Chapter 5

Analytic Trigonometry

5.5 Double-Angle, Power-Reducing, Half-Angle, Sum to Product, and Product to Sum Formulas

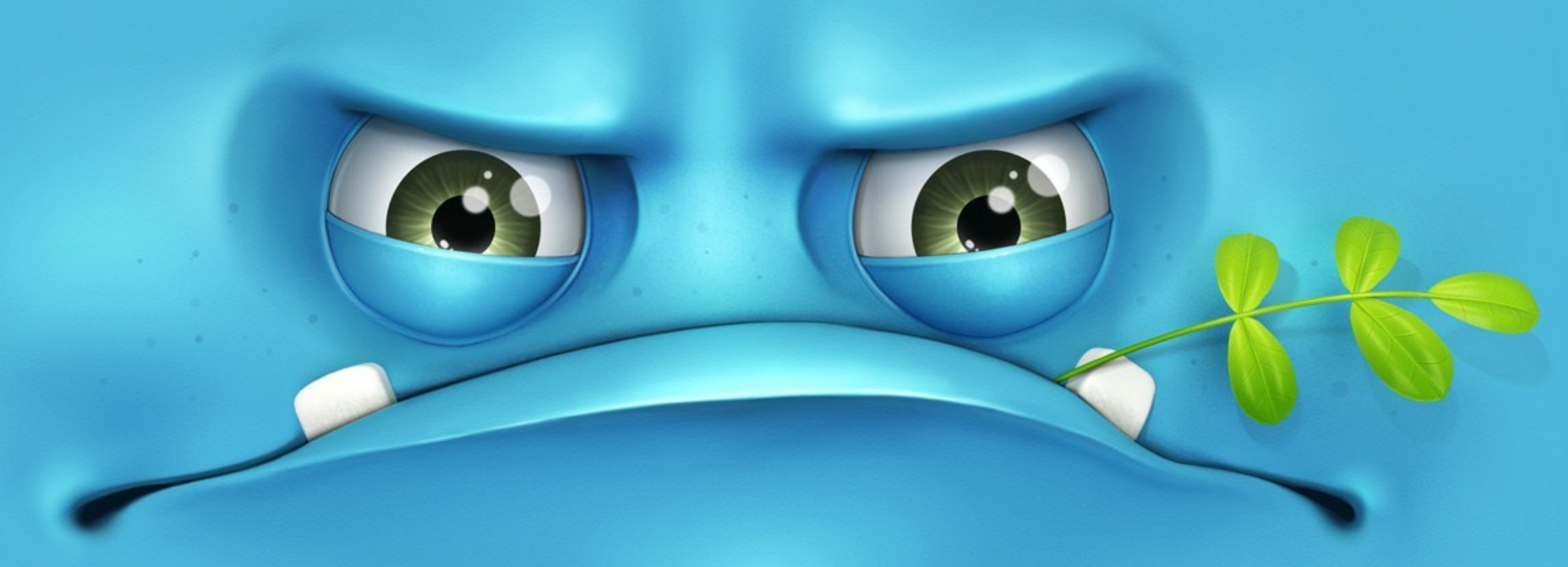
Chapter 5

Homework

- 5.5 Read Sec 5.5
- p415 1-121 every other odd

Chapter 5.5

Objectives



- Use multiple angle, power reducing, half-angle formulas to rewrite and evaluate trigonometric functions.
- Use product to sum and sum to product formulas to rewrite and evaluate trigonometric functions.

Double-Angle Formulas

$\sin 2\alpha = \sin(\alpha + \alpha)$

$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

$= \sin\alpha \cos\alpha + \cos\alpha \sin\alpha$

$= 2\sin\alpha \cos\alpha$

$$\sin 2\alpha = 2\sin\alpha \cos\alpha$$

Double-Angle Formulas

$\cos 2\alpha = \cos(\alpha + \alpha)$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

$= \cos^2\alpha - \sin^2\alpha$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

Double-Angle Formulas

$$\tan 2\alpha = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$$

$$= \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

Double-Angle Formulas

Be Careful

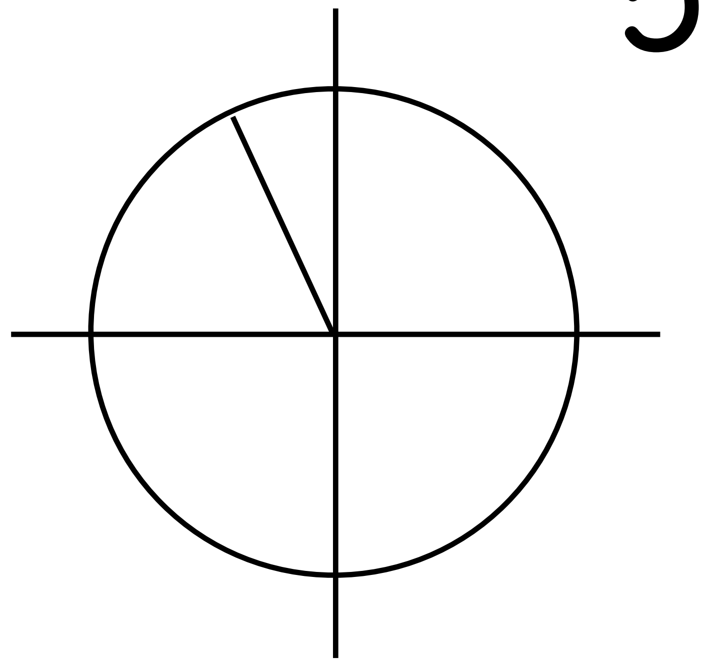
$\cos 2\theta \neq 2\cos\theta$
$\sin 2\theta \neq 2\sin\theta$
$\tan 2\theta \neq 2\tan\theta$

$\cos 2\theta \neq \cos 2\cos\theta$
$\sin 2\theta \neq \sin 2\sin\theta$
$\tan 2\theta \neq \tan 2\tan\theta$

Double-Angle Formulas

Using Double-Angle Formulas to Find Exact Values

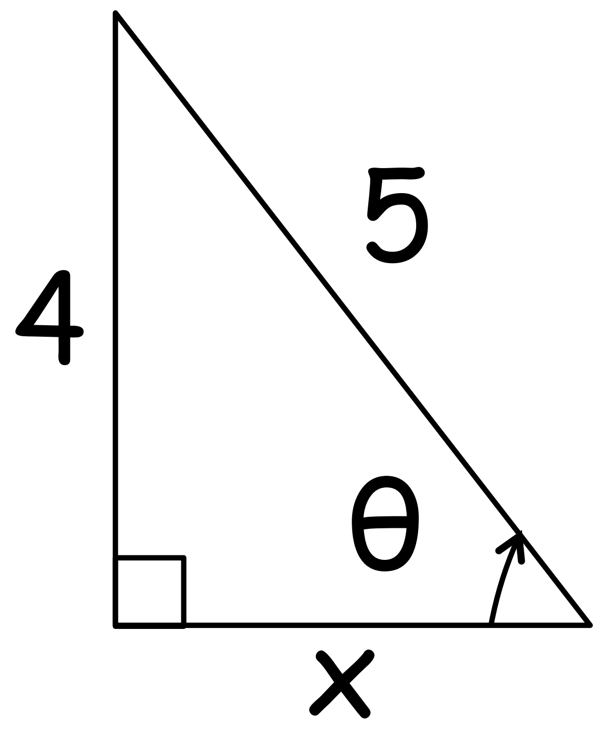
If $\sin\theta = \dfrac{4}{5}$ and θ lies in QII, find the exact value of $\cos 2\theta$.



