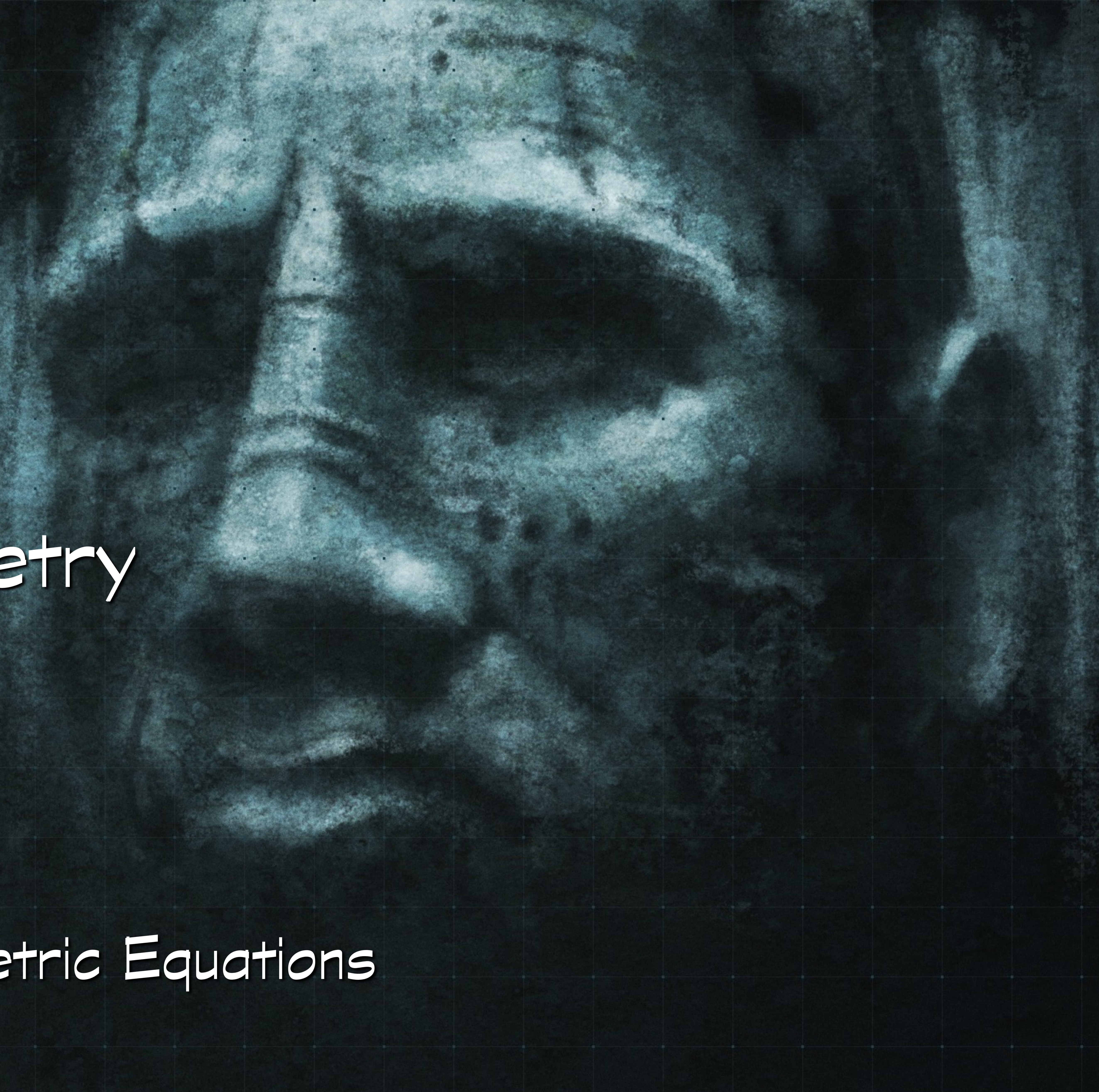
Chapter 5



- Analytic Trigonometry

5.3 Solving Trigonometric Equations

Homework

- Read Sec 5.3, complete notes
- Do p397 1-61 odd, 73, 75

Objectives:

- Find all solutions of a trigonometric equation.
- Solve equations with multiple angles.
- Solve trigonometric equations quadratic in form.
- Use factoring to separate different functions in trigonometric equations.
- Use identities to solve trigonometric equations.
- Use a calculator to solve trigonometric equations.

Trigonometric Equations and Their Solutions

A **trigonometric equation** is an equation that contains a trigonometric expression with a variable, such as sin x.

The values that satisfy such an equation are its **solutions**.

(There are trigonometric equations that have no solution.)

When an equation includes multiple angles, the period of the function plays an important role in ensuring that we do not leave out any solutions.

As with any function, the first step should be to isolate the function when possible.

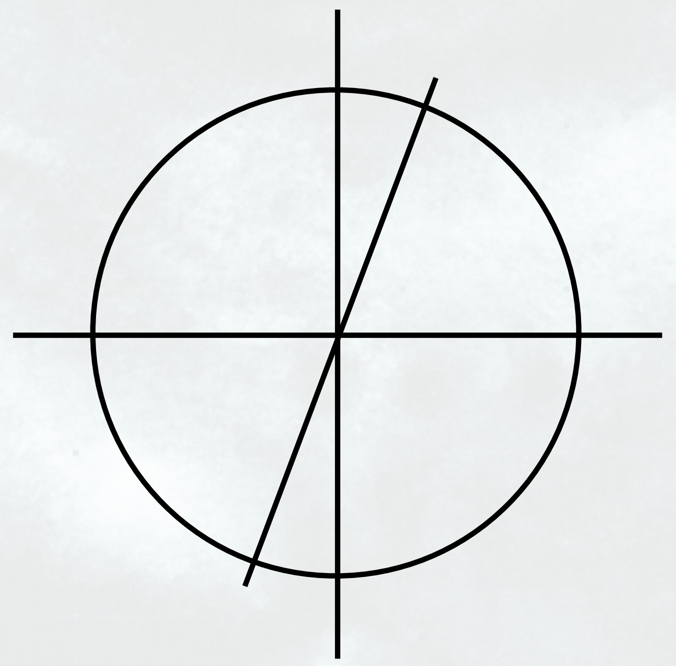
It will be the rare occasion when you will find a single solution!

Solving Trigonometric Equations with a Calculator

Solve the equation: $\tan x = 3.1044$ on the interval $[0,2\pi)$.

$\tan^{-1} 3.1044 \approx 1.259168376$

tan has period π, thus repeats every π

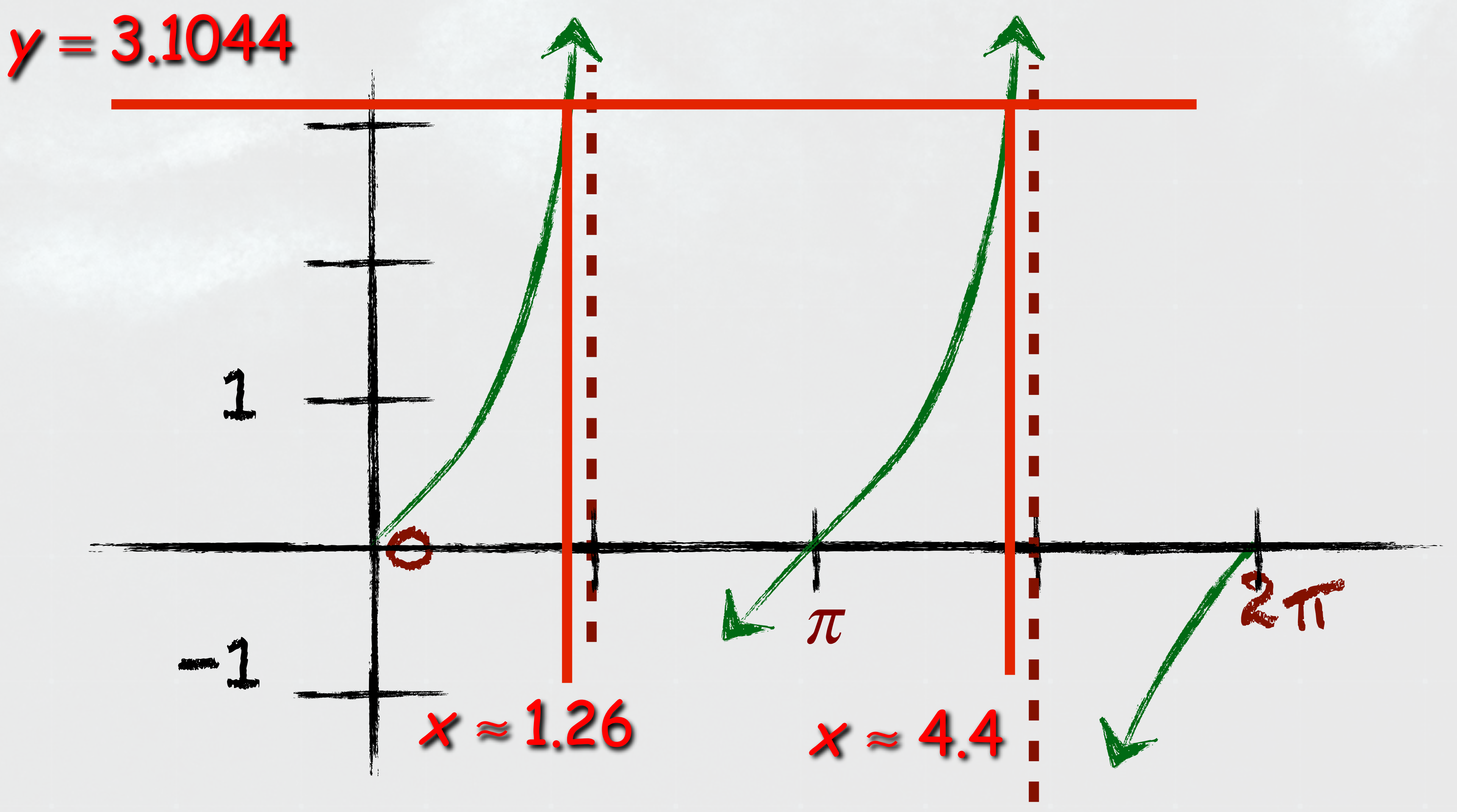


tan is also positive in QIII

$x \approx 1.259168376 + \pi$

$x \approx 4.400761029$

$x \approx 1.2592, 4.4008$

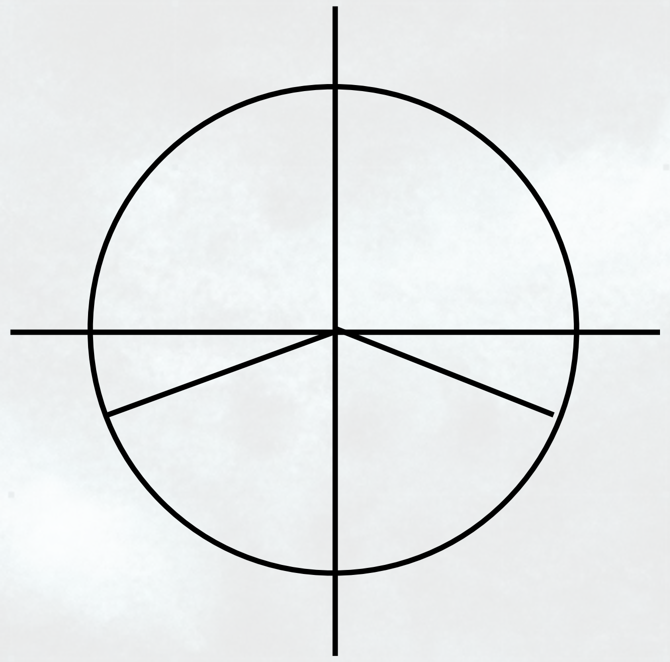


Solving Trigonometric Equations with a Calculator

Solve the equation: $\sin x = -0.2315, 0 \le x < 2\pi$

Using the calculator gives us:

