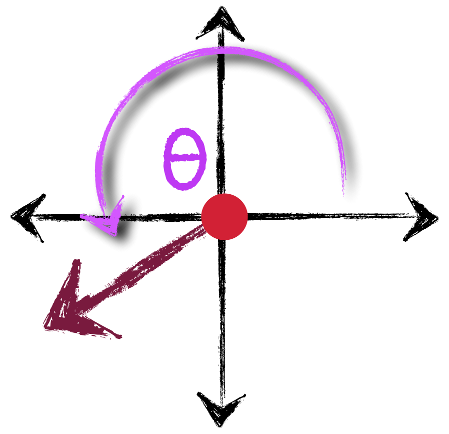
$$\frac{3\pi}{2} < \theta < 2\pi$$



Objectives: Use the trig ratios for any measure angle. Find the reference angle.

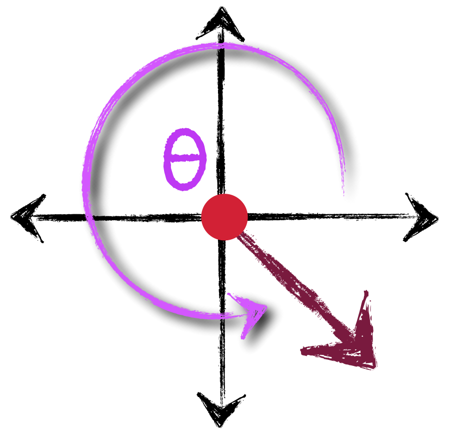
Find the reference angle, θ' for each of the following angles:

a. $\theta = 210^\circ$



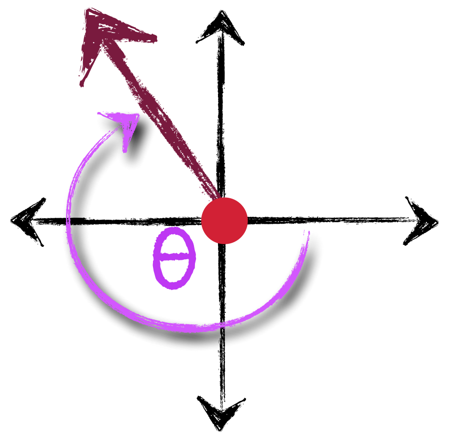
$\theta' = 210^\circ - 180^\circ = 30^\circ$

b. $\theta = \dfrac{7\pi}{4}$



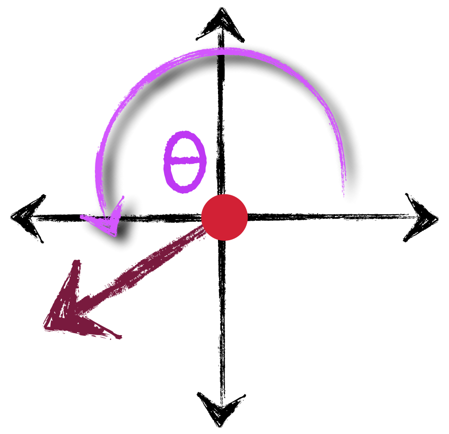
$\theta' = 2\pi - \dfrac{7\pi}{4} = \dfrac{\pi}{4}$

c. $\theta = -240^\circ$

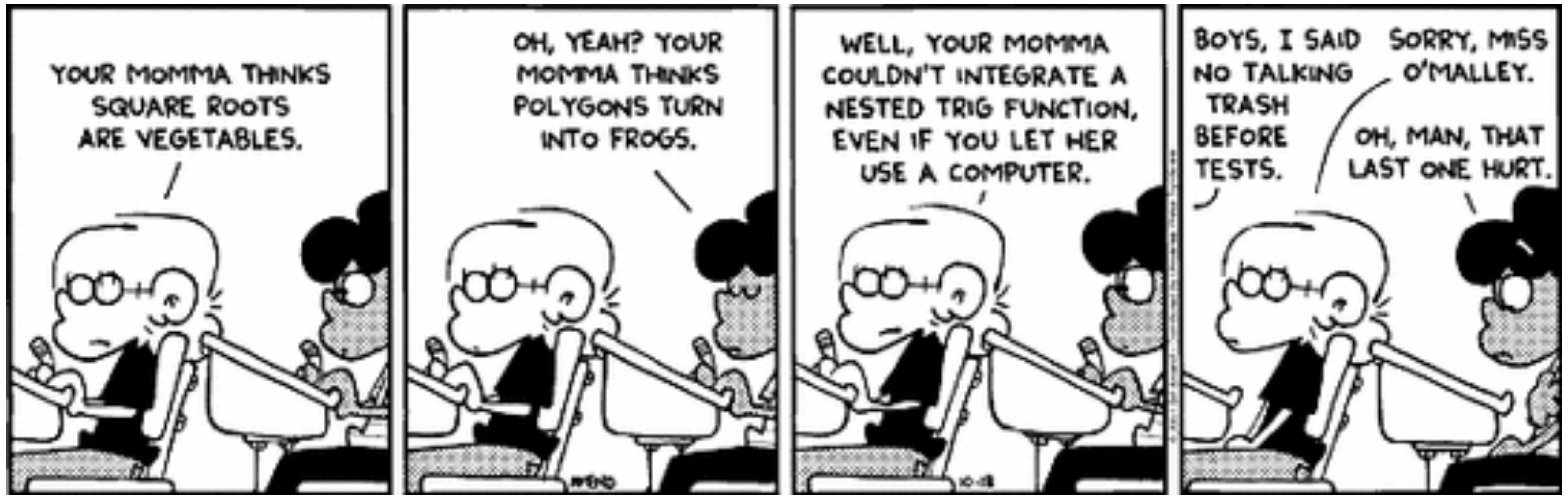


$\theta' = -240^\circ + 180^\circ = 60^\circ$

d. $\theta = 3.6$



$\theta' = 3.6 - \pi \approx 3.6 - 3.1416 \approx .4584$



Finding Reference Angles for Angles Greater Than 360°(2π) or Less Than -360°

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

∡ To find **reference angles** for angles > 360° (2π) or < -360° (-2π) follow these steps:

