Chapter 5

Analytic Trigonometry

5.2 Verifying Trigonometric Identities

Verifying Identities

Homework

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Verifying Identities

Objective

Verify trigonometric identities.

Fundamental Trigonometric Identities

- Once again, verifying Trig Identities starts with:

$$\sin^2 x + \cos^2 x = 1$$

When you are not sure what to do, and you are about to give up, one first step that is often useful is to convert everything to sine and cosine.

Fundamental Trigonometric Identities

Reciprocal Identities

$\csc x = \dfrac{1}{\sin x}$ $\quad \sin x = \dfrac{1}{\csc x}$

$\sec x = \dfrac{1}{\cos x}$ $\quad \cos x = \dfrac{1}{\sec x}$

$\cot x = \dfrac{1}{\tan x}$ $\quad \tan x = \dfrac{1}{\cot x}$

Quotient Identities

$\tan x = \dfrac{\sin x}{\cos x}$ $\quad \cot x = \dfrac{\cos x}{\sin x}$

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$1 - \cos^2 x = \sin^2 x$ $\quad 1 + \tan^2 x = \sec^2 x$

$1 - \sin^2 x = \cos^2 x$ $\quad 1 + \cot^2 x = \csc^2 x$

Co-function Identities

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$

$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x$

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

Even

$\cos(-x) = \cos x$

Odd

$\sin(-x) = -\sin x$

$\tan(-x) = -\tan x$

Verifying Trigonometric Identities

Using Fundamental Identities to Verify Other Identities

To **verify an identity**, we show that one side of the identity can be simplified so that it is **identical** to the other side.

There is no one single path to verifying an identity. This is your opportunity to show creativity in finding the steps to verifying a trigonometric identity.

Each side of the equation is manipulated independently of the other side of the equation.

Start with the side containing the more complicated expression.

Work with only ONE side of the identity but keep in mind the goal.

Substitute fundamental identities until you can rewrite the original expression in a form identical to the other side.

Verifying Trigonometric Identities

Study Tip:

Verifying an identity is different from solving an equation. You do not use the properties of equality to verify an identity. Each side is **manipulated independently** until you get a match.

You cannot be sure an identity is actually an identity until that identity has been verified.

When solving an equation we are finding values of a variable that make the equation a true statement. When verifying an identity, we are stating that the statement is true for **all** values in the domain of the function.

STUDY TIP

Remember that an identity is only true for all real values in the domain of the variable. For instance, in Example 1 the identity is not true when $\theta = \pi/2$ because $\sec^2 \theta$ is not defined when $\theta = \pi/2$.

Verifying Trigonometric Identities

When working with these identities, do not get lazy (as I do on occasion) and omit the argument (x or θ). Do not write sintan when you mean to write sin x tan x. The functions are meaningless without an argument (variable or angle measure).

You would not write $5\sqrt{} = 42$ because it makes no sense. Similarly, you do not write 5sin = 42.

Absolutely, positively never distribute or cancel a trig designation.

$\sin(x + 2) \neq \sin x + \sin 2$; $\dfrac{\sin x}{\sin y} \neq \dfrac{x}{y}$

The designations for the trig ratios are functions and have no meaning without an argument.

From Your Text

Guidelines for Verifying Trigonometric Identities

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.

2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.

3. Look for opportunities to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.

4. If the preceding guidelines do not help, try converting all terms to sines and cosines.

5. Always try *something*. Even paths that lead to dead ends provide insights.

Verifying Trigonometric Identities

We can start by simplifying a trigonometric expression by substitution.

Verify: $\cos x(\sec x - \cos x) = \sin^2 x$

$$\cos x(\sec x - \cos x) = \cos x\left(\frac{1}{\cos x} - \cos x\right)$$

$$\sec x = \frac{1}{\cos x}$$

$$= \frac{\cos x}{\cos x} - \cos^2 x$$

$$= 1 - \cos^2 x$$

$$= \sin^2 x$$

$$1 - \cos^2 x = \sin^2 x$$

Verifying Trigonometric Identities

Verify: $\csc x \tan x = \sec x$

$$\csc x \tan x = \sec x$$

$$\csc x \tan x = \frac{1}{\sin x} \bullet \frac{\sin x}{\cos x}$$

$$= \frac{1}{\cos x}$$

$$= \sec x$$

$$\csc x = \frac{1}{\sin x}$$

$$\tan x = \frac{\sin x}{\cos x}$$

The identity be verified.

Verifying Trigonometric Identities

Verify: $\cot^2 x + \cos^2 x + \sin^2 x = \csc^2 x$

$$\cot^2 x + \cos^2 x + \sin^2 x = \cot^2 x + 1$$

$$= \csc^2 x$$

$\cos^2 x + \sin^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

Example: Using Factoring to Verify an Identity

Verify: $\sin x - \sin x \cos^2 x = \sin^3 x$

Factor sin x from the two terms.

$$\sin x - \sin x \cos^2 x = \sin x(1 - \cos^2 x)$$

$$= \sin x(\sin^2 x)$$

$1 - \cos^2 x = \sin^2 x$

$$= \sin^3 x$$

Verifying Trigonometric Identities

Verify: $\dfrac{1}{\sec x - 1} - \dfrac{1}{\sec x + 1} = 2\cot^2 x$

$$\frac{1}{\sec x - 1} - \frac{1}{\sec x + 1} = \frac{\sec x + 1}{(\sec x - 1)(\sec x + 1)} - \frac{\sec x - 1}{(\sec x - 1)(\sec x + 1)}$$