$$\sin^{-1}(-0.2315) = -0.233619286$$

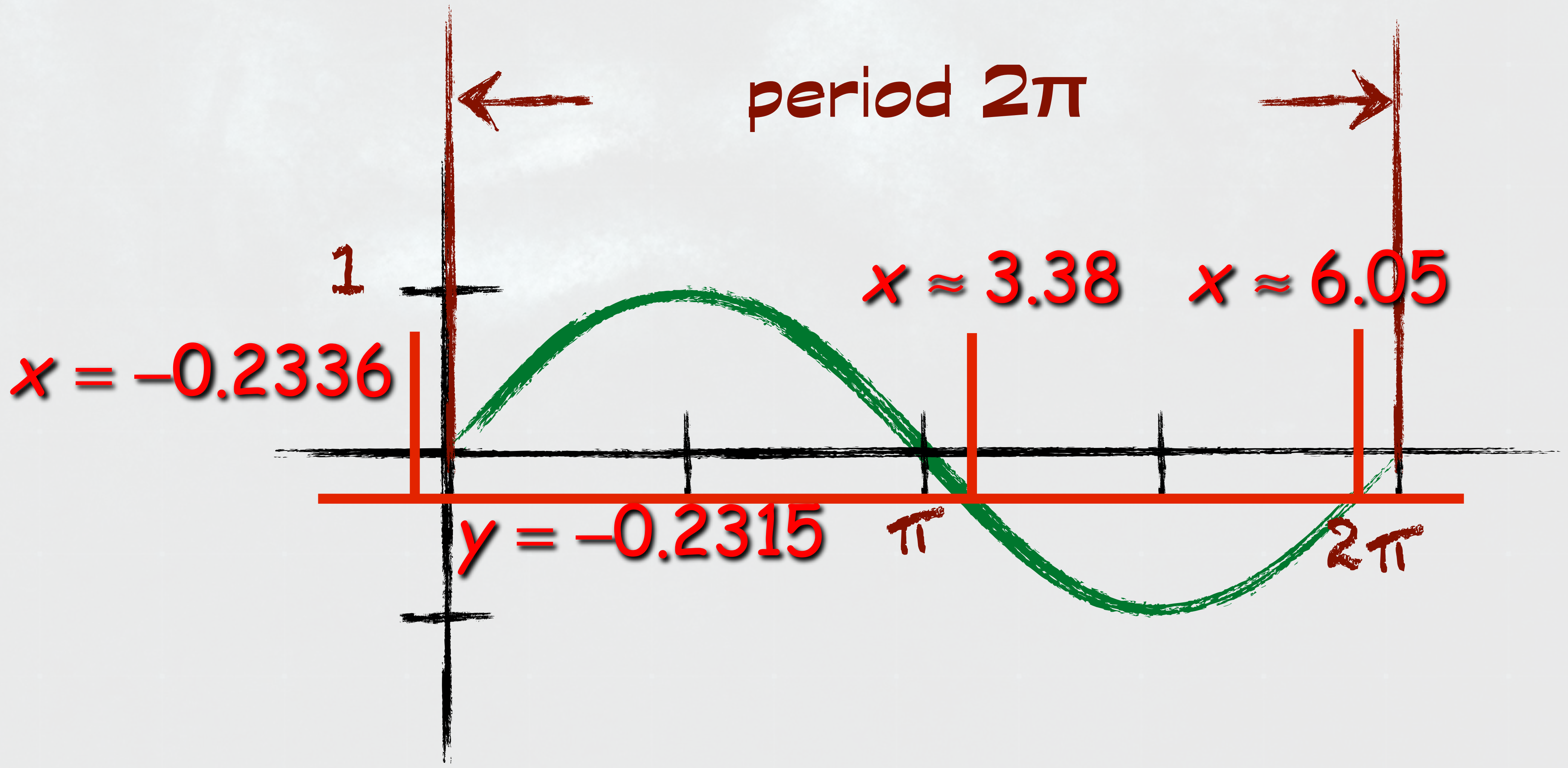


sinx is negative in QIII & QIV

$$-0.233619286 + 2\pi = 6.049566021$$

$$0.233619286 + \pi = 3.37521194$$

$$x = 3.3752, 6.0496$$



Finding all Solutions of a Trigonometric Equation

Solve the equation: $5 \sin x = 3 \sin x + \sqrt{3}$

Step 1 Isolate the **function** on one side of the equation.

$$5\sin x = 3\sin x + \sqrt{3}$$

$$5\sin x - 3\sin x = \sqrt{3}$$

$$2\sin x = \sqrt{3}$$

$$\sin x = \frac{\sqrt{3}}{2}$$

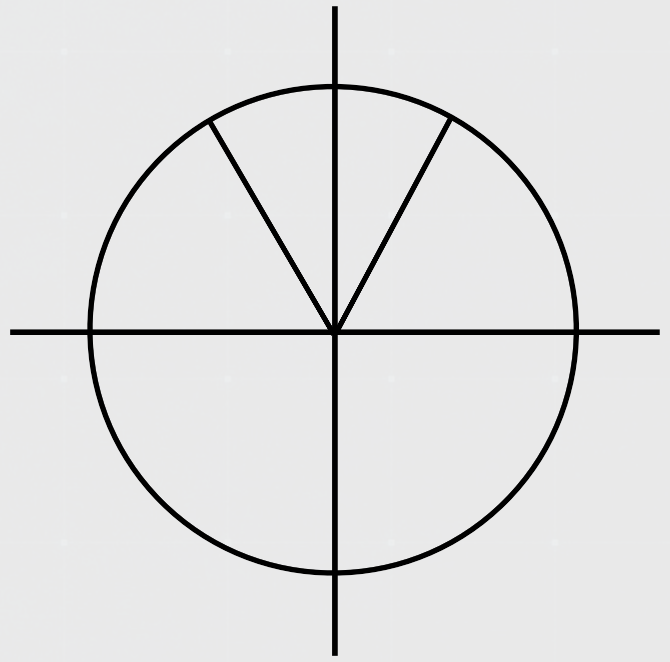
Step 2 Solve for x

Solutions for this equation in [0, 2π) are:

$$\frac{\pi}{3} \text{ and } \frac{2\pi}{3}$$

But we are not limited to [0, 2π), and the period is 2π so:

$$x = \left(\frac{\pi}{3}\right) + n2\pi, \left(\frac{2\pi}{3}\right) + n2\pi$$

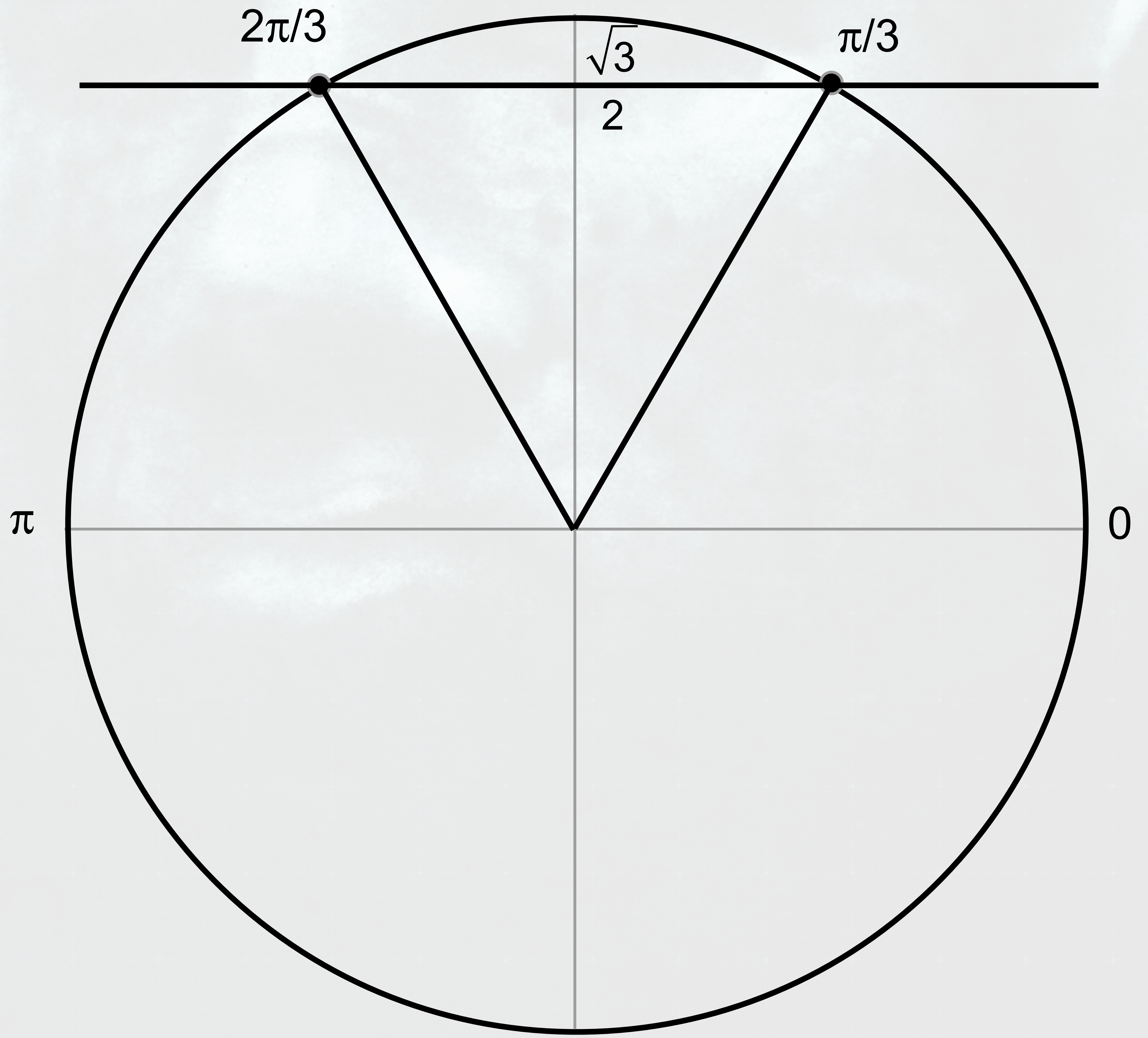


Unit Circle Representation

Solve the equation: $\mathbf{5\sin x = 3\sin x + \sqrt{3}}$

$$\sin x = \frac{\sqrt{3}}{2}$$

$$x = \left(\frac{\pi}{3}\right) + n2\pi,\ \left(\frac{2\pi}{3}\right) + n2\pi$$



TI-84

Solve the equation: $5\sin x = 3\sin x + \sqrt{3}$ $\sin x = \frac{\sqrt{3}}{2}$

Graph two equations: $y = \sin x$ and $y = \frac{\sqrt{3}}{2}$

To better view your results set the window parameters:

Xmin=-7
Xmax=7
Xscl=1
Ymin=-1.5
Ymax=1.5
Yscl=1

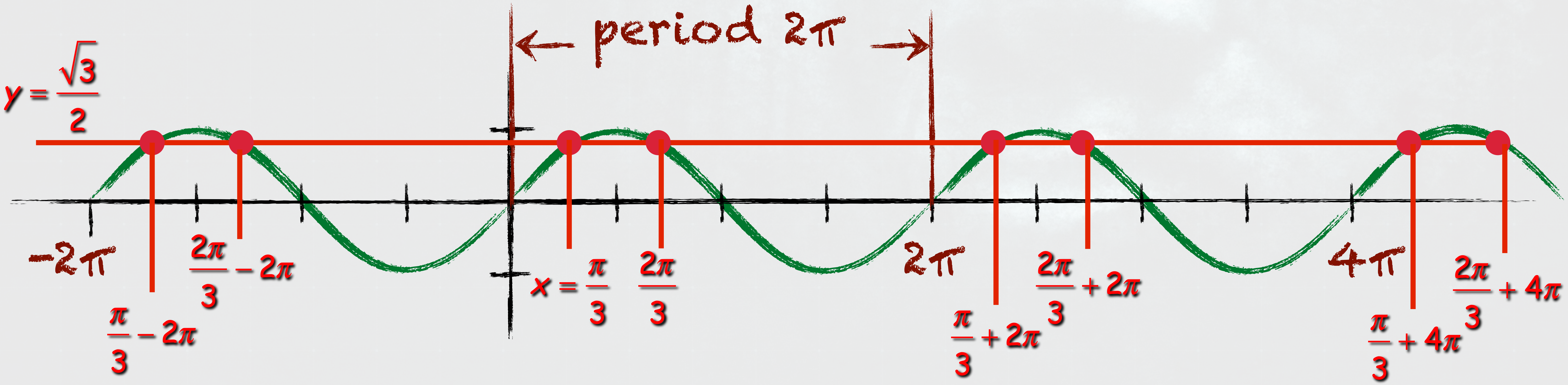
Using the intersect function, note where the graphs intersect.

Please keep in mind that I will NOT accept approximations when the exact solution is available (π/3).