$$4^2 + x^2 = 5^2$$

$$x = -3$$

$$\cos\theta = -\frac{3}{5}$$



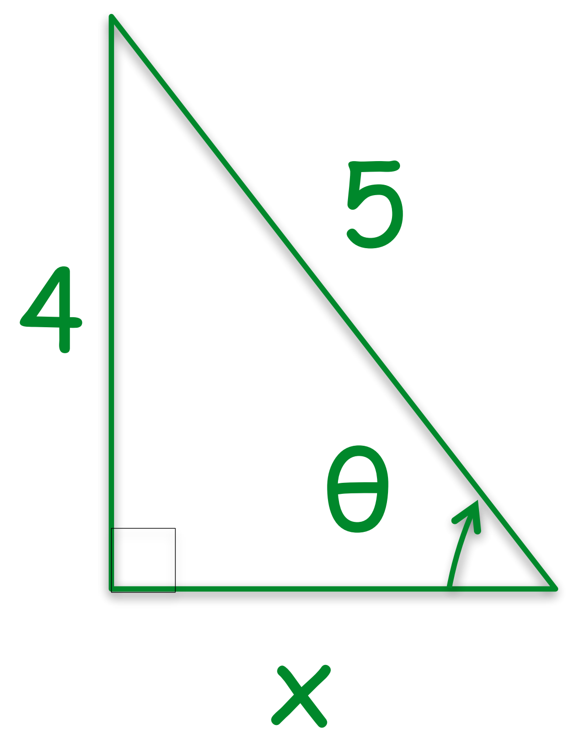
cos2α = cos² α – sin² α

$$= \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

Double-Angle Formulas

Using Double-Angle Formulas to Find Exact Values

If $\sin\theta = \dfrac{4}{5}$ and θ lies in QII, find the exact value of $\sin 2\theta$.

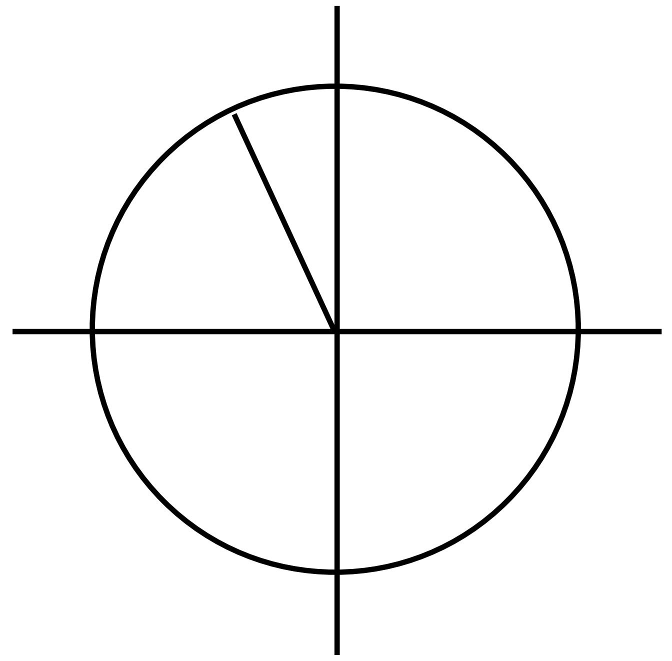


$$\cos\theta = -\frac{3}{5}$$

sin2α = 2sin α cos α

$$\sin 2\theta = 2\sin\theta\cos\theta$$

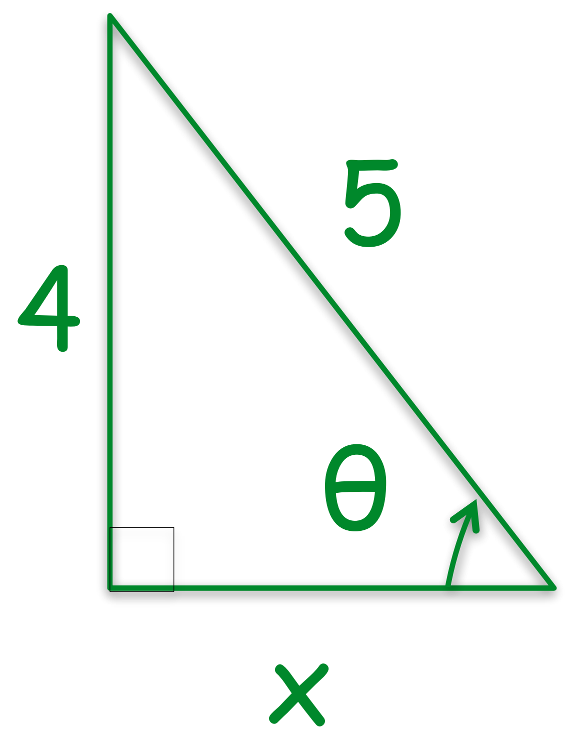
$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

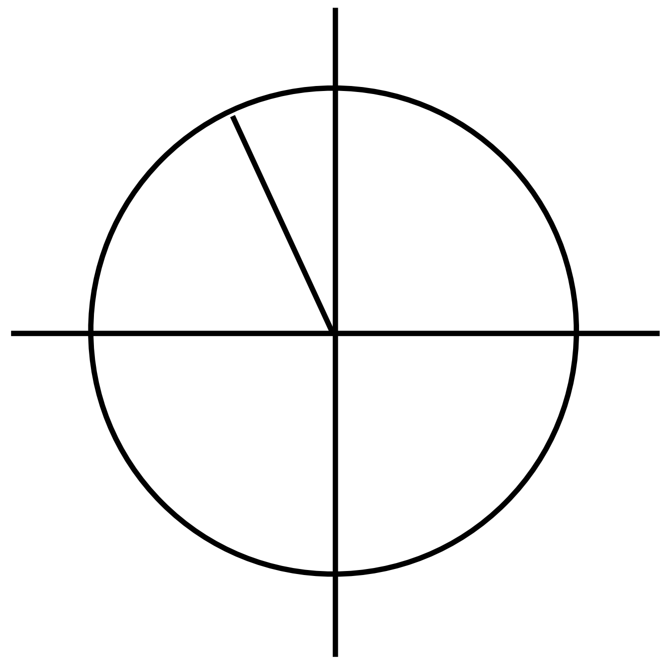


Double-Angle Formulas

Using Double-Angle Formulas to Find Exact Values

If $\sin\theta = \frac{4}{5}$ and θ lies in QII, find the exact value of $\tan 2\theta$





$$\cos\theta = -\frac{3}{5} \qquad \tan\theta = -\frac{4}{3}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = \frac{24}{7}$$