Common denominator

$$= \frac{2}{\sec^2 x - 1}$$

$$= \frac{2}{\tan^2 x}$$

$\tan^2 x = \sec^2 x - 1$

$$= 2\cot^2 x$$

Combining Fractional Expressions to Verify an Identity

Verify: $\dfrac{\sin x}{1+\cos x}+\dfrac{1+\cos x}{\sin x} = 2\csc x$

$$\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x} = \frac{\sin^2 x}{\sin x(1+\cos x)}+\frac{(1+\cos x)(1+\cos x)}{\sin x(1+\cos x)}$$

Common denominator

$$= \frac{\sin^2 x}{\sin x(1+\cos x)}+\frac{(1+2\cos x+\cos^2 x)}{\sin x(1+\cos x)}$$

$$= \frac{\sin^2 x+1+2\cos x+\cos^2 x}{\sin x(1+\cos x)}$$

Verifying Trigonometric Identities

Combining Fractional Expressions to Verify an Identity

Verify: $$\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x} = 2\csc x$$

$$\frac{\sin x}{1+\cos x}-\frac{1+\cos x}{\sin x} = \frac{\sin^2 x+1+2\cos x+\cos^2 x}{\sin x\left(1+\cos x\right)}$$

$$= \frac{\sin^2 x+\cos^2 x+1+2\cos x}{\sin x\left(1+\cos x\right)}$$

$$= \frac{1+1+2\cos x}{\sin x\left(1+\cos x\right)} = \frac{2+2\cos x}{\sin x\left(1+\cos x\right)}$$

$$= \frac{2(1+\cos x)}{\sin x\left(1+\cos x\right)} = \frac{2}{\sin x} = 2\csc x$$

$$\sin^2 x+\cos^2 x = 1$$

We did it.

Verifying Trigonometric Identities

Example: Working with Both Sides Separately

Verify: $\dfrac{1}{1+\sin\theta}+\dfrac{1}{1-\sin\theta} = 2+2\tan^2\theta$

$$\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta} = \frac{(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}+\frac{(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$$

Common denominator

$$= \frac{(1-\sin\theta)+(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} = \frac{2}{1-\sin^2\theta} = \frac{2}{\cos^2\theta}$$

$$= 2\sec^2\theta \quad = 2(1+\tan^2\theta) \quad = 2+2\tan^2\theta$$

$1+\tan^2 x = \sec^2 x$

Verifying Trigonometric Identities

Verify: $\dfrac{1+\cos\theta}{\sin\theta} = \csc\theta + \cot\theta$

$$\frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \csc\theta + \cot\theta$$

$$\frac{1}{\sin\theta} = \csc\theta$$

$$\frac{\cos\theta}{\sin\theta} = \cot\theta$$

Verifying Trigonometric Identities

Verify: $$\frac{\sec x + \csc(-x)}{\sec x \csc x} = \sin x - \cos x$$

$$\frac{\sec x + \csc(-x)}{\sec x \csc x} = \frac{\sec x + -\csc(x)}{\sec x \csc x}$$

$$\csc(-x) = -\csc x$$

$$= \frac{\sec x}{\sec x \csc x} - \frac{\csc x}{\sec x \csc x}$$

$$\cos x = \frac{1}{\sec x}$$

$$\sin x = \frac{1}{\csc x}$$

$$= \frac{1}{\csc x} - \frac{1}{\sec x}$$

$$= \sin x - \cos x$$

Verifying Trigonometric Identities

Verify: $\left(1+\cot^2 x\right)\left(1-\sin^2 x\right)=\cot^2 x$

$$\left(1+\cot^2 x\right)\left(1-\sin^2 x\right)=\left(\csc^2 x\right)\left(\cos^2 x\right)$$

$$=\left(\frac{1}{\sin^2 x}\right)\left(\cos^2 x\right)$$

$$=\frac{\cos^2 x}{\sin^2 x}$$

$$=\cot^2 x$$

$$\left(1+\cot^2 x\right)=\csc^2 x$$

$$\left(1-\sin^2 x\right)=\cos^2 x$$

$$\csc^2 x=\frac{1}{\sin^2 x}$$

Verifying Trigonometric Identities

Verify: $\sec u + \tan u = \dfrac{1}{\sec u - \tan u}$

$$\sec u + \tan u = \frac{1}{\cos u} + \frac{\sin u}{\cos u} = \frac{1 + \sin u}{\cos u} = \frac{1 + \sin u}{\cos u}\left(\frac{1 - \sin u}{1 - \sin u}\right) = \frac{1 - \sin^2 u}{\cos u\left(1 - \sin u\right)}$$

$$= \frac{\cos^2 u}{\cos u\left(1 - \sin u\right)} = \frac{\cos u}{1 - \sin u} = \frac{\cos u}{1 - \sin u}\left(\frac{\sec u}{\sec u}\right) = \frac{1}{\sec u - \sin u \sec u}$$

$$= \frac{1}{\sec u - \sin u \dfrac{1}{\cos u}} = \frac{1}{\sec u - \dfrac{\sin u}{\cos u}} = \frac{1}{\sec u - \tan u}$$

Verifying Trigonometric Identities

Verify: $\cot t \cos t = \csc t - \sin t$

$$\cot t \cos t = \frac{\cos t}{\sin t}\cos t = \frac{\cos^2 t}{\sin t} = \frac{1-\sin^2 t}{\sin t} = \frac{1}{\sin t} - \frac{\sin^2 t}{\sin t} = \csc t - \sin t$$