Graphical Representation

Solve the equation: $\mathbf{5\sin x = 3\sin x + \sqrt{3}}$

$$\sin x = \frac{\sqrt{3}}{2} \qquad x = \left(\frac{\pi}{3}\right) + n2\pi,\ \left(\frac{2\pi}{3}\right) + n2\pi$$



Solving an Equation with a Multiple Angle

Objective: Solving Trigonometric Equations

Solve the equation: $\tan 2x = \sqrt{3},\ 0 \le x < 2\pi$ **Note the restriction on the domain.**

Step 1 $\tan 2x = \sqrt{3}$

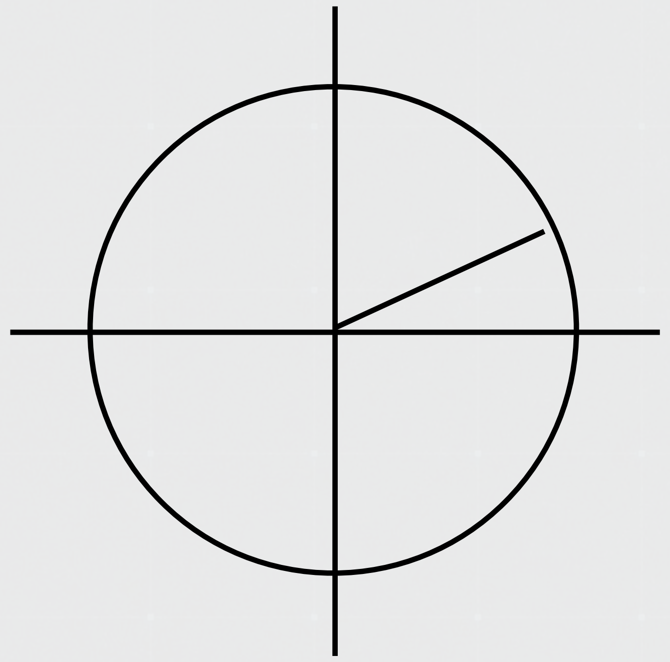
$\tan^{-1}\sqrt{3} = 2x \qquad 2x = \frac{\pi}{3}$

The period for tanx is **π**, so

$$2x = \frac{\pi}{3} + n\pi$$

But we are not looking for 2x

$$x = \frac{\pi}{6} + \frac{n\pi}{2}$$



Step 2 Solve for x

We are looking for solutions from 0 ≤ x < 2π,

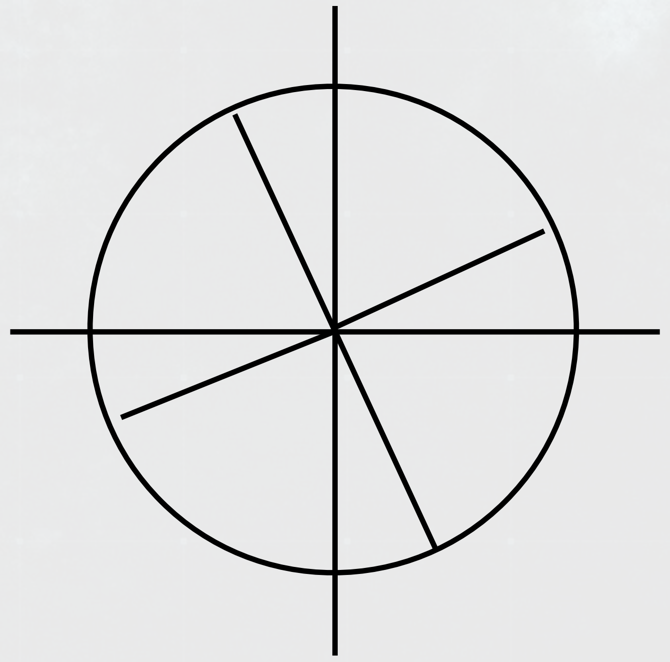
$$x = \frac{\pi}{6} + \frac{0\pi}{2} = \frac{\pi}{6}$$

$$x = \frac{\pi}{6} + \frac{1\pi}{2} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$x = \frac{\pi}{6} + \frac{2\pi}{2} = \frac{7\pi}{6}$$

$$x = \frac{\pi}{6} + \frac{3\pi}{2} = \frac{10\pi}{6} = \frac{5\pi}{3}$$

$$x = \frac{\pi}{6} + \frac{4\pi}{2} = \frac{13\pi}{6}$$ Oops, too big



Graphical Representation

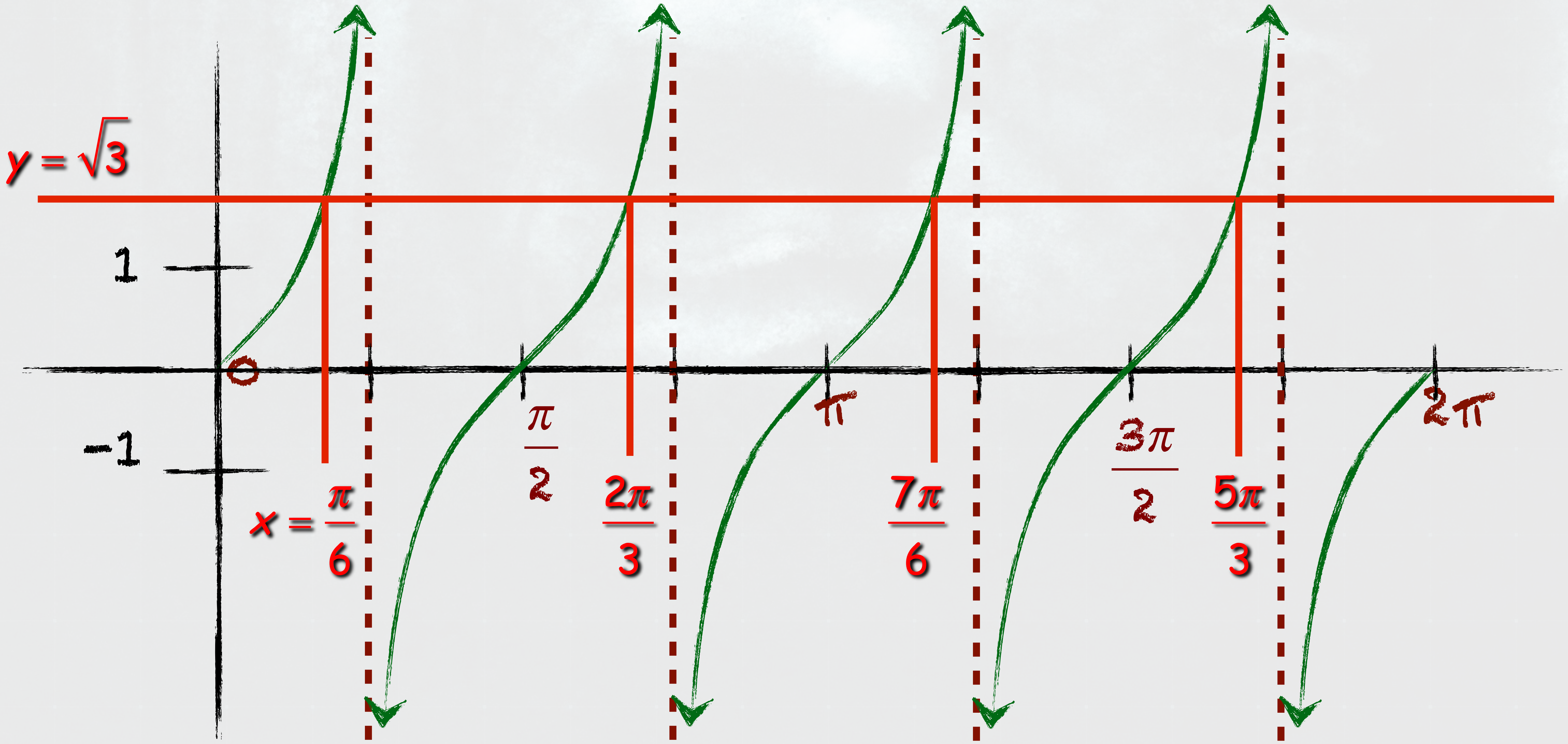
Solve the equation: $\tan 2x = \sqrt{3},\ 0 \le x < 2\pi$

Remember: We are solving for x, but graphing **tan 2x**.

What is the period of tan 2x? **The new period is π/2**

$$x = \frac{\pi}{6} + \frac{n\pi}{2}$$

$$\frac{\pi}{6} \quad \frac{2\pi}{3} \quad \frac{7\pi}{6} \quad \frac{5\pi}{3}$$

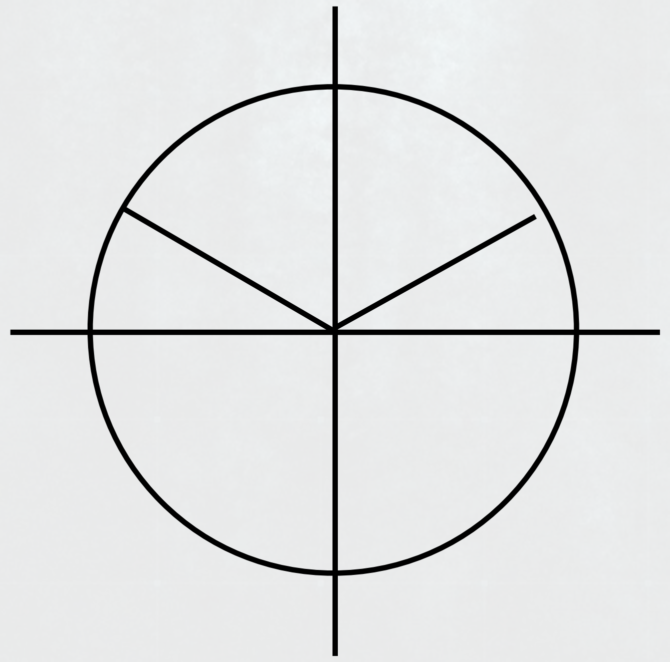


Solving Equations with a Single Trigonometric Function

Solve the equation: $2\sin\frac{x}{2} = 1$ on the interval $[0,2\pi)$.

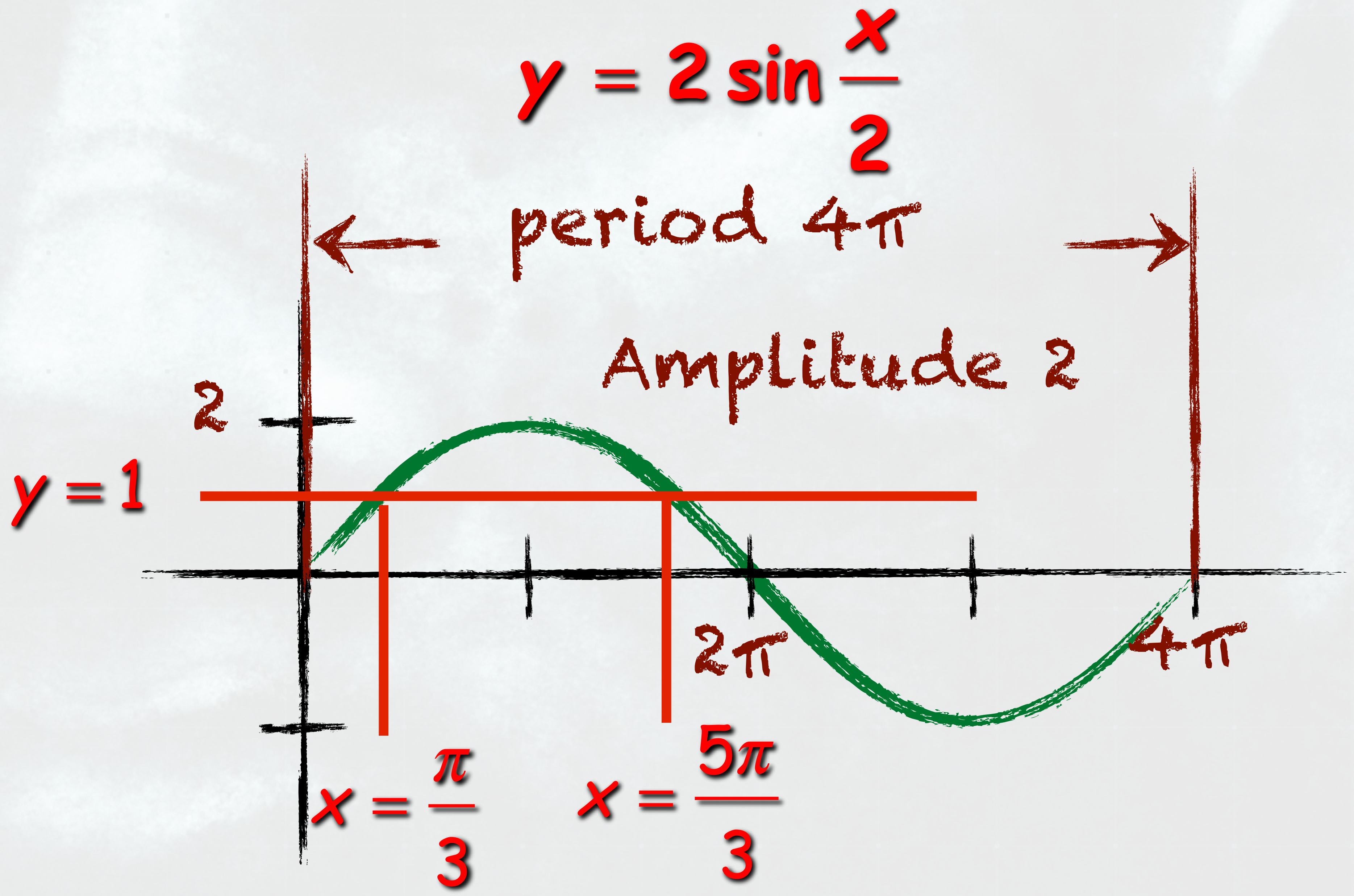
Graph $y = 2\sin\frac{x}{2}$ and $y = 1$ on TI-84