Of course

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \frac{-\frac{24}{25}}{-\frac{7}{25}} = \frac{24}{7}$$

Double-Angle Formulas

Three Forms of the Double-Angle Formula for $\cos 2\theta$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= \cos^2\alpha - (1 - \cos^2\alpha)$$

$$= \cos^2\alpha - 1 + \cos^2\alpha)$$

$$= 2\cos^2\alpha - 1$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

Double-Angle Formulas

Three Forms of the Double-Angle Formula for cos2θ

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= 1 - \sin^2\alpha - \sin^2\alpha$$

$$= 1 - 2\sin^2\alpha$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

Double-Angle Formulas

Double Angle Formulae

$\sin 2\alpha = 2 \sin\alpha \cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha = 2\cos^2\alpha - 1$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

Example

Solve: sin 2x – cos x = 0

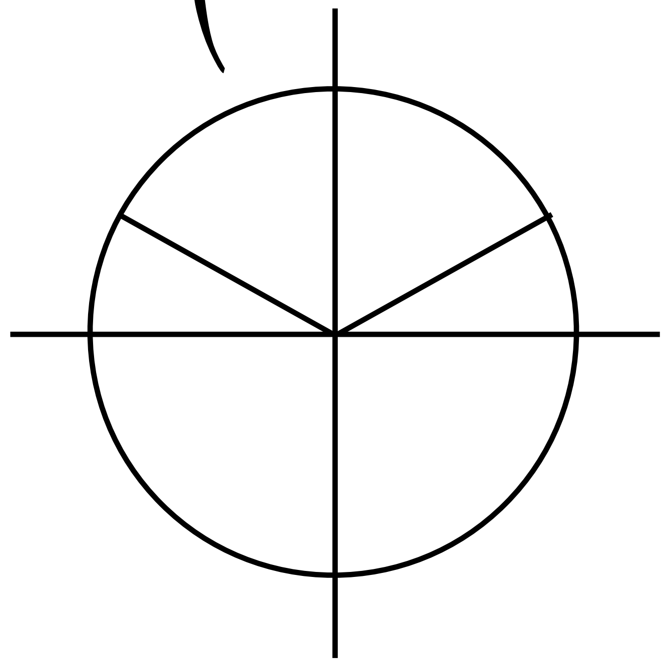
$$\sin 2x - \cos x = 0$$

sin2α = 2sin α cos α

$$2\sin x\cos x - \cos x = 0$$

$$\cos x\left(2\sin x - 1\right) = 0$$

$$\cos x = 0$$



$$2\sin x - 1 = 0$$

$$\sin x = \frac{1}{2}$$

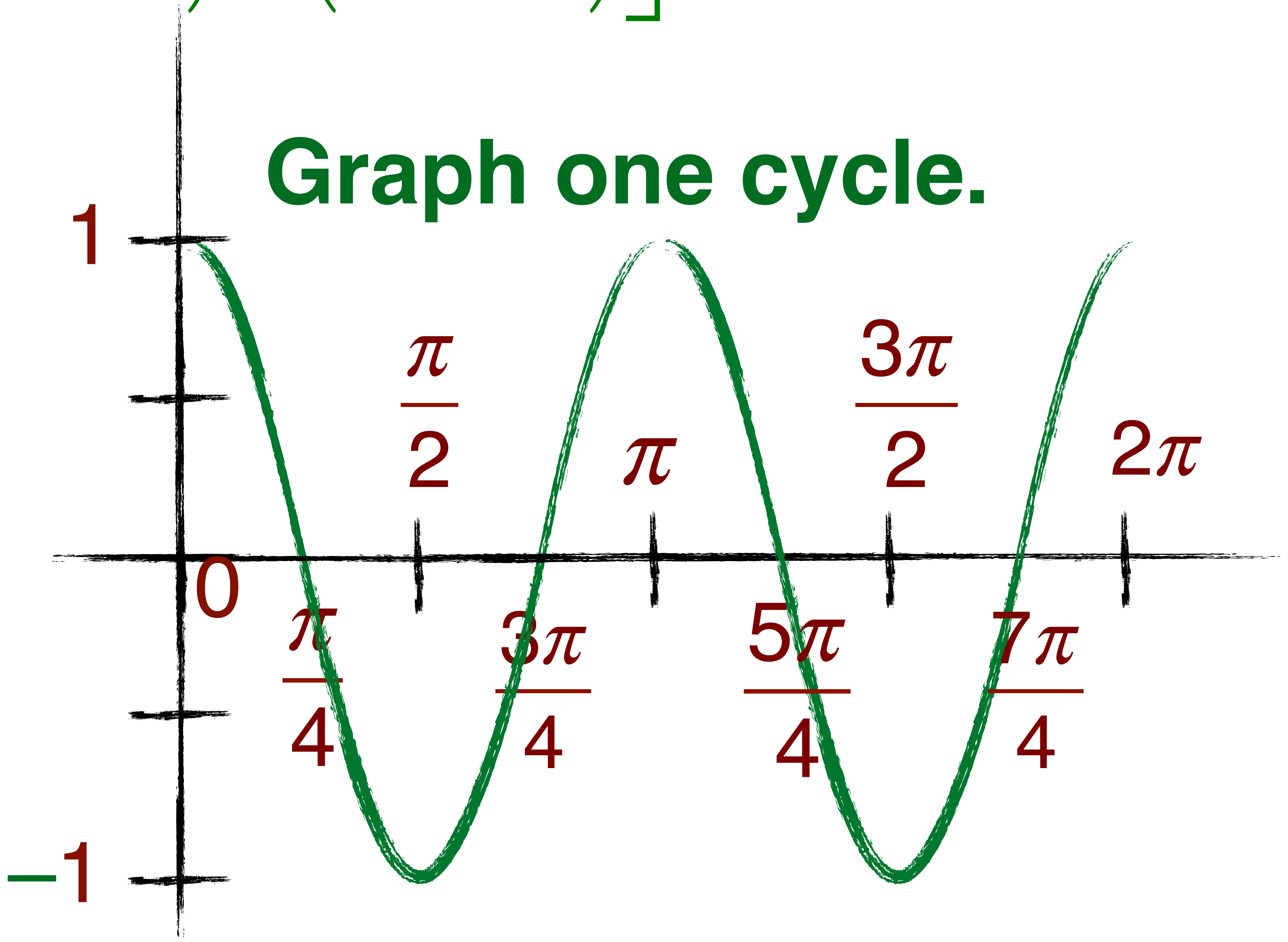
$$x = \frac{\pi}{2} + n2\pi \quad x = \frac{3\pi}{2} + n2\pi \qquad x = \frac{\pi}{6} + n2\pi \quad x = \frac{5\pi}{6} + n2\pi$$