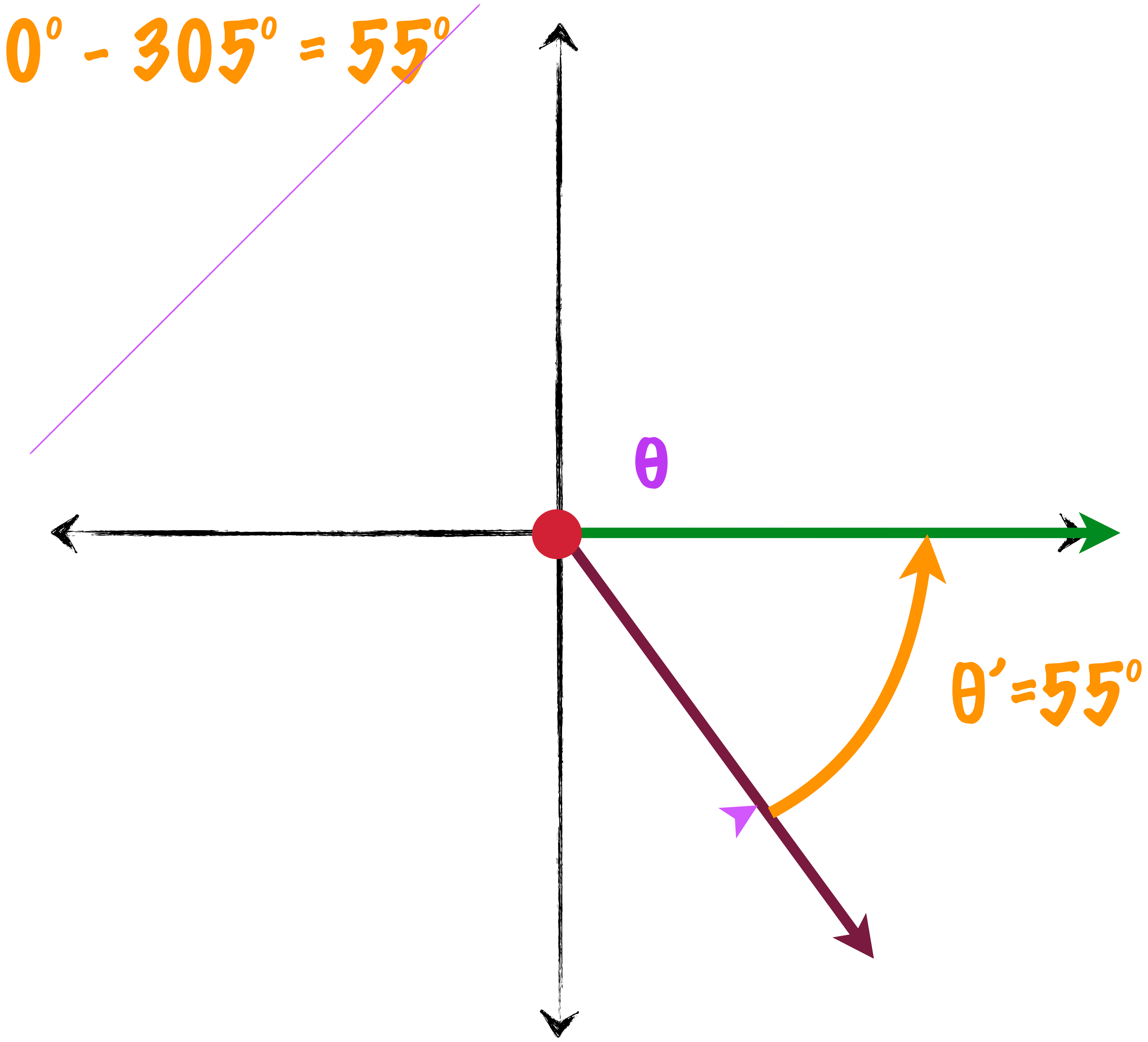
1. Find a positive angle α less than 360° (2π) that is co-terminal with the given angle.
2. Draw α in standard position.
3. Use the drawing to find the **reference angle** for the given angle. The positive acute angle formed by the terminal side of α and the x-axis is the reference angle.

Example: Finding Reference Angles

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Find the reference angle θ' for each of the following angles:

  - a. $\theta = 665^{\circ}$ $665^{\circ} - 360^{\circ} = 305^{\circ}$ $\theta' = 360^{\circ} - 305^{\circ} = 55^{\circ}$