$\sin\frac{x}{2} = \frac{1}{2}$ $\quad \sin\frac{\pi}{6} = \frac{1}{2}$ $\quad \sin\frac{5\pi}{6} = \frac{1}{2}$



$\frac{x}{2} = \frac{\pi}{6} + 2\pi n$ $\quad \frac{x}{2} = \frac{5\pi}{6} + 2\pi n$

$x = \frac{\pi}{3} + 4\pi n$ $\quad x = \frac{5\pi}{3} + 4\pi n$



The only values within the restricted domain are:

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Solving an Equation with a Multiple Angle

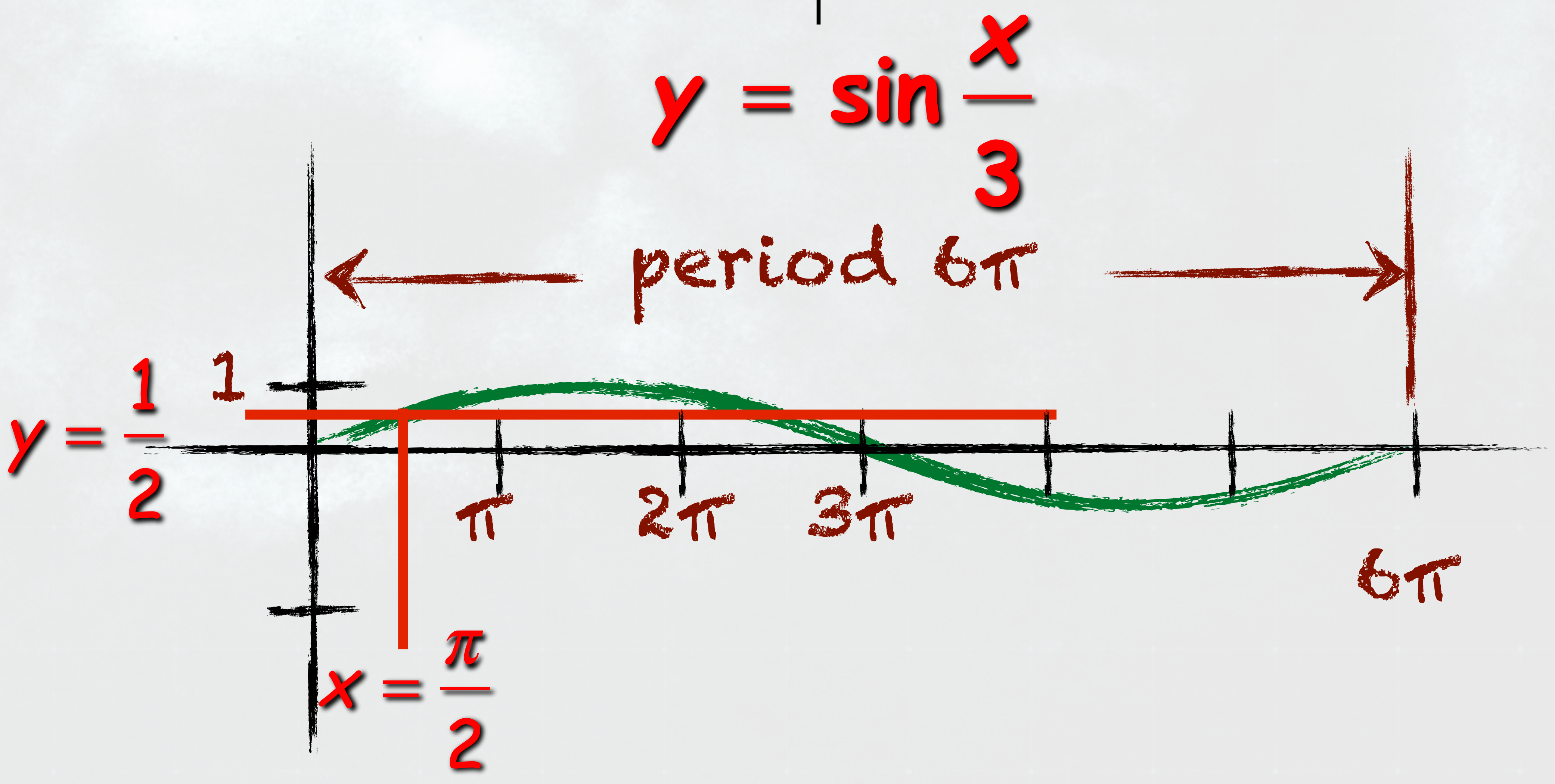
Solve the equation: $\sin\frac{x}{3}=\frac{1}{2},\ 0 \le x < 2\pi$

$$\sin\frac{x}{3}=\frac{1}{2}$$

$$\sin\frac{\pi}{6}=\frac{1}{2} \qquad \sin\frac{5\pi}{6}=\frac{1}{2}$$

$$\frac{x}{3}=\frac{\pi}{6}+2\pi n \qquad \frac{x}{3}=\frac{5\pi}{6}+2\pi n$$

$$x=\frac{\pi}{2}+6\pi n \qquad x=\frac{5\pi}{2}+6\pi n$$



The only values within the restricted domain are: $x=\frac{\pi}{2}$

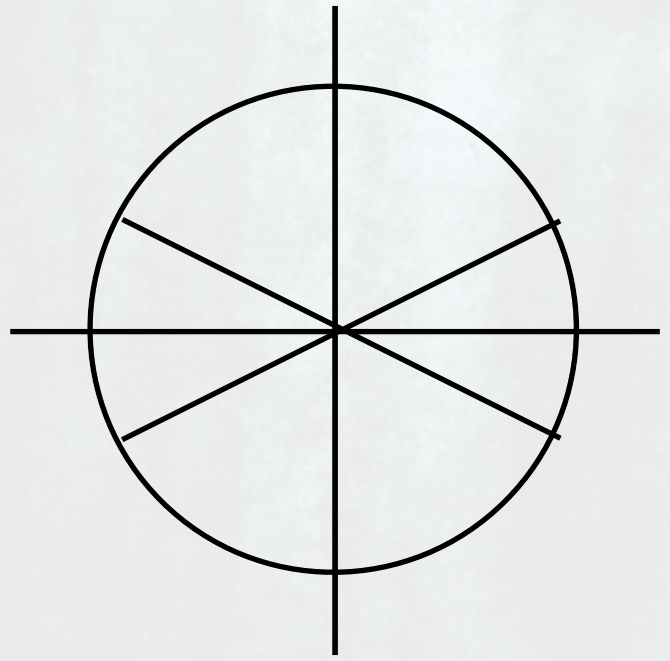
Solving a Trigonometric Equation Quadratic in Form

Solve the equation: $4\cos^2 x - 3 = 0$ on the interval $[0,2\pi)$.

$$4\cos^2 x - 3 = 0$$

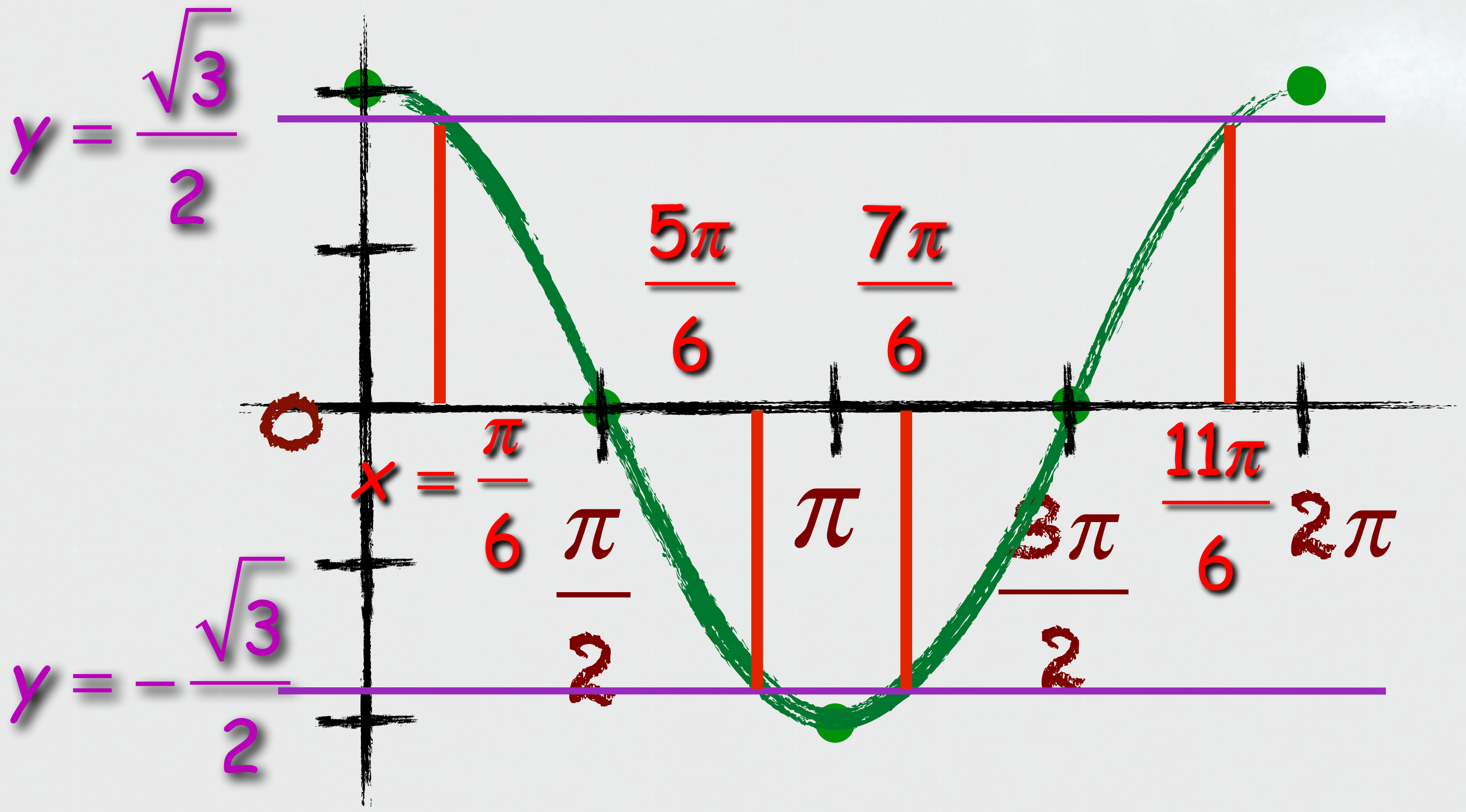
$$\cos^2 x = \frac{3}{4}$$

$$\cos x = \pm\frac{\sqrt{3}}{2}$$



$$\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6} \qquad \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6} \qquad \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$