Graph

- Simplify and graph y = $\cos^4 x - \sin^4 x$ on [0, 2π)

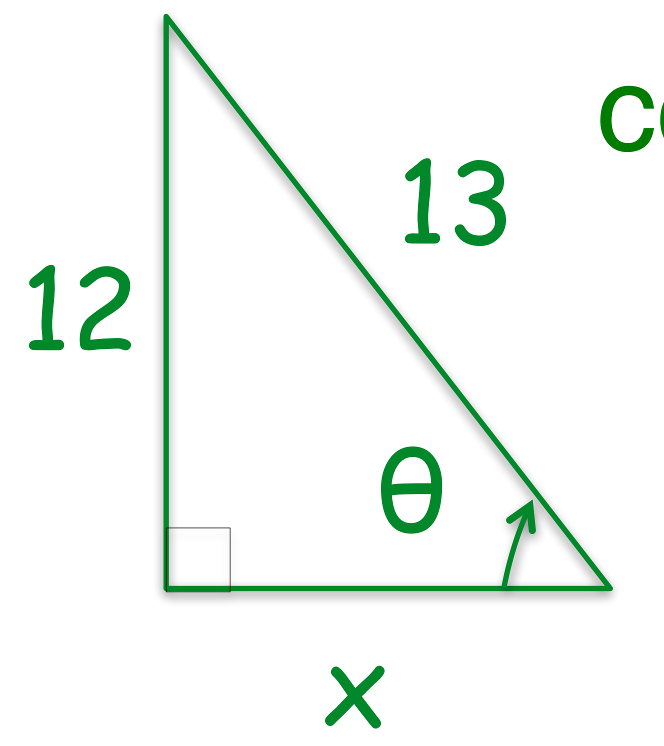
$$\cos^4 x - \sin^4 x = \left(\cos^2 x\right)^2 - \left(\sin^2 x\right)^2$$
$$= \left[\left(\cos^2 x\right) - \left(\sin^2 x\right)\right]\left[\left(\cos^2 x\right) + \left(\sin^2 x\right)\right]$$
$$= \left[\left(\cos^2 x\right) - \left(\sin^2 x\right)\right][1]$$
$$= \cos 2x$$

Graph one cycle.

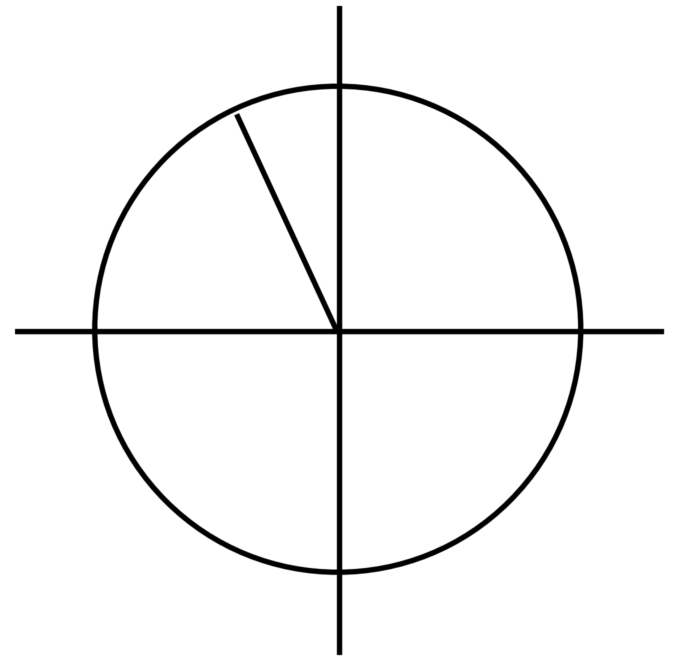


x	0	π/2	π	3π/	2π
y=2cosx	1	-1	1	-1	1

- Given $\sin\theta = \dfrac{12}{13}$ and $\dfrac{\pi}{2} < \theta < \pi$, find sin 2θ, cos 2θ, and tan 2θ.



$$\cos\theta = -\frac{5}{13}$$



sin2α = 2sin α cos α

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$

$$\sin 2\theta = 2\left(\frac{12}{13}\right)\left(-\frac{5}{13}\right) = -\frac{120}{169}$$

cos2α = cos² α – sin² α

$$\cos 2\theta = \left(-\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = -\frac{119}{169}$$

$$\tan 2\theta = \frac{-\frac{120}{169}}{-\frac{119}{169}} = \frac{120}{119}$$

Verifying an Identity

Verify the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\sin 3\theta = \sin(2\theta + \theta)$$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$= (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$$