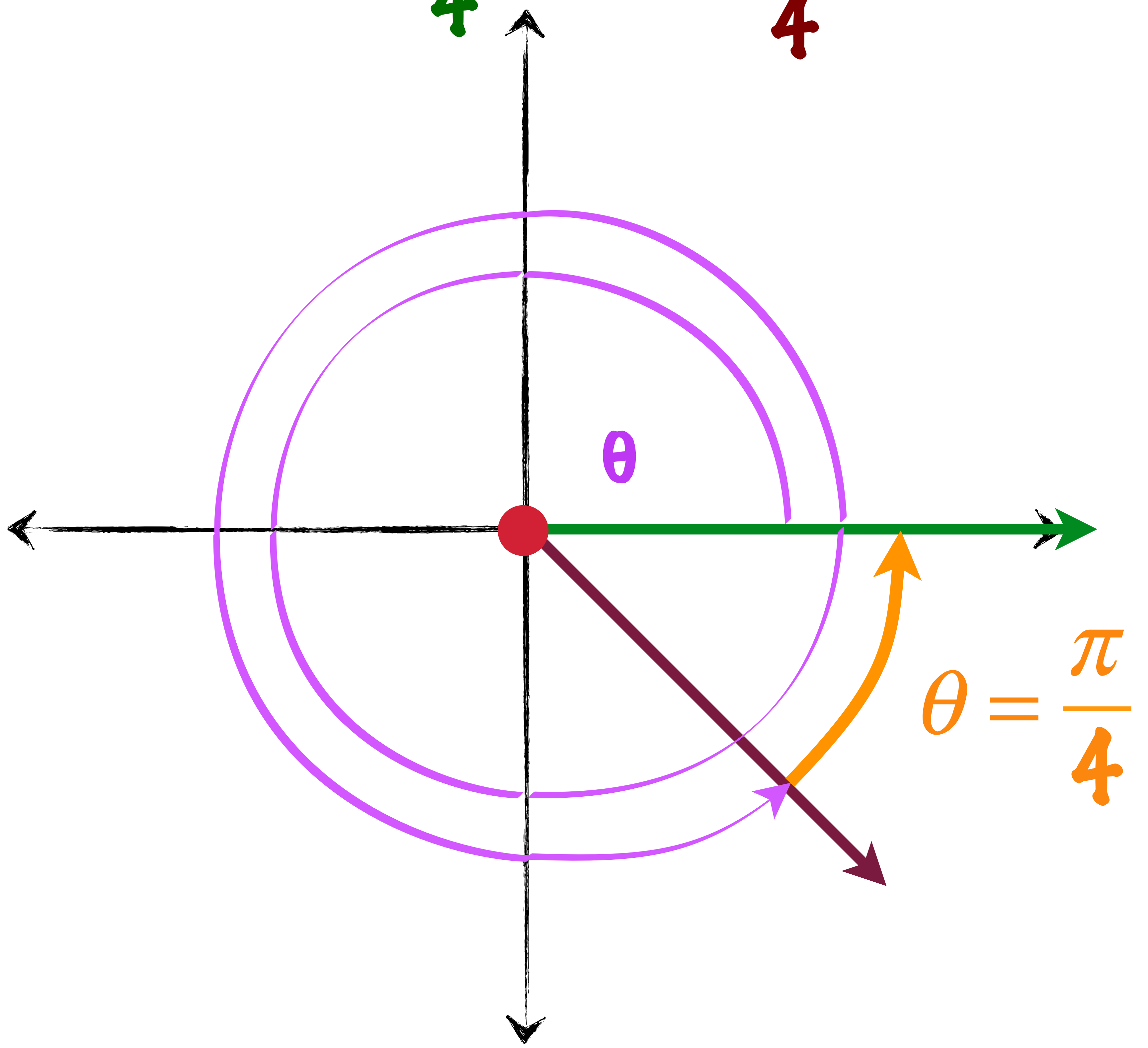


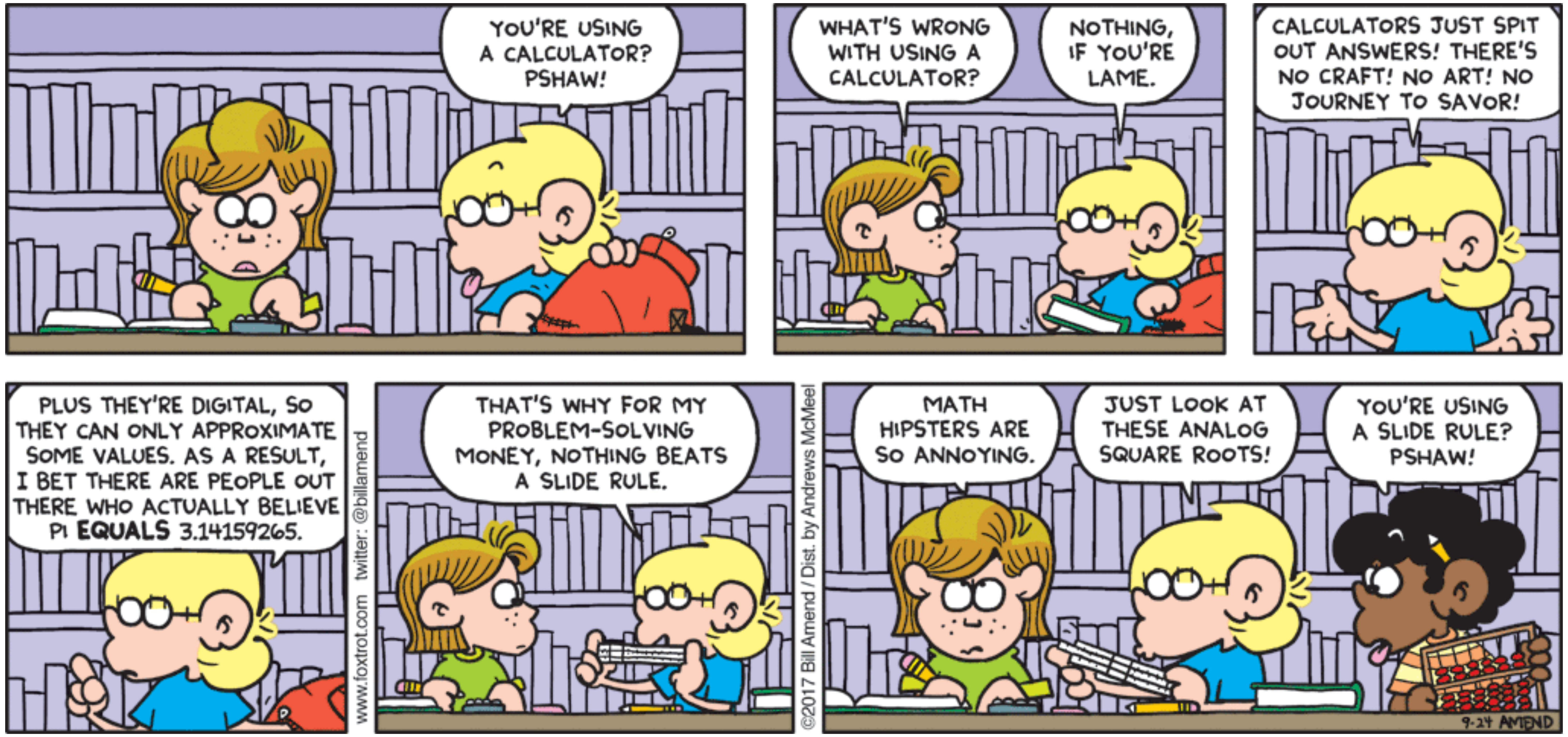
Example: Finding Reference Angles

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Find the reference angle θ' for each of the following angles:

b. $\theta = \dfrac{15\pi}{4}$ $\qquad \dfrac{15\pi}{4} - 2\pi = \dfrac{7\pi}{4}$ $\qquad \theta' = 2\pi - \dfrac{15\pi}{4} = \dfrac{\pi}{4}$