$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

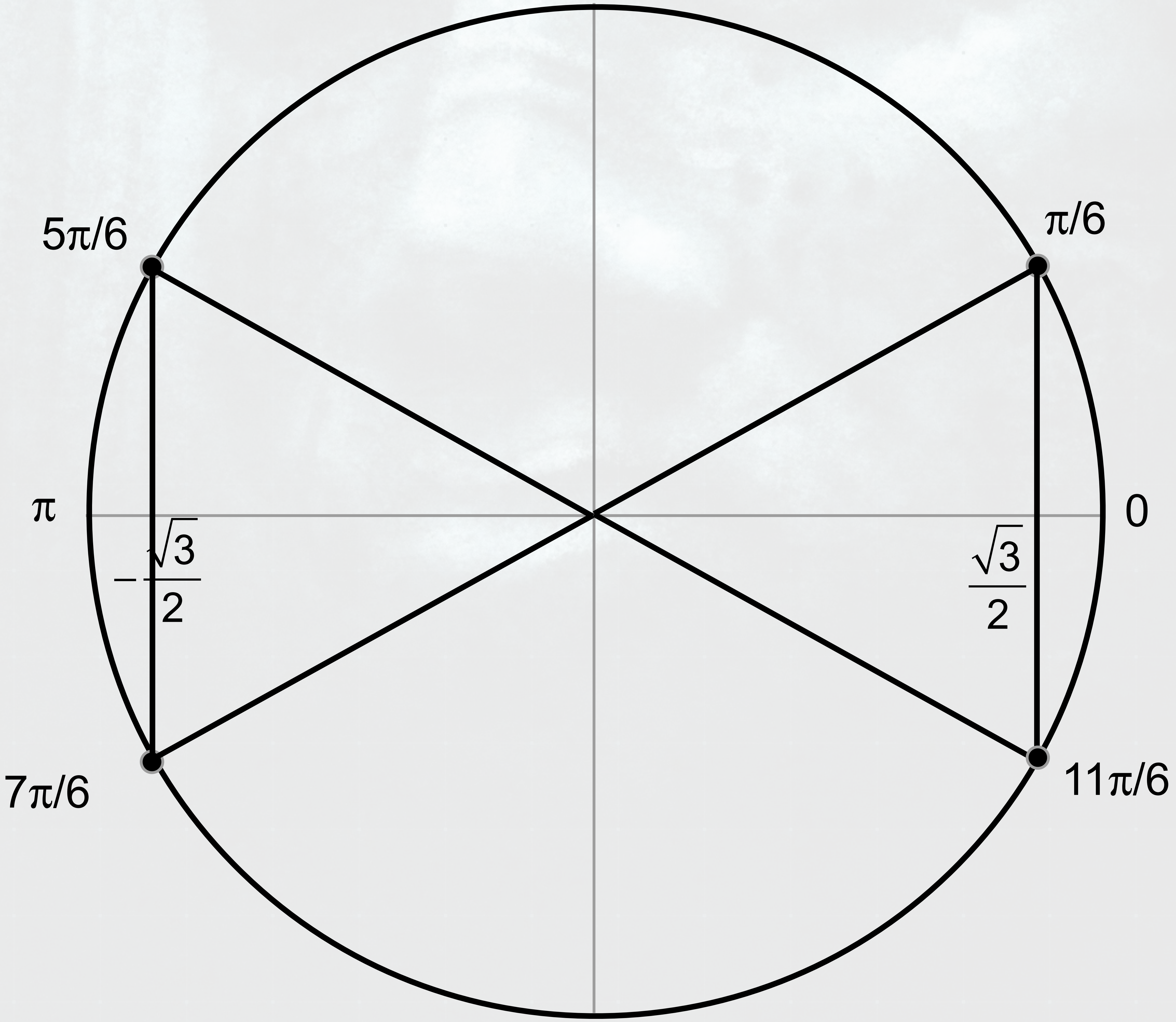
Solving a Trigonometric Equation Quadratic in Form

Solve the equation: $4\cos^2 x - 3 = 0$ on the interval $[0, 2\pi)$.

$$\cos x = \pm\frac{\sqrt{3}}{2}$$

$$x = \left(\frac{\pi}{6}\right), \left(\frac{11\pi}{6}\right)$$

$$x = \left(\frac{5\pi}{6}\right), \left(\frac{7\pi}{6}\right)$$



Solving Trig Equations

1. Isolate the trig function(s)

   – Simplify and/or factor

2. Determine the angle(s) that return(s) the final ratio.

   Unit circle values or inverse trig functions.

3. Add multiples of $(n2\pi)$.

4. Solve for the variable over the appropriate interval.

Using Factoring to Separate Different Functions

Solve the equation: $\sin x \tan x = \sin x, 0 \le x < 2\pi$

Caution: This is trickier than it looks.

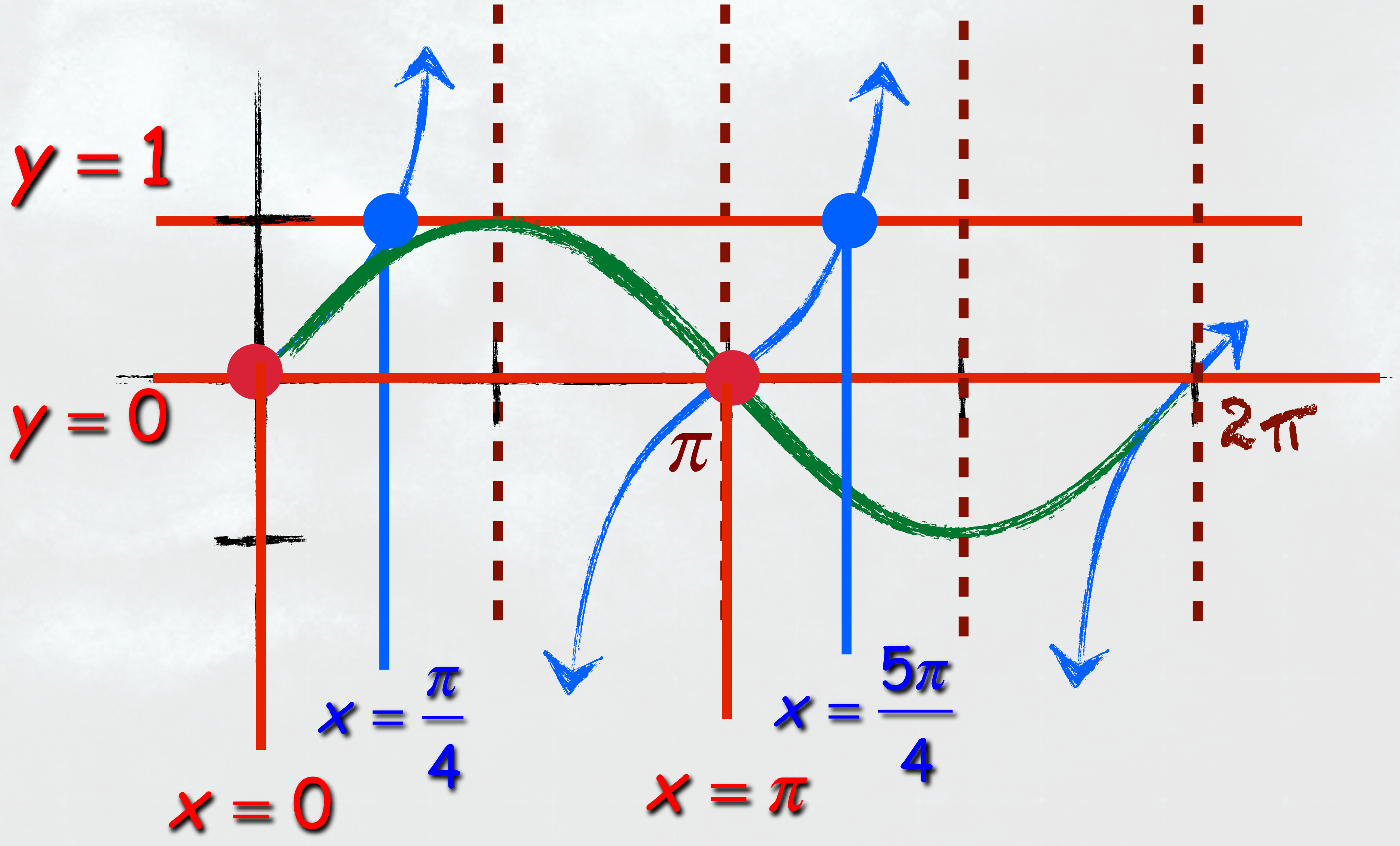
$$\sin x \tan x = \sin x, 0 \le x < 2\pi$$

$$\sin x \tan x - \sin x = 0$$

$$\sin x(\tan x - 1) = 0$$

$\sin x = 0$ $\qquad$ $\tan x = 1$

$x = 0, \pi$ $\qquad$ $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$



$$x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$$

Caution

Solve the equation: $\sin x \tan x = \sin x, 0 \le x < 2\pi$

For the equation $\sin x \tan x = \sin x, 0 \le x < 2\pi$ we cannot divide by sinx. Explain why.

sinx = 0 was a possible solution and dividing by sinx would lose those solutions. If sinx = 0 you cannot divide by sinx (0).

Using an Identity to Solve a Trigonometric Equation

Solve the equation: $\cos x + \sin x = 1, 0 \le x < 2\pi$

$$\cos x - 1 = -\sin x$$

$$\cos^2 x - 2\cos x + 1 = \sin^2 x$$

$$\cos^2 x - 2\cos x + 1 = 1 - \cos^2 x$$

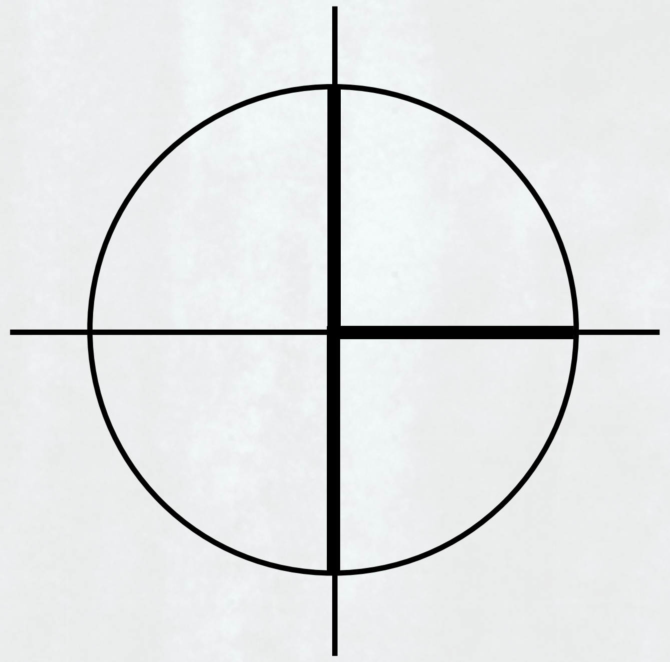
$$2\cos^2 x - 2\cos x = 0$$

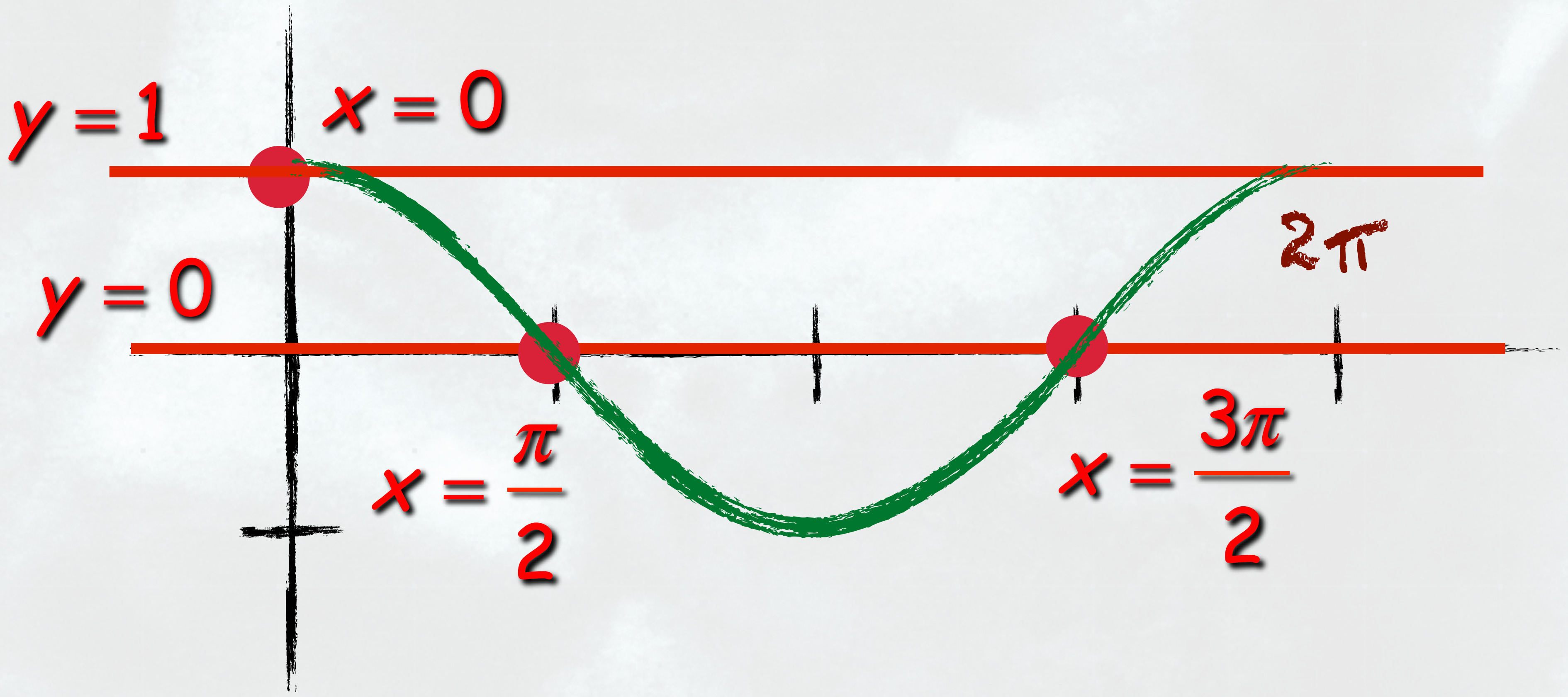
$$2\cos x(\cos x - 1) = 0$$

$$2\cos x = 0 \qquad \cos x - 1 = 0$$

$$\cos x = 0 \qquad \cos x = 1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = 0$$