$$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$$

$$= 3\sin\theta\cos^2\theta - \sin^3\theta$$

$$= 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta$$

$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

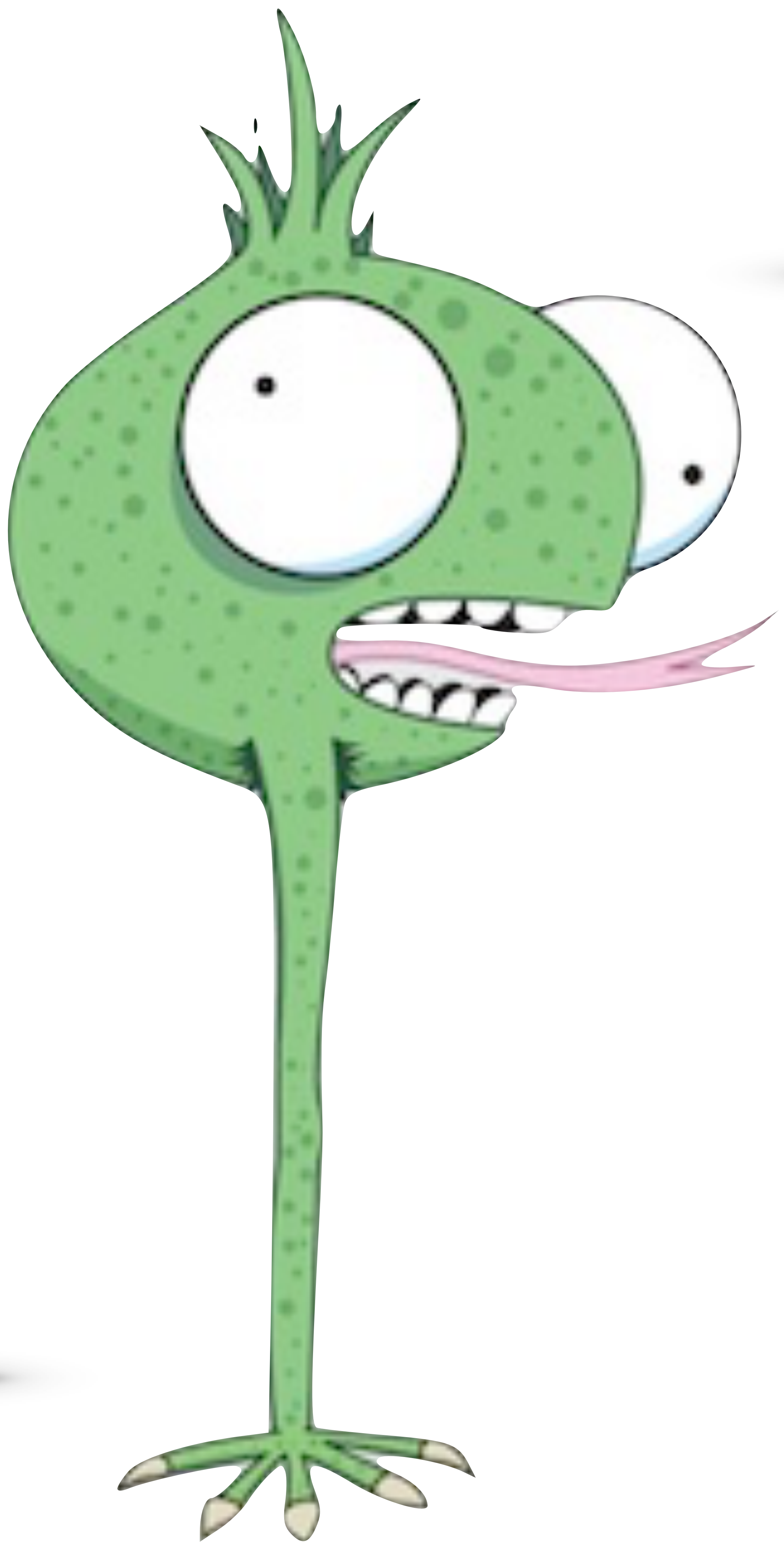
Power-Reducing Formulas

$\cos 2\alpha = 2\cos^2\alpha - 1$

$$2\cos^2\alpha = \cos 2\alpha + 1$$

$$\cos^2\alpha = \frac{\cos 2\alpha + 1}{2}$$

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}$$



$\cos 2\alpha = 1 - 2\sin^2\alpha$

$$2\sin^2\alpha = 1 - \cos 2\alpha$$

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$$

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$$

Power-Reducing Formulas

$\tan^2 \alpha$

$$\tan^2 \alpha = \frac{\sin^2 \alpha}{\cos^2 \alpha}$$

$$= \frac{\dfrac{1-\cos 2\alpha}{2}}{\dfrac{1+\cos 2\alpha}{2}} = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

$$\boldsymbol{\tan^2 \alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}}$$

All Together Now

$$\sin 2\alpha = 2 \sin\alpha \cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

Power-Reducing Formulas

Reducing the Power of a Trigonometric Function

Write an equivalent expression for **sin⁴ x** that does not contain powers of trigonometric functions greater than 1.

$$\sin^4 x = \left(\sin^2 x\right)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1-2\cos 2x+\cos^2 2x}{4}$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= \frac{1-2\cos 2x+\left(\frac{1+\cos 4x}{2}\right)}{4} = \frac{2-4\cos 2x+1+\cos 4x}{8}$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= \frac{3-4\cos 2x+\cos 4x}{8}$$

Half-Angle Formulas

Deriving a Half-Angle Formula

$\cos 2\alpha = 2\cos^2\alpha - 1$

$$\cos\left(\frac{2\alpha}{2}\right) = 2\cos^2\left(\frac{\alpha}{2}\right) - 1$$

$$\cos\alpha = 2\cos^2\left(\frac{\alpha}{2}\right) - 1$$

$$\cos\alpha + 1 = 2\cos^2\left(\frac{\alpha}{2}\right)$$

$$\frac{\cos\alpha + 1}{2} = \cos^2\left(\frac{\alpha}{2}\right)$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{\cos\alpha + 1}{2}} \quad or \quad -\sqrt{\frac{\cos\alpha + 1}{2}}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$$

Know the Quadrant!

Half-Angle Formulas

Deriving a Half-Angle Formula

$\cos 2\alpha = 1 - 2\sin^2\alpha$

$$\cos\left(\frac{2\alpha}{2}\right) = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$\cos\alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$2\sin^2\left(\frac{\alpha}{2}\right) = 1 - \cos\alpha$$

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}$$

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{2}} \text{ or } -\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Know the Quadrant!

Half-Angle Formulas

Deriving a Half-Angle Formula

$$\tan\left(\frac{\alpha}{2}\right)$$

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$= \frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Know the Quadrant!

All Together Once Again

Double-Angle Formulas

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha = 2\cos^2\alpha - 1$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Formula Reduction

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

Half-Angle Formulas

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Caution

Keep in mind that with half angle formulae, the sign outside the radical (the sign of the half angle function) is determined by the quadrant in which $\frac{\alpha}{2}$ is found. The sign of the cos function (inside the radical) is determined by the quadrant in which α is found.

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Example

Verify: $\tan\theta = \dfrac{\sin 2\theta}{1+\cos 2\theta}$

$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+(1-2\sin^2\theta)}$$

$$= \frac{2\sin\theta\cos\theta}{2-2\sin^2\theta} = \frac{2\sin\theta\cos\theta}{2(1-\sin^2\theta)}$$

$$= \frac{2\sin\theta\cos\theta}{2(\cos^2\theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Example

Verify: $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta}$

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-\left(2\cos^2\theta - 1\right)}{2\sin\theta\cos\theta}$$

$$= \frac{2-2\cos^2\theta}{2\sin\theta\cos\theta} = \frac{2(1-\cos^2\theta)}{2\sin\theta\cos\theta}$$

$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Half-Angle Formulas

$$\tan\theta = \frac{\sin 2\theta}{1+\cos 2\theta}$$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{2\theta}{2}}{1+\cos\frac{2\theta}{2}} = \frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$$

$$\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta}$$

$$\tan\frac{\theta}{2} = \frac{1-\cos\frac{2\theta}{2}}{\sin\frac{2\theta}{2}} = \frac{1-\cos\theta}{\sin\theta}$$

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

Example

Find the exact value of cos105°.