Example: Finding Reference Angles

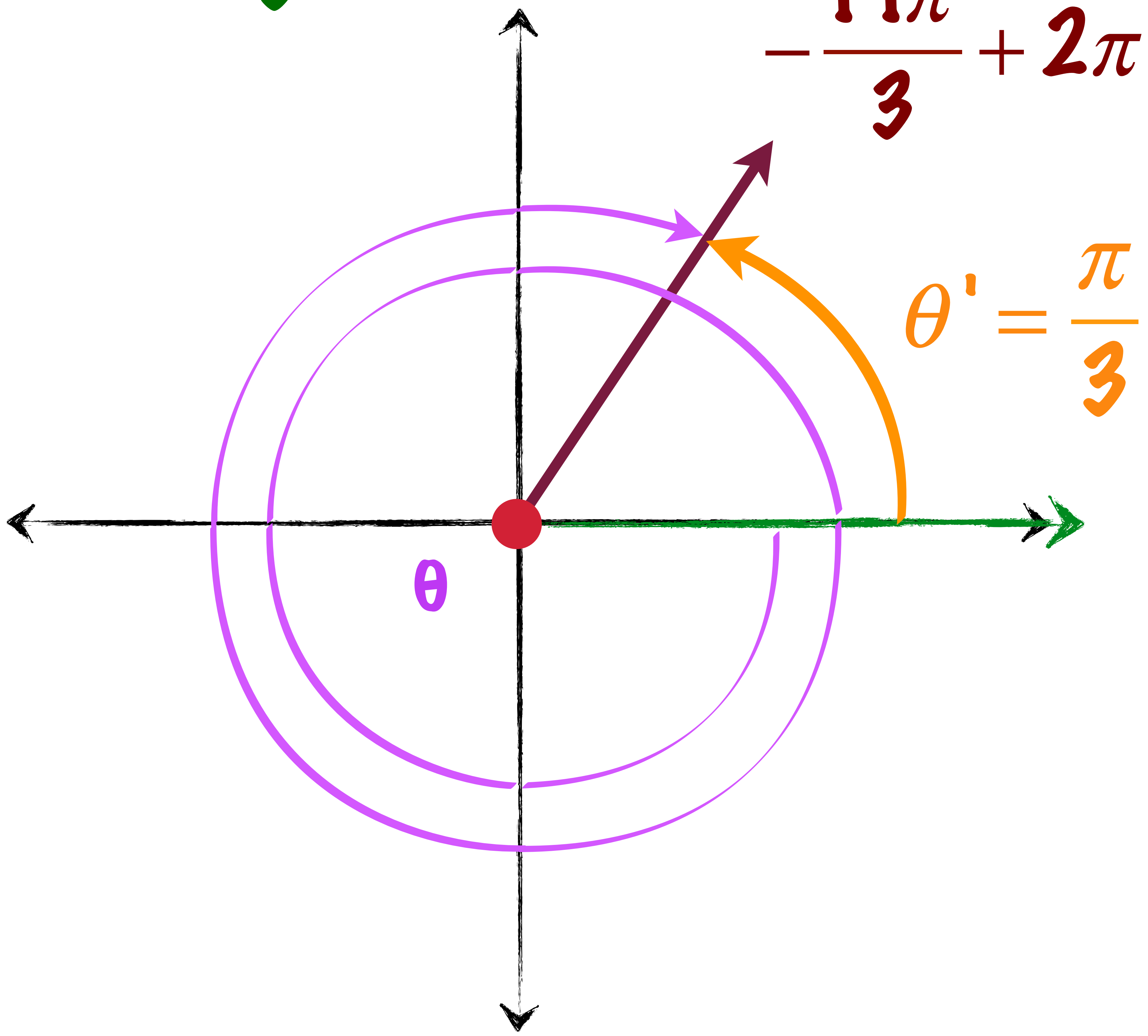
Objectives: Use the trig ratios for any measure angle. Find the reference angle.

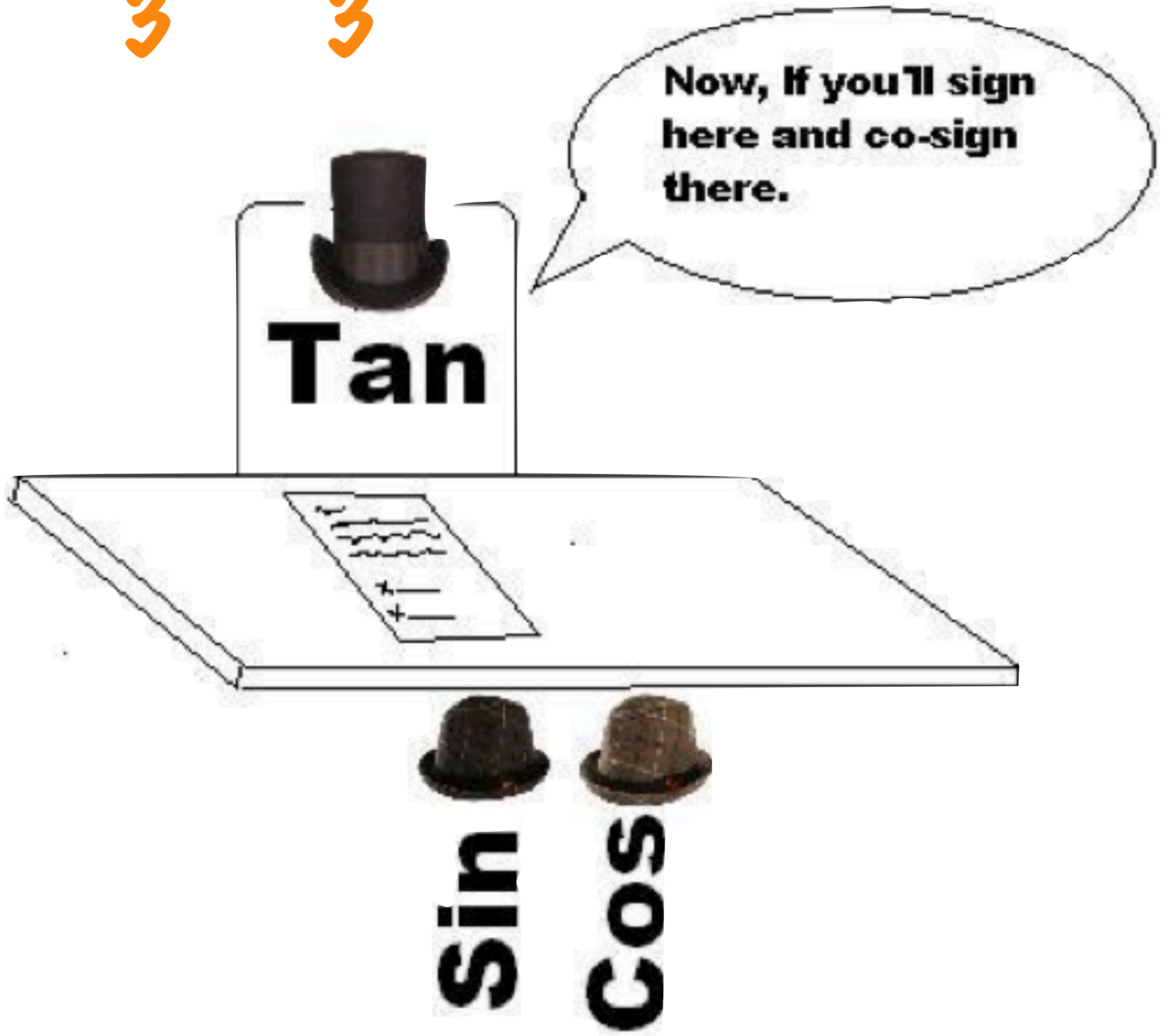
Find the reference angle θ' for each of the following angles:

c. $\theta = -\dfrac{11\pi}{3}$

$$-\frac{11\pi}{3} + 2\pi = -\frac{5\pi}{3}$$

$$\theta' = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$





Using Reference Angles to Evaluate Trigonometric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- The values of the trigonometric functions of a given angle, θ, are the same as the values of the trigonometric function of the reference angle, θ’, except, possibly, for the sign.
  - A function value of the acute reference angle, θ’, is always positive. However, the same function value for θ may be positive or negative.
    - To determine the sign of the function value for θ, simply determine the Quadrant in which θ falls.

A Procedure for Using Reference Angles to Evaluate Trigonometric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- In other words. To determine the value of a trigonometric function of any angle θ is found by:

1. Find the associated reference angle, θ', and the function value for θ'.

2. Use the quadrant in which θ lies to determine the appropriate **sign** for the function value of θ.

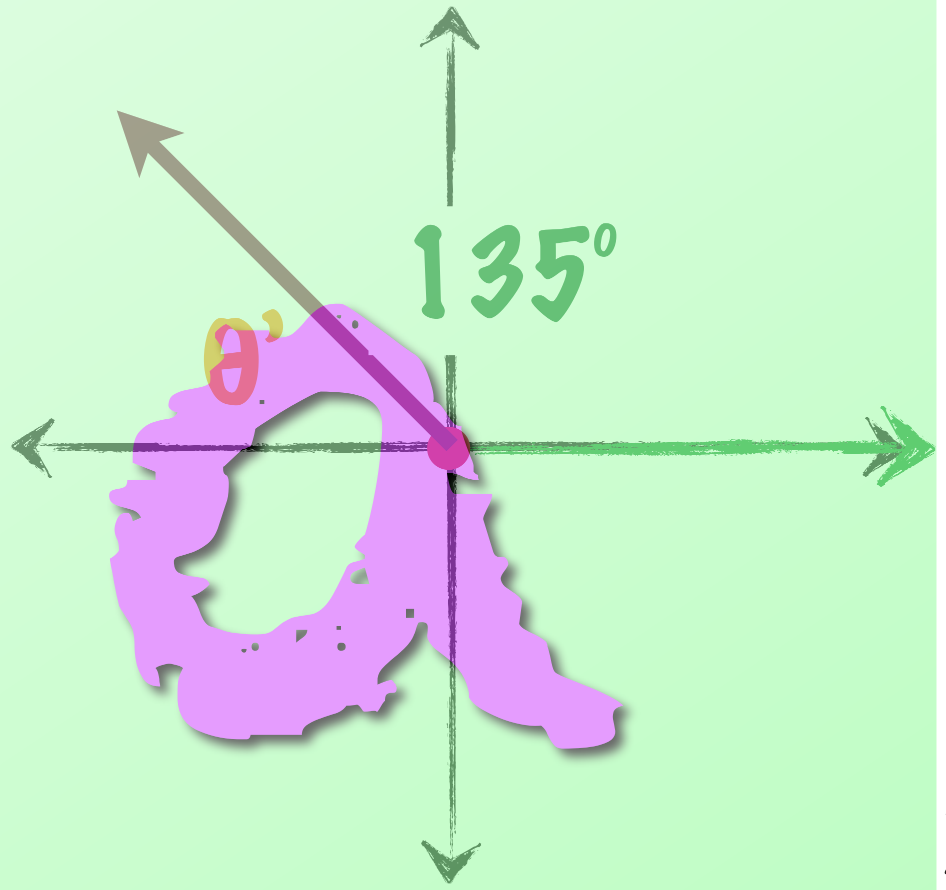
Using Reference Angles to Evaluate Trigonometric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Use reference angles to find the exact value of sin 135°.

  - Step 1 Find the reference angle, θ' and sinθ'.

    $\theta' = 180° - 135° = 45°$ $\qquad \sin 45° = \frac{\sqrt{2}}{2}$



Draw a Picture

  - Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value.