Careful: We squared the equation, so we must check our results for extraneous solutions:

$$\cos 0 + \sin 0 = 1 + 0 = 1$$

$$\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$$

~~$\cos\frac{3\pi}{2} + \sin\frac{3\pi}{2} = 0 + -1 \neq 1$~~

$$x = 0, \frac{\pi}{2}$$

Example

Solve the equation: $\tan 2x = -1, 0 \le x < 2\pi$

$$\tan 2x = -1$$

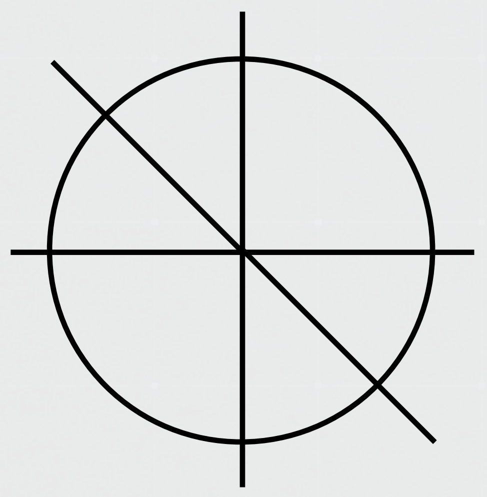
$$\tan\frac{3\pi}{4} = -1 \qquad \tan\frac{7\pi}{4} = -1$$

The period for tanx is π

The period for tan2x is π/2

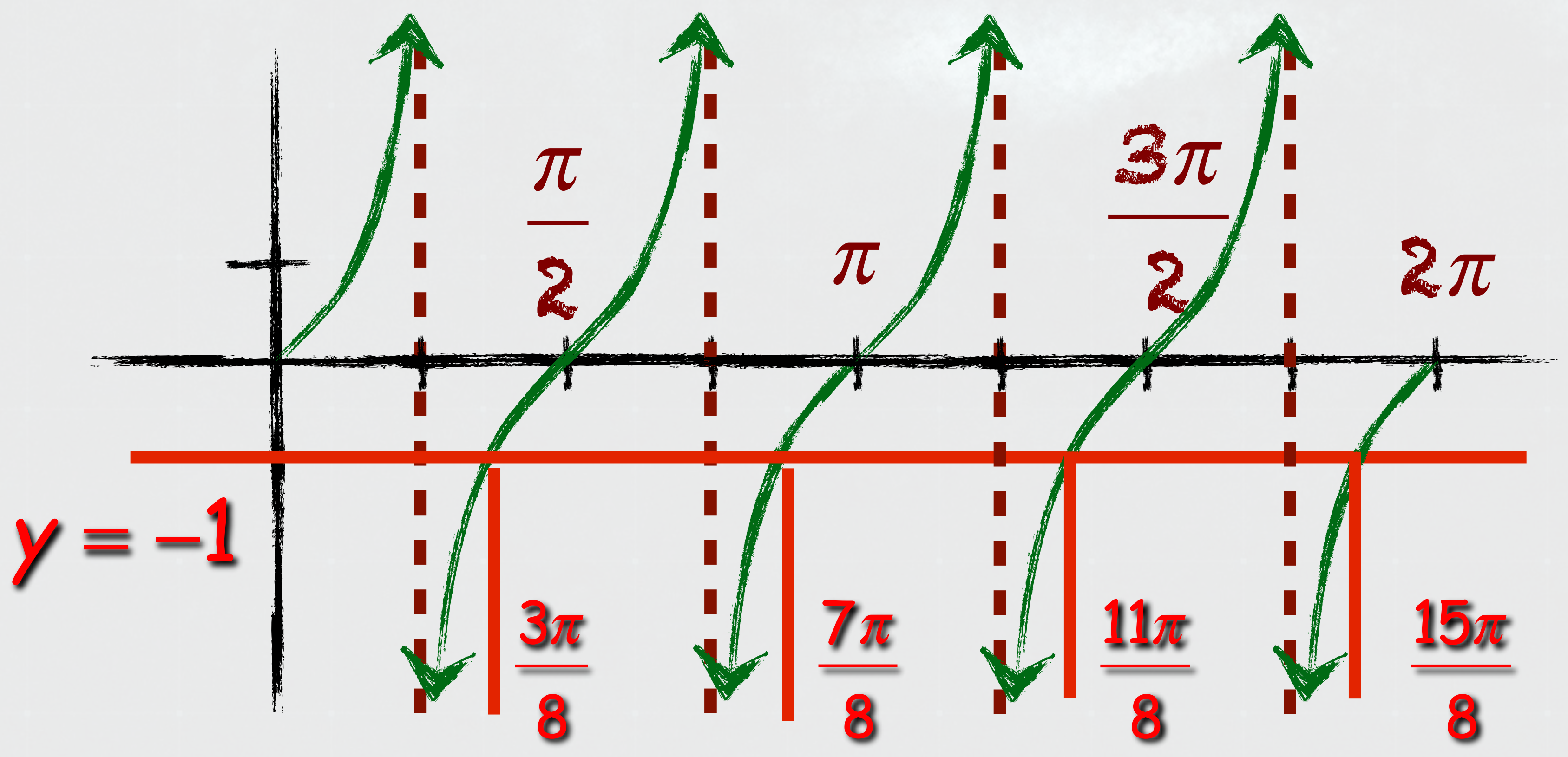
$$2x = \frac{3\pi}{4} + n\pi$$

$$x = \frac{3\pi}{8} + \frac{n\pi}{2}$$



$$x = \frac{3\pi}{8} + \frac{0\pi}{2} \qquad x = \frac{3\pi}{8} \qquad x = \frac{3\pi}{8} + \frac{1\pi}{2} \qquad x = \frac{7\pi}{8}$$

$$x = \frac{3\pi}{8} + \frac{2\pi}{2} \qquad x = \frac{11\pi}{8} \qquad x = \frac{3\pi}{8} + \frac{3\pi}{2} \qquad x = \frac{15\pi}{8}$$



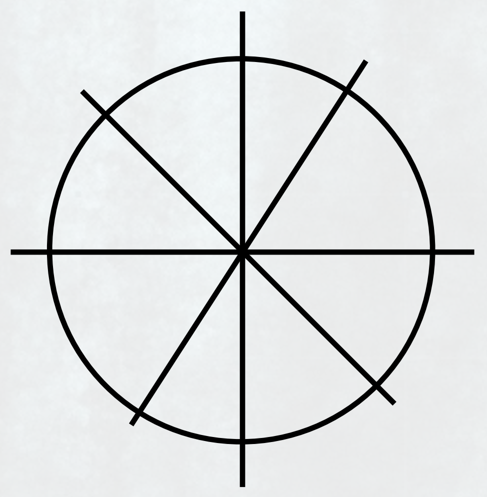
$$x = \frac{3\pi}{8}, \frac{7\pi}{8}. \frac{11\pi}{8}, \frac{15\pi}{8}$$

Example

Solve the equation: $\tan^2 x - \tan x - 2 = 0,\ 0 \le x < 2\pi$

$$\tan^2 x - \tan x - 2 = 0$$

$$(\tan x - 2)(\tan x + 1) = 0$$



$\tan x = 2$ $\qquad$ $\tan x = -1$

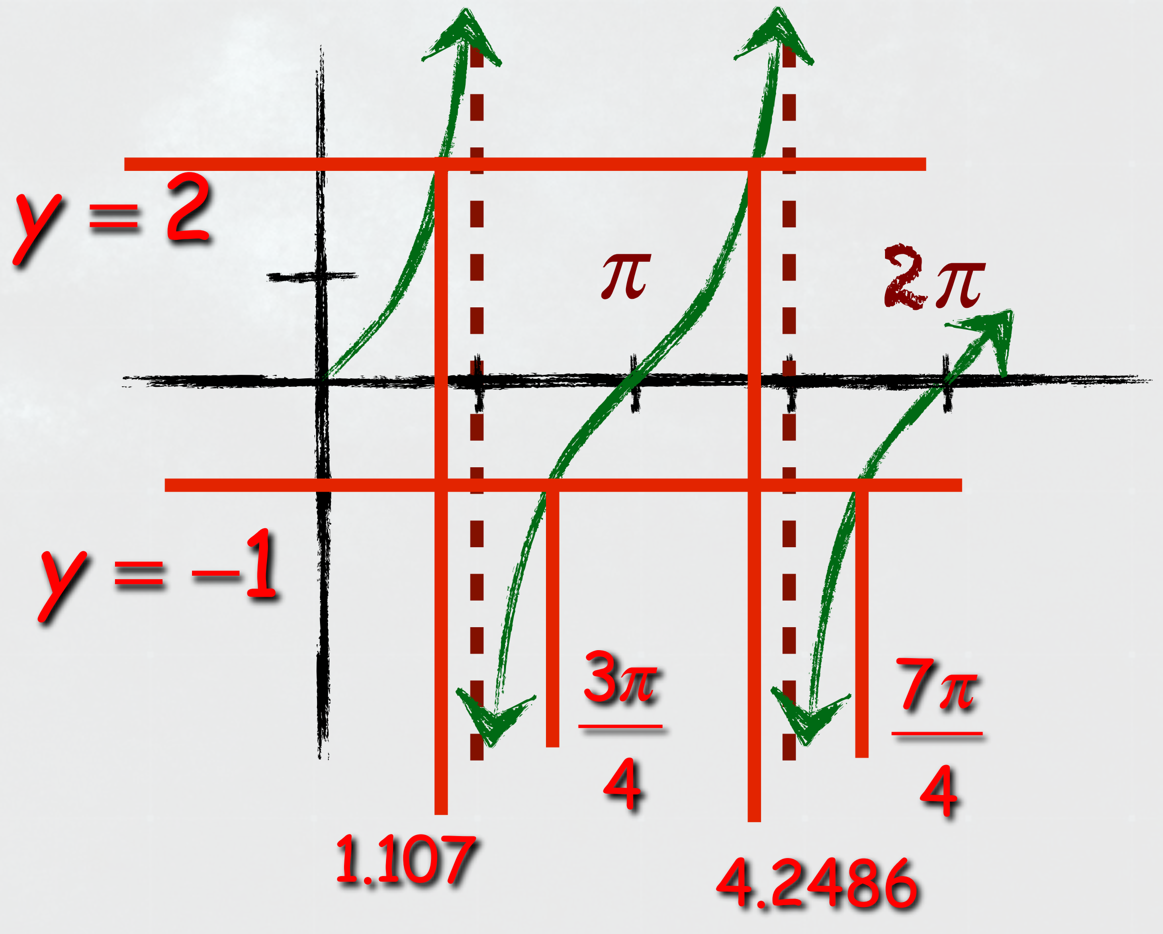
$x = \tan^{-1} 2 \approx 1.107$ $\qquad$ $x = \tan^{-1}(-1) \approx \frac{3\pi}{4}$

The period for tan is π

$x \approx 1.107 + n\pi$ $\qquad$ $x = \frac{3\pi}{4} + n\pi$

n = 0 $\quad x = 1.107 \quad x = \frac{3\pi}{4}$

n = 1 $\quad x = 4.2486 \quad x = \frac{7\pi}{4}$



$$x \approx 1.107, \frac{3\pi}{4}, 4.2486, \frac{7\pi}{4}$$

Example

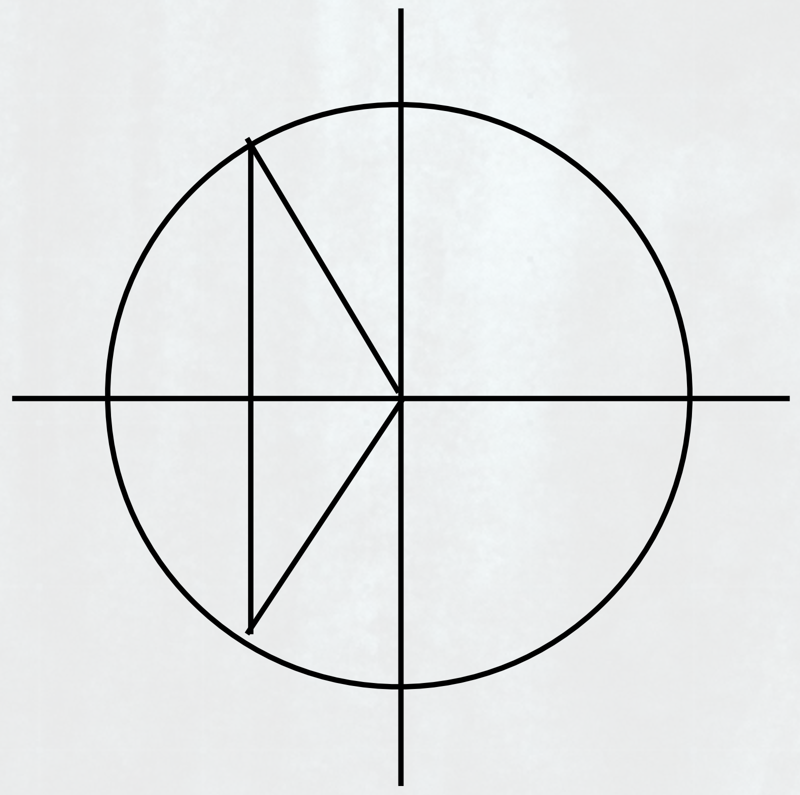
Solve the equation: $\mathbf{4\cos^2 x + 4\cos x = -1}$ on the interval $[0,2\pi)$.