$$\cos 105^\circ = \cos\left(\frac{210^\circ}{2}\right)$$

$$= -\sqrt{\frac{1+\cos 210^\circ}{2}}$$

105° is in QII, 210° is in QIII

$$\cos 210^\circ = -\frac{\sqrt{3}}{2}$$

$$= -\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}} = -\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{4}} = -\sqrt{\frac{2-\sqrt{3}}{4}}$$

Example

Find the exact value of $\cos^2 15° - \sin^2 15°$

$$\cos^2 15° - \sin^2 15° = \cos(2 \cdot 15°)$$

$$= \cos 30°$$

$$= \frac{\sqrt{3}}{2}$$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

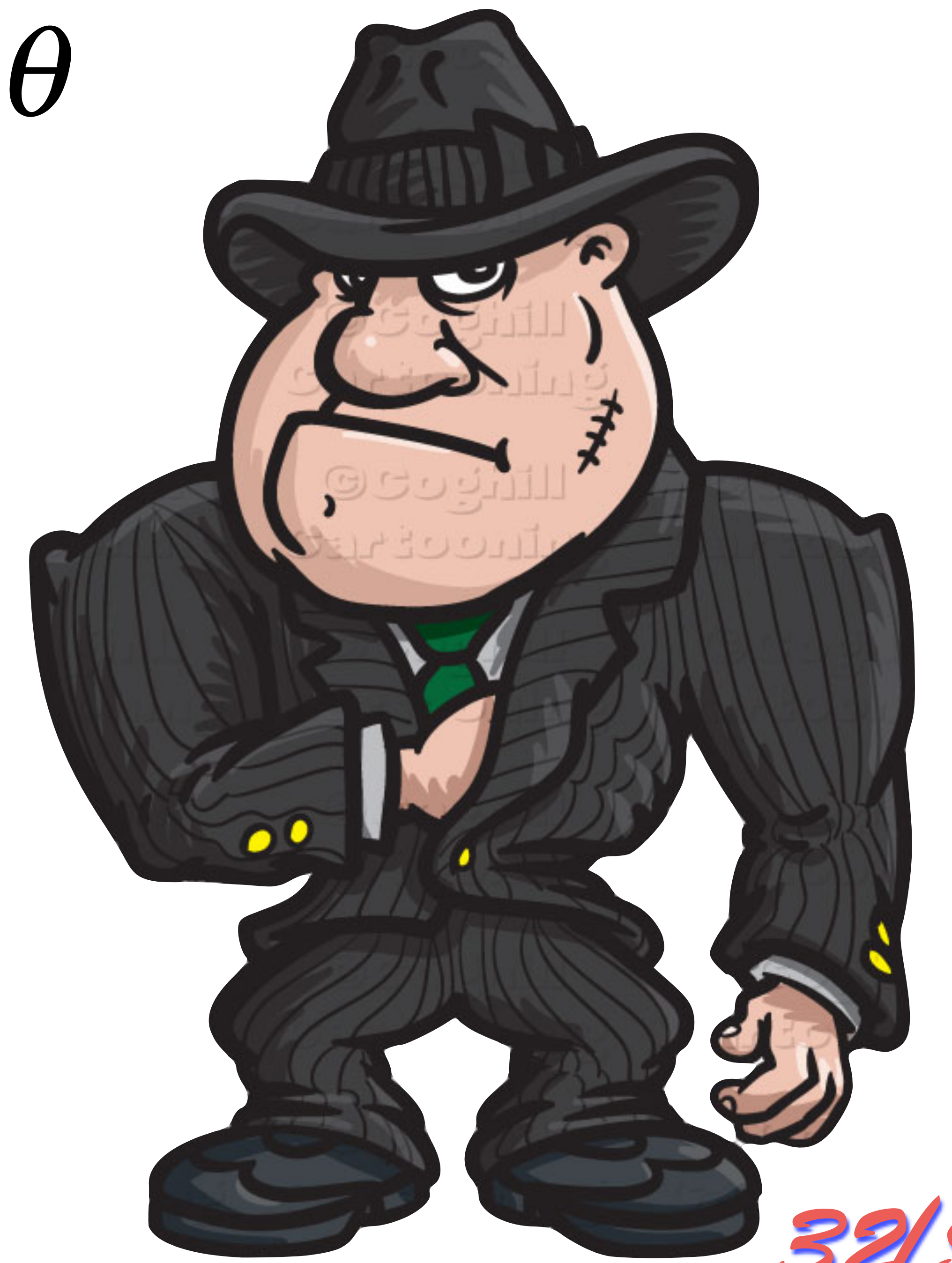
30° is in QI

Example

Verify: $\tan\dfrac{\theta}{2} = \dfrac{\sec\theta}{\sec\theta\csc\theta + \csc\theta}$

$$\frac{\sec\theta}{\sec\theta\csc\theta + \csc\theta} = \frac{\dfrac{1}{\cos\theta}}{\dfrac{1}{\cos\theta\sin\theta} + \dfrac{1}{\sin\theta}} = \frac{\dfrac{1}{\cos\theta}}{\dfrac{1}{\cos\theta\sin\theta} + \dfrac{1}{\sin\theta}} \bullet \frac{\cos\theta\sin\theta}{\cos\theta\sin\theta}$$

$$= \frac{\sin\theta}{1 + \cos\theta} = \tan\frac{\theta}{2}$$



Example

Verify: $\sin 4x = 4\sin x\cos^3 x - 4\sin^3 x\cos x$

$$\sin 4x = \sin 2(2x) = 2\sin 2x\cos 2x$$

$$= 2(2\sin x\cos x)(\cos^2 x - \sin^2 x)$$

$$= (4\sin x\cos x)(\cos^2 x - \sin^2 x)$$

$$= (4\sin x\cos x)(\cos^2 x) - (4\sin x\cos x)(\sin^2 x)$$

$$= (4\sin x\cos^3 x) - (4\sin^3 x\cos x)$$

Example

Verify: $\cos 3x = 4\cos^3 x - 3\cos x$

$$\begin{aligned}
\cos 3x = \cos(2x + x) &= \cos 2x \cos x - \sin 2x \sin x \\
&= (\cos^2 x - \sin^2 x)\cos x - (2\sin x \cos x)\sin x \\
&= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x \\
&= \cos^3 x - 3\sin^2 x \cos x \\
&= \cos^3 x - 3(1 - \cos^2 x)\cos x \\
&= \cos^3 x - 3(\cos x - \cos^3 x) \\
&= \cos^3 x - 3\cos x + 3\cos^3 x \\
&= 4\cos^3 x - 3\cos x
\end{aligned}$$