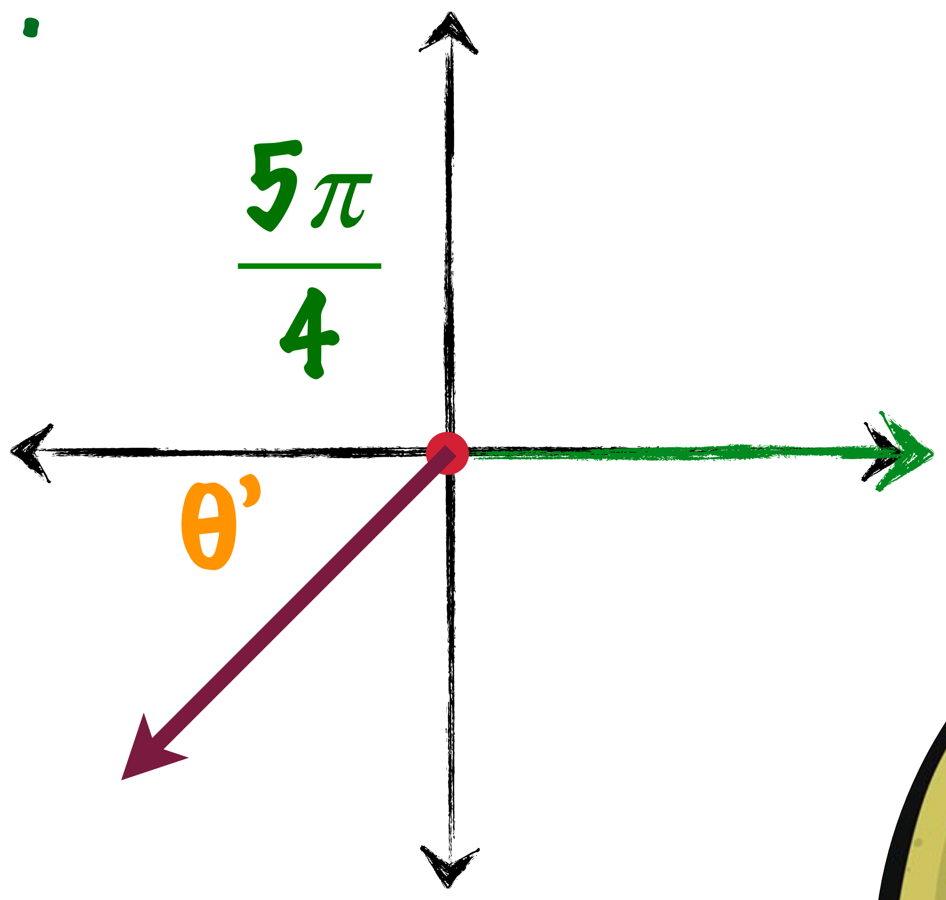
    135° is in QII, sin > 0 $\qquad \sin 135° = \frac{\sqrt{2}}{2}$

Using Reference Angles to Evaluate Trigonometric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Use reference angles to find the exact value of $\tan\frac{5\pi}{4}$.
  - Step 1 Find the reference angle, θ' and sinθ'.

$$\theta' = \frac{5\pi}{4} - \pi = \frac{\pi}{4} \qquad \tan\frac{\pi}{4} = 1$$



  - Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value.

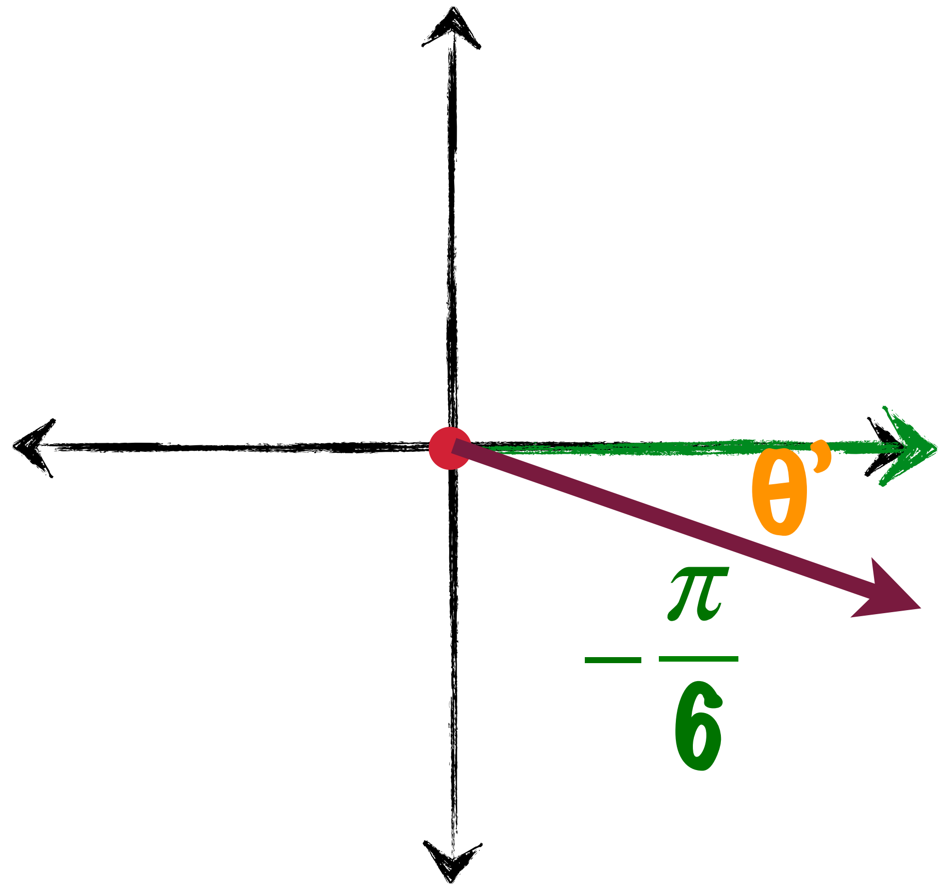
$$\frac{5\pi}{4} \text{ is in QII, } \tan > 0 \qquad \tan\frac{5\pi}{4} = 1$$

Using Reference Angles to Evaluate Trigonometric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Use reference angles to find the exact value of $\sec\left(-\dfrac{\pi}{6}\right)$.
  - Step 1 Find the reference angle, θ′ and sinθ′.

$$\theta' = \frac{\pi}{6} \qquad \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$



  - Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value.

$$-\frac{\pi}{6} \text{ is in QIV, } \sec > 0 \qquad \sec\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$$