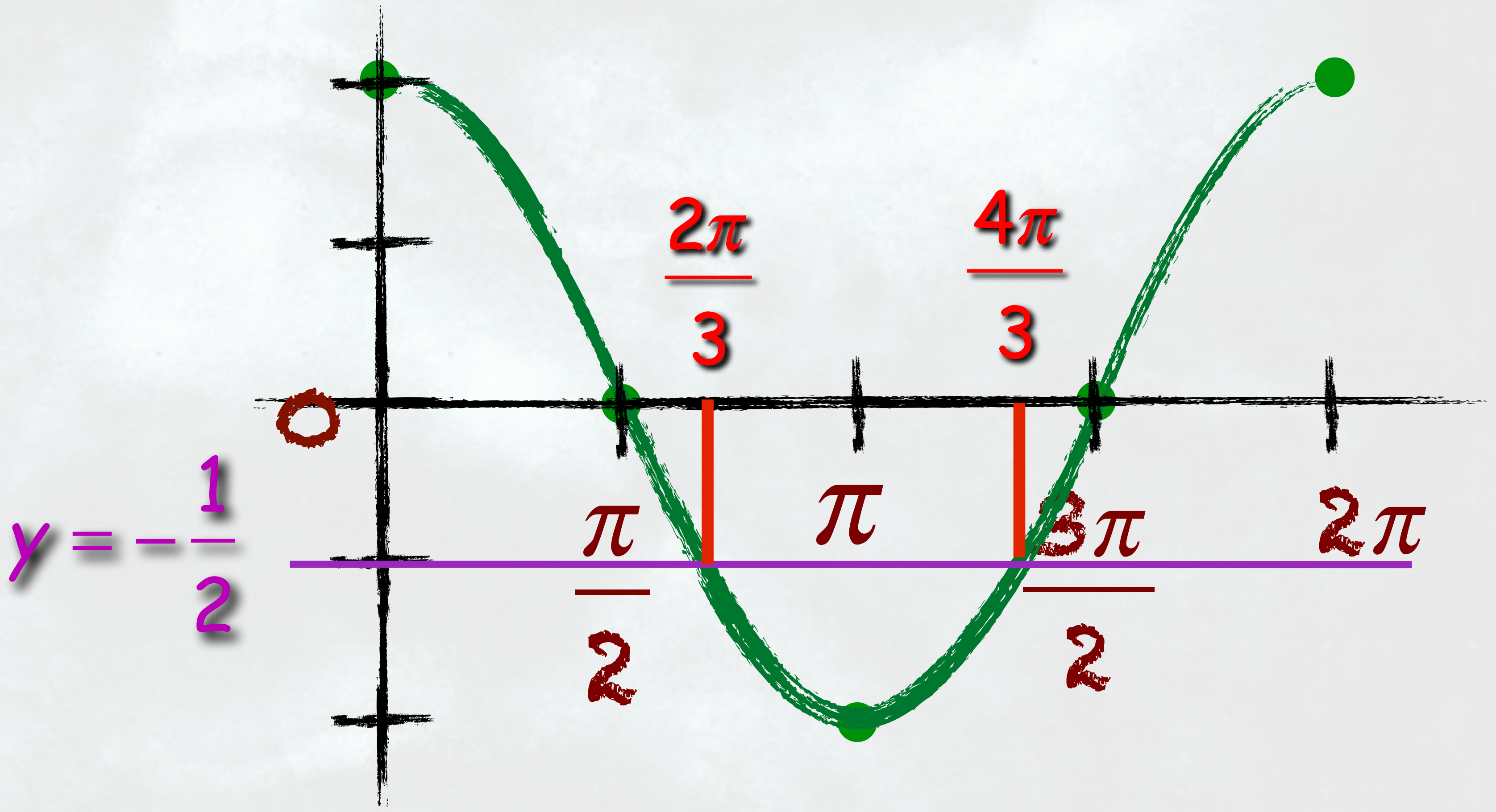
$$4\cos^2 x + 4\cos x + 1 = 0$$

$$(2\cos x + 1)^2 = 0$$

$$2\cos x = -1$$

$$\cos x = -\frac{1}{2}$$





$$x = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3} \qquad x = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{4\pi}{3}$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Example (Inverse Function)

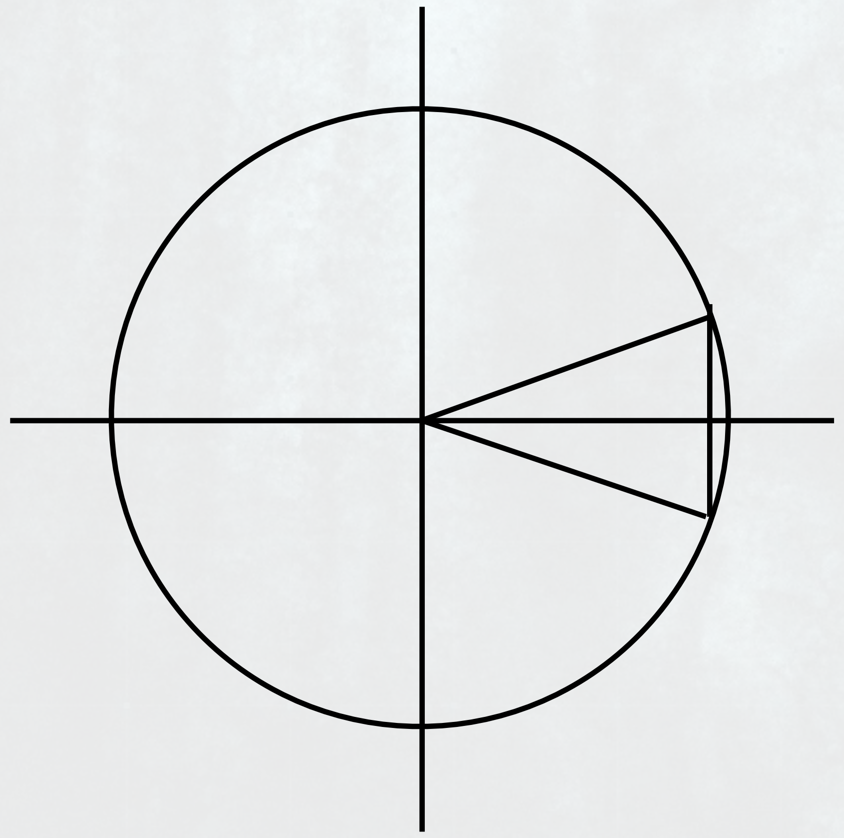
Solve the equation: $\mathbf{5\sec^2 x = 6\sec x,\ 0^\circ \le x < 360^\circ}$

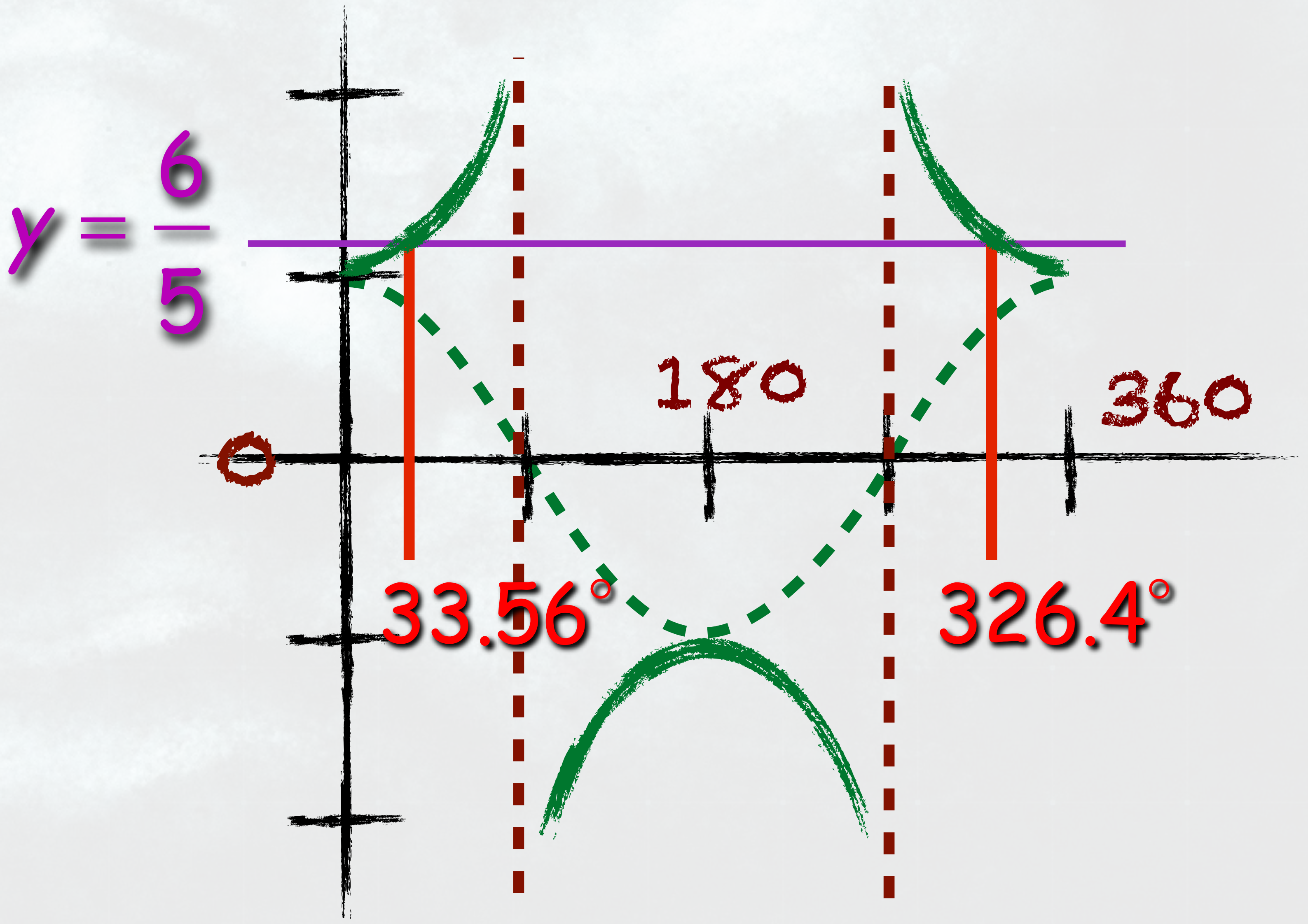
$$5\sec^2 x - 6\sec x = 0$$

$$\sec x(5\sec x - 6) = 0$$

~~$\sec x = 0$~~ $\quad 5\sec x = 6$

$$\sec x = \frac{6}{5}$$



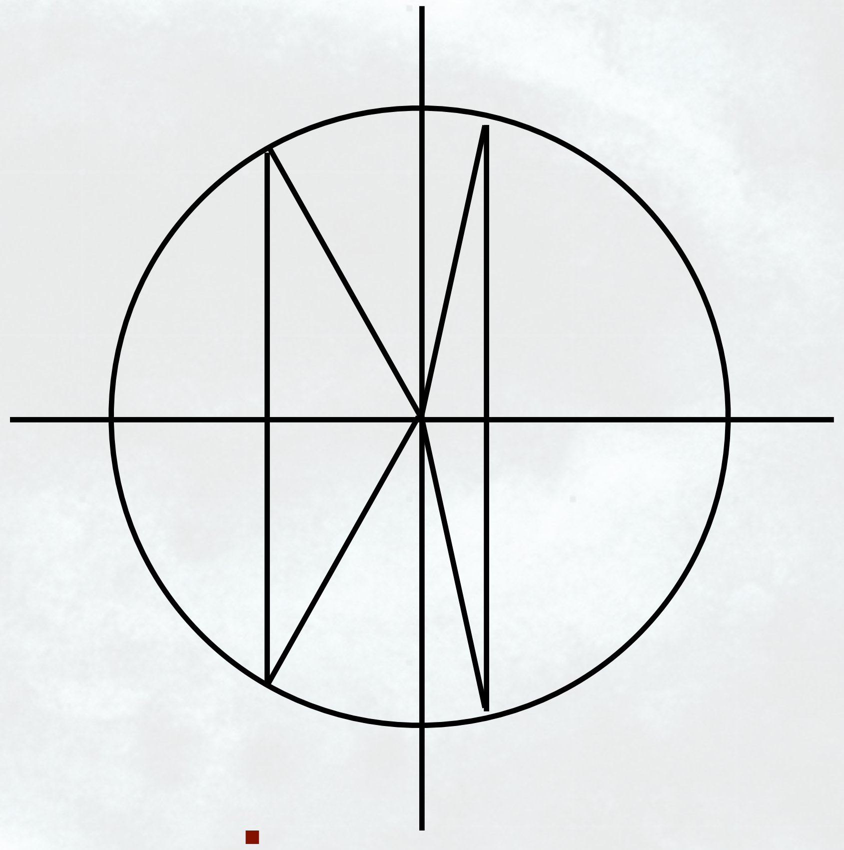


$$x = \cos^{-1}\left(\frac{5}{6}\right) \approx 33.5573^\circ$$

$$x \approx 33.5573, 326.4427$$

Example (Inverse Function)