Example

Find the exact value of $\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

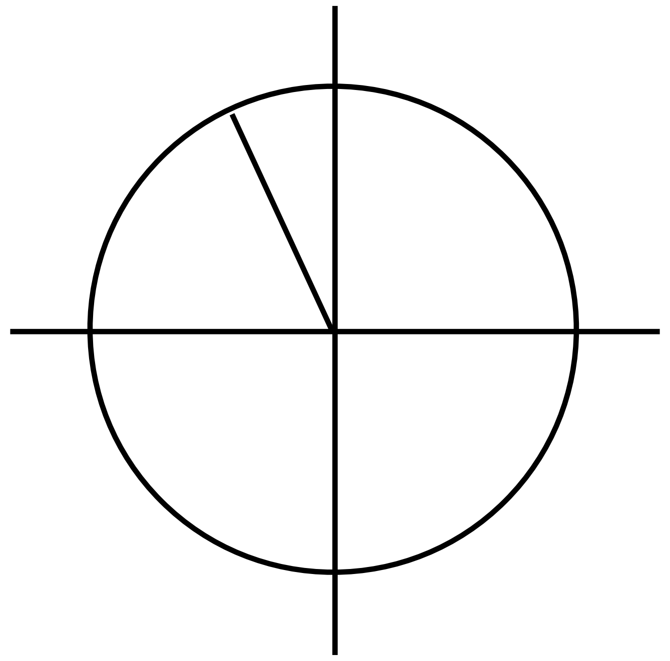
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}} = \tan 2\left(\frac{\pi}{8}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

Example

Find the exact value of $\cos \frac{5\pi}{8}$

$\frac{5\pi}{8}$ is in QII



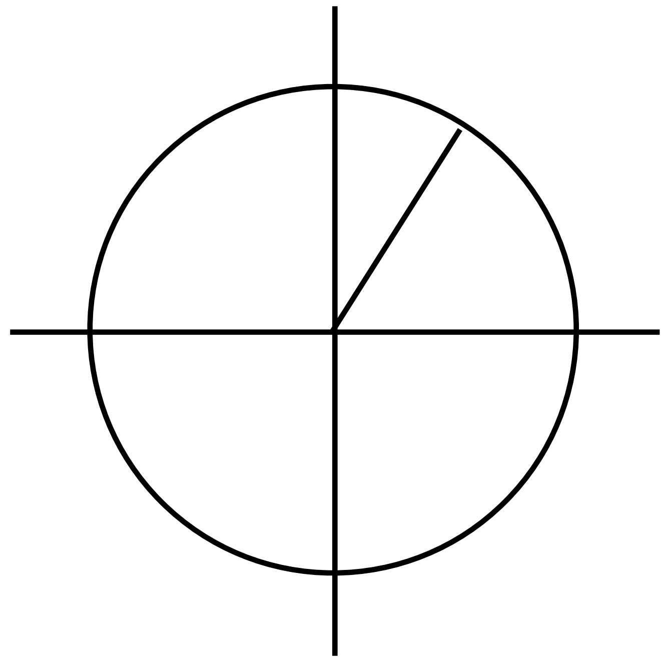
$$\cos \frac{5\pi}{8} = \cos\left(\frac{\frac{5\pi}{4}}{2}\right) = -\sqrt{\left(\frac{1+\cos\frac{5\pi}{4}}{2}\right)} = -\sqrt{\left(\frac{1+-\frac{\sqrt{2}}{2}}{2}\right)}$$

$$= -\sqrt{\frac{1}{2} - \frac{\sqrt{2}}{4}}$$

Example

Find the exact value of $\sin 75°$

75° is in QI



$$\sin 75^\circ = \sin\frac{150^\circ}{2} = \sqrt{\frac{1-\cos 150^\circ}{2}}$$

$$= \sqrt{\frac{1 - -\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}$$

Solve

- Solve $2\sin^2\frac{x}{2} = \cos x$ on [0, 2π).

$$2\sin^2\left(\frac{x}{2}\right) - \cos x = 0$$

$$2\left(\sin\left(\frac{x}{2}\right)\right)^2 - \cos x = 0$$

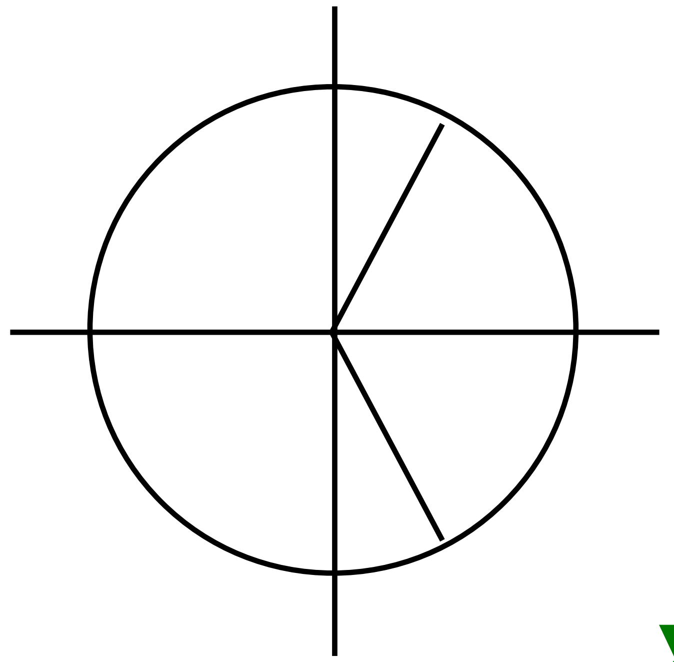
$$2\left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2 - \cos x = 0$$

$$2\left(\frac{1-\cos x}{2}\right) - \cos x = 0$$

$$1 - \cos x - \cos x = 0$$

$$1 - 2\cos x = 0$$

$$\cos x = \frac{1}{2}$$



$$x = \frac{\pi}{3} \quad x = \frac{5\pi}{3}$$

Product to Sum Formulae

Verify

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

$$= \frac{1}{2}\left[(\cos\alpha\cos\beta+\sin\alpha\sin\beta)-(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\right]$$

$$= \frac{1}{2}\left[\cancel{\cos\alpha\cos\beta}+\sin\alpha\sin\beta-\cancel{\cos\alpha\cos\beta}+\sin\alpha\sin\beta\right]$$

$$= \frac{1}{2}\left[2\sin\alpha\sin\beta\right]$$

$$= \sin\alpha\sin\beta$$

Product to Sum Formulae

Verify $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$

$$= \frac{1}{2}\left[(\cos\alpha\cos\beta+\sin\alpha\sin\beta)+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\right]$$

$$= \frac{1}{2}\left[\cos\alpha\cos\beta+\cancel{\sin\alpha\sin\beta}+\cos\alpha\cos\beta-\sin\alpha\sin\beta\right]$$

$$= \frac{1}{2}\left[2\cos\alpha\cos\beta\right]$$

$$= \cos\alpha\cos\beta$$

Product to Sum Formulae

Verify $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$$= \frac{1}{2}\left[(\sin\alpha\cos\beta + \cos\alpha\sin\beta) + (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\right]$$