Using Reference Angles to Evaluate Trigonometric Functions

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Evaluate $\cos\frac{13\pi}{3}\sec\frac{5\pi}{3}-\cot\frac{\pi}{3}$
  - Step 1 Find the reference angles, θ':

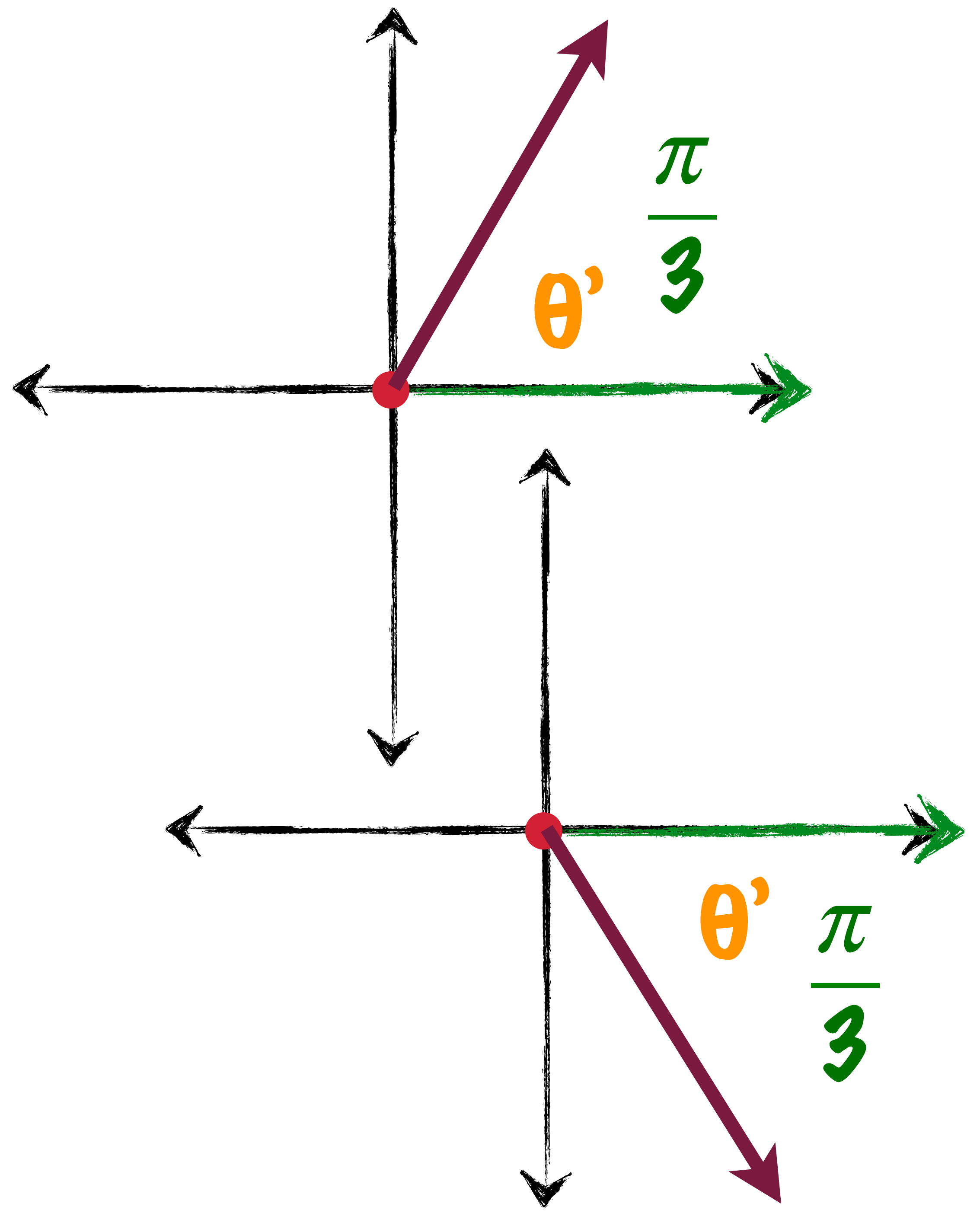
$$\theta'=\frac{13\pi}{3}-\frac{12\pi}{3}=\frac{\pi}{3}$$

$$\theta'=\frac{6\pi}{3}-\frac{5\pi}{3}=\frac{\pi}{3}$$

$$\cos\frac{13\pi}{3}=\frac{1}{2}$$

$$\sec\frac{5\pi}{3}=2$$

$$\cot\frac{\pi}{3}=\frac{1}{\sqrt{3}}$$



- Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value.

$$\cos\frac{13\pi}{3}\sec\frac{5\pi}{3}-\cot\frac{\pi}{3}=\frac{1}{2}\cdot\frac{2}{1}-\frac{1}{\sqrt{3}}=1-\frac{1}{\sqrt{3}}$$