Solve the equation: $\sec^2 x - 3\sec x - 10 = 0$



$$\sec^2 x - 3\sec x - 10 = 0$$

$$(\sec x - 5)(\sec x + 2) = 0$$

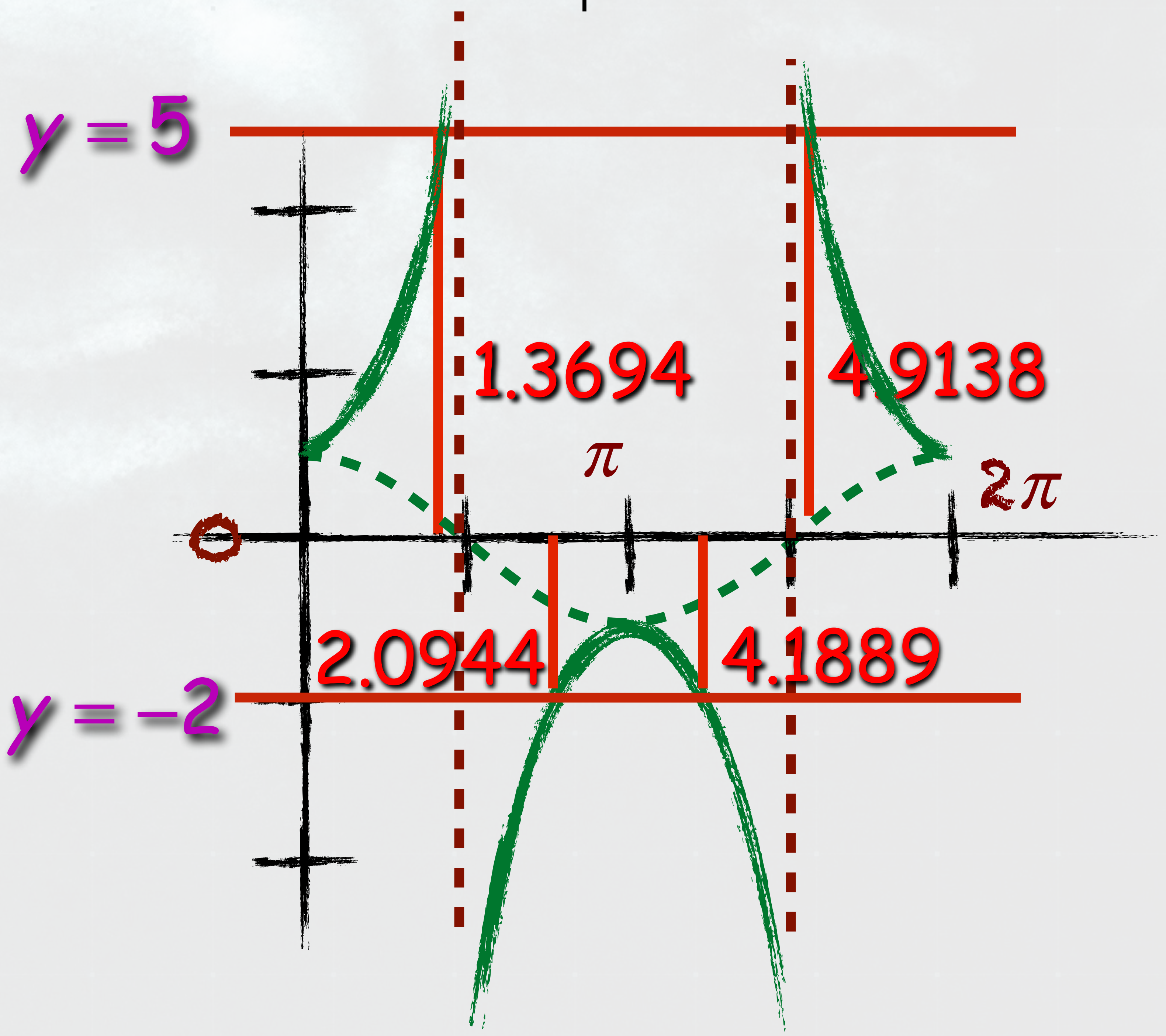
$$\sec x = 5 \qquad \sec x = -2$$

$$\cos x = \frac{1}{5} \qquad \cos x = -\frac{1}{2}$$

$$x = \cos^{-1}\left(\frac{1}{5}\right) \approx 1.3694 \qquad x = \cos^{-1}\left(-\frac{1}{2}\right) \approx 2.0944$$

$$x \approx 1.3694 + n2\pi \qquad x \approx 2.0944 + n2\pi$$

$$x \approx 4.9138 + n2\pi \qquad x \approx 4.1889 + n2\pi$$



Example

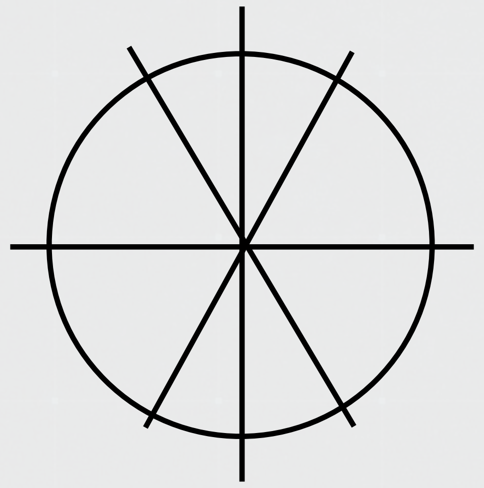
- Solve $3\cot^2 x - 1 = 0$ for all values of x.

$$3\cot^2 x - 1 = 0$$

$$\cot^2 x = \frac{1}{3}$$

$$\cot x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$$

$$\tan x = \pm\sqrt{3}$$



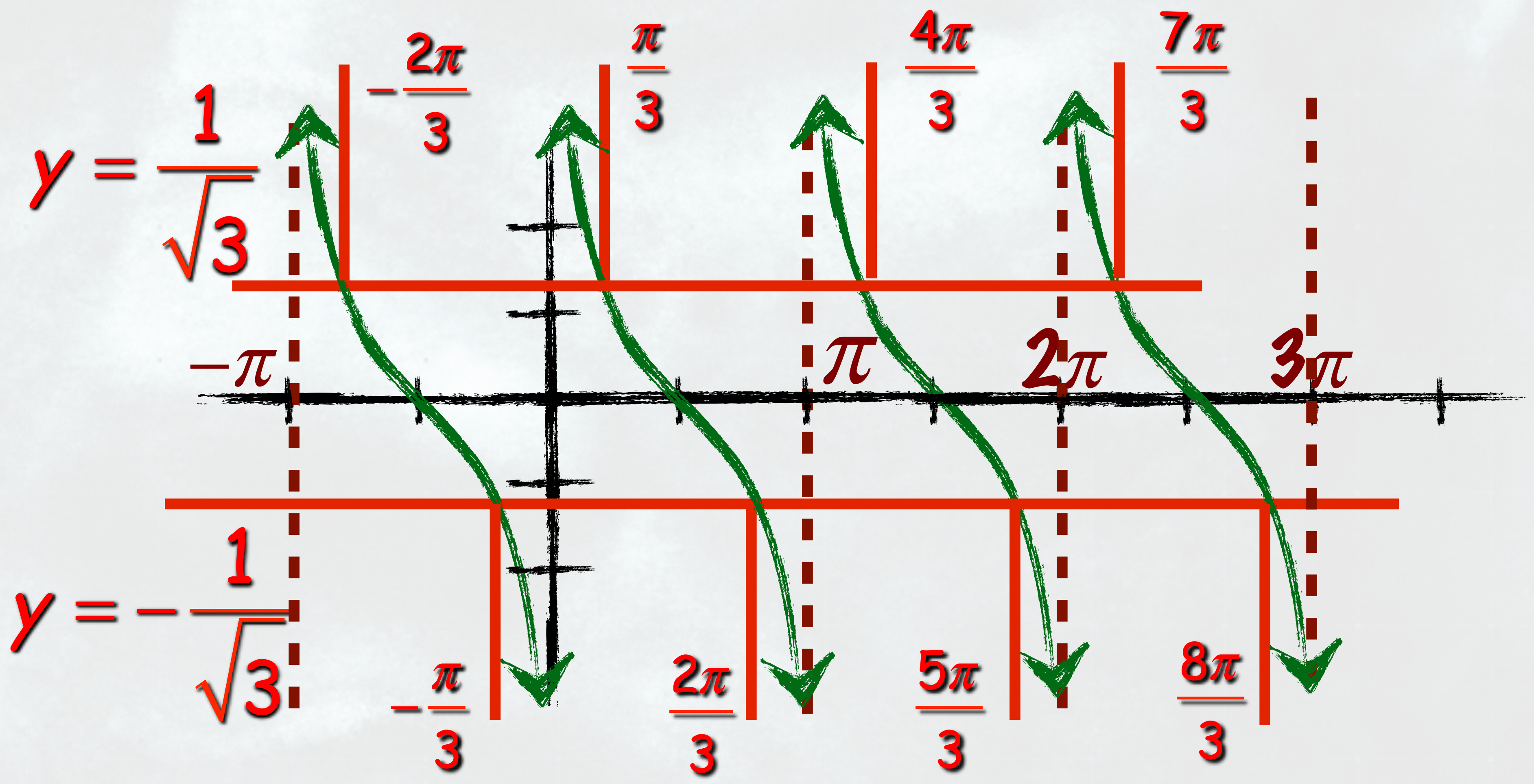
$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$x = \frac{\pi}{3} \pm n\pi$$

$$x = \frac{2\pi}{3} \pm n\pi$$

$$x = \frac{4\pi}{3} \pm n\pi$$

$$x = \frac{5\pi}{3} \pm n\pi$$



$$x = \frac{\pi}{3} \pm n\pi \text{ or } x = \frac{2\pi}{3} \pm n\pi$$

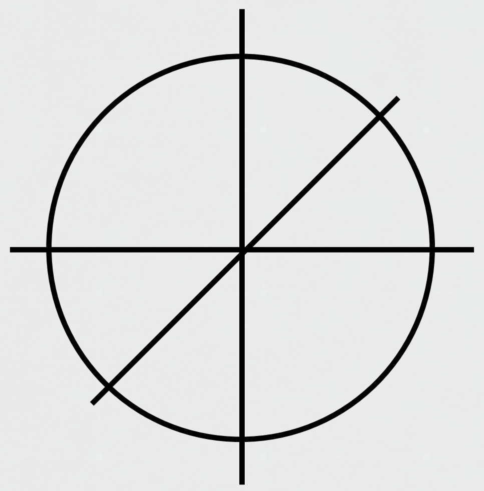
- So $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$ are redundant.

Example

∿ Solve $\tan\left(\frac{\theta}{2}-\frac{\pi}{3}\right)-1=0$ for all values of θ.

$$\tan\left(\frac{\theta}{2}-\frac{\pi}{3}\right)-1=0$$

$$\tan\left(\frac{\theta}{2}-\frac{\pi}{3}\right)=1$$



$$\left(\frac{\theta}{2}-\frac{\pi}{3}\right)=\frac{\pi}{4}\pm n\pi,\frac{5\pi}{4}\pm n\pi$$

$$\frac{\pi}{4}+\pi=\frac{5\pi}{4}$$

∿ So $\frac{5\pi}{4}$ is redundant.

$$\frac{\theta}{2}-\frac{\pi}{3}=\frac{\pi}{4}\pm n\pi$$

$$\frac{\theta}{2}=\frac{\pi}{4}+\frac{\pi}{3}\pm n\pi$$

$$\frac{\theta}{2}=\frac{7\pi}{12}\pm n\pi$$

$$\theta=\frac{7\pi}{6}\pm n2\pi$$