$$= \frac{1}{2}\left[2\sin\alpha\cos\beta\right]$$

$$= \sin\alpha\cos\beta$$

Product to Sum Formulae

Verify $\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$

$$= \frac{1}{2}\left[(\cancel{\sin\alpha\cos\beta} + \cos\alpha\sin\beta) - (\cancel{\sin\alpha\cos\beta} - \cos\alpha\sin\beta)\right]$$

$$= \frac{1}{2}\left[2\cos\alpha\sin\beta\right]$$

$$= \cos\alpha\sin\beta$$

The Product to Sum Formulae

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

Using the Product-to-Sum Formulas


Express the product as a sum or difference: $\sin 5x \sin 2x$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

$$\sin 5x \sin 2x = \frac{1}{2}\left[\cos(5x-2x)-\cos(5x+2x)\right]$$

$$= \frac{1}{2}\left[\cos 3x - \cos 7x\right]$$

Using the Product-to-Sum Formulas

Express the product as a sum or difference: $\cos 7x \cos x$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

$$\cos 7x \cos x = \frac{1}{2}\left[\cos(7x - x) + \cos(7x + x)\right]$$

$$= \frac{1}{2}\left[\cos 6x + \cos 8x\right]$$

Sum to Product Formulae

Verify $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\bullet\frac{1}{2}\left[\sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)+\sin\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)\right]$$

$$=\left[\sin\left(\frac{2\alpha}{2}\right)+\sin\left(\frac{2\beta}{2}\right)\right] = \sin\alpha+\sin\beta$$

Sum to Product Formulae

Verify $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} = 2\bullet\frac{1}{2}\left[\sin\left(\frac{\alpha-\cancel{\beta}}{2}+\frac{\alpha+\cancel{\beta}}{2}\right)+\sin\left(\frac{\cancel{\alpha}-\beta}{2}-\frac{\cancel{\alpha}+\beta}{2}\right)\right]$$

$$=\left[\sin\left(\frac{2\alpha}{2}\right)+\sin\left(-\frac{2\beta}{2}\right)\right]=\left[\sin(\alpha)+\sin(-\beta)\right]$$

$$=\sin\alpha-\sin\beta$$

Sum to Product Formulae

Verify $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\bullet\frac{1}{2}\left[\cos\left(\frac{\cancel{\alpha}+\beta}{2}-\frac{\cancel{\alpha}-\beta}{2}\right)+\cos\left(\frac{\alpha+\cancel{\beta}}{2}+\frac{\alpha-\cancel{\beta}}{2}\right)\right]$$

$$=\left[\cos\left(\frac{2\beta}{2}\right)+\cos\left(\frac{2\alpha}{2}\right)\right]$$

$$=\cos\alpha+\cos\beta$$

Sum to Product Formulae

 Verify $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = -2\bullet\frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)\right]$$

$$= -1\left[\cos\left(\frac{2\beta}{2}\right)-\cos\left(\frac{2\alpha}{2}\right)\right] \quad = -1\left[\cos(\beta)-\cos(\alpha)\right]$$

$$= \cos\alpha - \cos\beta$$

The Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Today's Menu

Product to Sum

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

Sum to Product

$$\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Using the Sum-to-Product Formulas

 Express the sum as a product: $\sin 7x + \sin 3x$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin 7x + \sin 3x = 2\sin\frac{7x+3x}{2}\cos\frac{7x-3x}{2}$$

$$= 2\sin 5x\cos 2x$$

Using the Sum-to-Product Formulas

Express the sum as a product: $\cos 3x + \cos 2x$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos 3x + \cos 2x = 2\cos\frac{3x+2x}{2}\cos\frac{3x-2x}{2}$$

$$= 2\cos\frac{5x}{2}\cos\frac{x}{2}$$

Example

- Express the product as a sum or difference: $\sin 2x \cos 3x$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\sin 2x \cos 3x = \frac{1}{2}\left[\sin(2x+3x)+\sin(2x-3x)\right]$$

$$= \frac{1}{2}\left[\sin 5x+\sin(-x)\right]$$

$$= \frac{1}{2}\left[\sin 5x-\sin x\right]$$

Ejemplo otra vez

 Express the product as a sum or difference: $2\sin 74^\circ \cos 114^\circ$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$2\sin 74^\circ \cos 114^\circ = 2\bullet\frac{1}{2}\left[\sin(74+114)^\circ + \sin(74-114)^\circ\right]$$

$$= 1\left[\sin 188^\circ + \sin(-40^\circ)\right]$$

$$= \sin 188^\circ - \sin 40^\circ$$

Example again

Verify: $\tan x = \dfrac{\sin 3x - \sin x}{\cos 3x + \cos x}$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \frac{2\sin\frac{3x-x}{2}\cos\frac{3x+x}{2}}{2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2}} = \frac{2\sin\frac{2x}{2}\cos\frac{4x}{2}}{2\cos\frac{4x}{2}\cos\frac{2x}{2}}$$

$$= \frac{2\sin x\cos 2x}{2\cos 2x\cos x} = \tan x$$

Another One

- Find the exact value of sin 195° - sin 105°.

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\sin 195^\circ - \sin 105^\circ = 2\left[\cos\left(\frac{195+105}{2}\right)^\circ \sin\left(\frac{195-105}{2}\right)^\circ\right]$$

$$= 2\left[\cos(150)^\circ \sin(45)^\circ\right]$$

$$= 2\left[-\frac{\sqrt{3}}{2}\bullet\frac{\sqrt{2}}{2}\right] = -\frac{\sqrt{6}}{2}$$

Last one

 Verify $\dfrac{\sin 7x - \sin 5x}{\cos 7x - \cos 5x} = -\cot 6x$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\frac{\sin 7x - \sin 5x}{\cos 7x - \cos 5x} = \frac{2\cos\left(\frac{7x+5x}{2}\right)\sin\left(\frac{7x-5x}{2}\right)}{-2\sin\left(\frac{7x+5x}{2}\right)\sin\left(\frac{7x-5x}{2}\right)}$$

$$= \frac{\cos(6x)}{-\sin(6x)} \quad = -\cot 6x$$

Solve

- Solve cos 3x + cos x = 0 on [0, 2π)

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos 3x + \cos x = 0$$

$$2\left(\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)\right) = 0$$

$$2(\cos(2x)\cos(x)) = 0$$

$$\cos(2x) = 0$$

$$2x = \frac{\pi}{2} \quad 2x = \frac{3\pi}{2} \quad 2x = \frac{5\pi}{2} \quad 2x = \frac{7\pi}{2}$$

$$x = \frac{\pi}{4} \quad x = \frac{3\pi}{4} \quad x = \frac{5\pi}{4} \quad x = \frac{7\pi}{4}$$

$$\cos(x) = 0$$

$$x = \frac{\pi}{2} \quad x = \frac{3\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$$