Using Reference Angles to Evaluate Trigonometric Functions

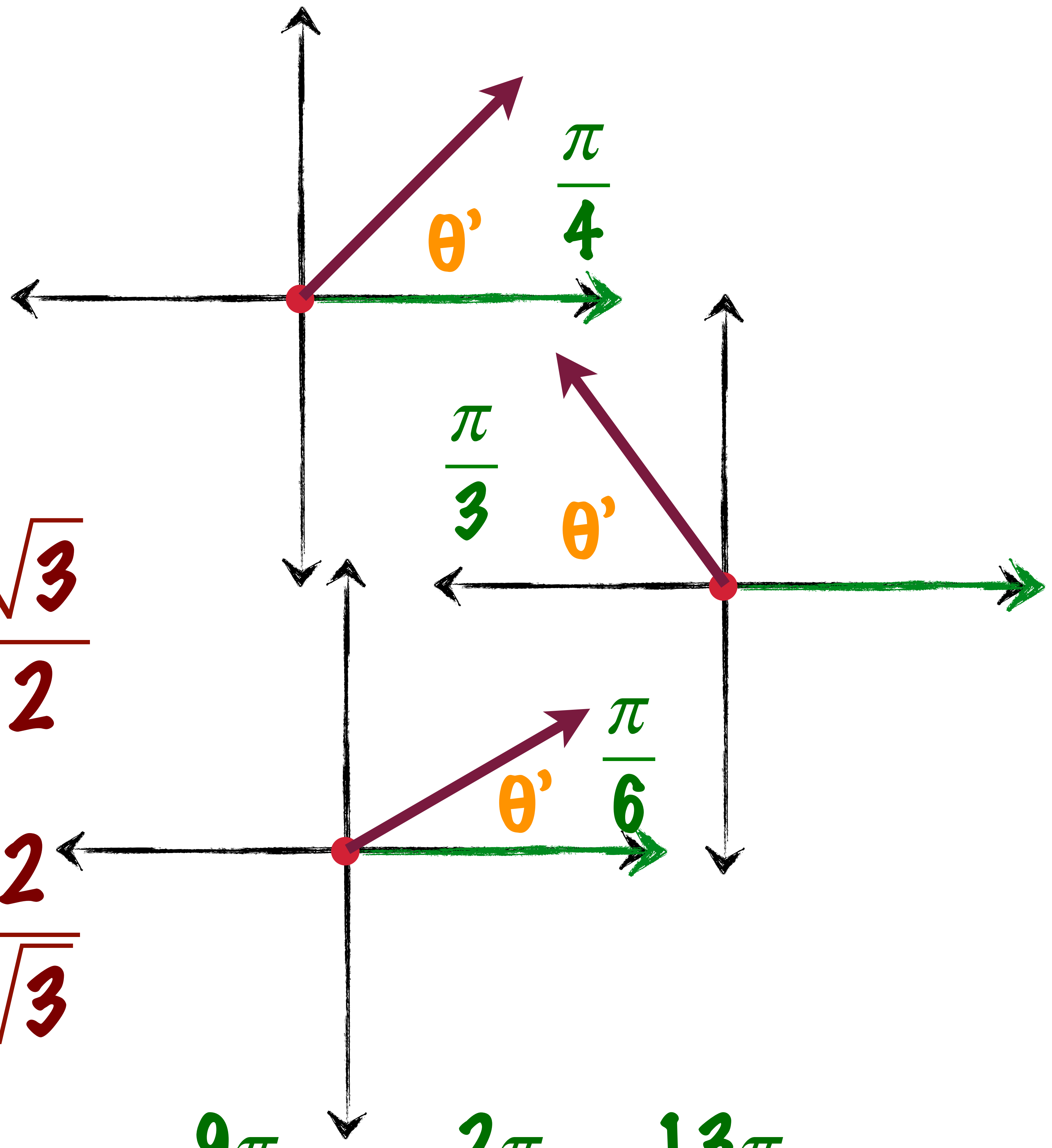
Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Evaluate $6\tan\frac{9\pi}{4}+\sin\frac{2\pi}{3}\sec\frac{13\pi}{6}$
  - Step 1 Find the reference angles, θ'

$$\theta'=\frac{9\pi}{4}-\frac{8\pi}{4}=\frac{\pi}{4} \qquad \tan\frac{9\pi}{4}=1$$

$$\theta'=\frac{3\pi}{3}-\frac{2\pi}{3}=\frac{\pi}{3} \qquad \sin\frac{2\pi}{3}=-\frac{\sqrt{3}}{2}$$

$$\theta'=\frac{13\pi}{6}-\frac{12\pi}{6}=\frac{\pi}{6} \qquad \sec\frac{13\pi}{6}=\frac{2}{\sqrt{3}}$$



- Step 2 Use the quadrant in which θ lies to prefix the appropriate sign to the function value.

$$6\tan\frac{9\pi}{4}+\sin\frac{2\pi}{3}\sec\frac{13\pi}{6}=6\cdot 1+\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{3}}=6+1=7$$

Finding the Measure of an Angle

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Let θ be an angle in the third quadrant such that cosθ = $-1/4$. Find θ, then find sinθ, and tanθ.

$$\cos\theta = \frac{x}{r} = -\frac{1}{4} \qquad x = -1,\ r = 4 \qquad y = -\sqrt{r^2 - x^2} = -\sqrt{4^2 - (-1)^2} = -\sqrt{15}$$

$$\sin\theta = -\frac{\sqrt{15}}{4} \qquad \text{(third quadrant)}$$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{15}}{1} \qquad \text{(third quadrant)}$$

- Now we get to find θ.

Finding the Measure of an Angle

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- Let θ be an angle in the third quadrant such that cosθ = −1/4. Find θ, then find sinθ, and tanθ.
  - To find the angle θ we must briefly meet some new friends, arccosine, arcsine, arctangent. Also known as inverse cosine ($\cos^{-1}$), inverse sine ($\sin^{-1}$), and inverse tangent ($\tan^{-1}$). These values are **angle measures** that give us the ratios.
  - The sine of an angle is a ratio, the inverse sine of a ratio give us an angle.

$$\sin\theta = \frac{y}{r} \qquad \sin^{-1}\frac{y}{r} = \theta$$

Finding the Measure of an Angle

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Let θ be an angle in the third quadrant such that cosθ = −1/4. Find θ, then find sinθ, and tanθ.
  - Fortunately, our calculator can do the heavy lifting, **BUT IT CANOT DO THE THINKING FOR YOU.**

$\cos^{-1}$

2nd | COS | (-) | 1 | ÷ | 4 |) = 1.823476582 radians

- But 1.8234 is not in the 3rd quadrant. To find the correct angle we need to find the reference angle.

$\pi - 1.8235 = 1.3181$ radians

- And add to π $\pi + 1.3181 = 4.4597$ radians

Finding the Measure of an Angle

Objectives: Use the trig ratios for any measure angle. Find the reference angle.

- Let θ be an angle in the third quadrant such that cosθ = −1/4. Find θ, then find sinθ, and tanθ.
  - Should you wish to know the angle in degrees you must set the calculator mode to degree.

MODE ⋎ RADIAN DEGREE 4.4597 2nd APPS 3: r ENTER 255.5220°

- Now we can find the rest of the values

$$\sin\theta = -\frac{\sqrt{15}}{4}$$

$$\tan\theta = \frac{-\sqrt{15}}{-1} = \sqrt{15}$$

Verify your results