Chapter 6

6.4 The Dot Product

Chapter 6.4

Homework

Read Sec 6-4

Do p467 1-73 odd

Chapter 6.4

Objectives

Find the dot product of two vectors.

Find the angle between two vectors.

Use the dot product to determine if two vectors are orthogonal.

Express a vector as the sum of two orthogonal vectors.

Compute work.

Chapter 6.4

Reminder

Draw a Picture

Dot Product

The Dot Product of Two Vectors

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

We have learned that the operations of vector addition and scalar multiplication result in **vectors**.

The **dot product** of two vectors results in a **scalar** (real number) value, rather than a vector.

If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$ are vectors, the **dot product** $\mathbf{v} \cdot \mathbf{w}$ is defined as:

$$\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2$$

The **dot product** of two vectors is the **sum of the products of the horizontal components and vertical components** of the two vectors.

Dot Product

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

From your text

Definition of the Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Definition of the Dot Product

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

If $\mathbf{v} = 7\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$, find each of the following dot products:

a. $\mathbf{v} \cdot \mathbf{w}$ $\quad \mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2 \quad = 7 \cdot 2 + -4 \cdot -1 = 18$

b. $\mathbf{w} \cdot \mathbf{v}$ $\quad = 2 \cdot 7 + -1 \cdot -4 = 18$

c. $\mathbf{w} \cdot \mathbf{w}$ $\quad = 2 \cdot 2 + -1 \cdot -1 = 5$

d. $\mathbf{v} \cdot \mathbf{v}$ $\quad = 7 \cdot 7 + -4 \cdot -4 = 65$

Properties of the Dot Product

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

If u, v and w are vectors, and k is a scalar, then:

1. $u \cdot v = v \cdot u$
2. $u \cdot (v + w) = u \cdot v + u \cdot w$
3. $0 \cdot v = 0$
4. $v \cdot v = \|v\|^2$
5. $(k\,u) \cdot v = k(u \cdot v) = u \cdot (kv)$

Example

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

If $\mathbf{u} = 5\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} - 7\mathbf{j}$, find $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w})$

Method 1	Method 2
$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$	$(\mathbf{v}+\mathbf{w}) = \mathbf{i} - \mathbf{j} + 3\mathbf{i} - 7\mathbf{j}$
$= [5(1)+2(-1)] + [5(3)+2(-7)]$	$(\mathbf{v}+\mathbf{w}) = \mathbf{i} - \mathbf{j} + 3\mathbf{i} - 7\mathbf{j}$
$= 3 + 1$	$(\mathbf{v}+\mathbf{w}) = 4\mathbf{i} - 8\mathbf{j}$
$= 4$	$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = [5(4)+2(-8)]$
	$= 20 + (-16)$
	$= 4$

Example

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

Let $u = \langle 2, 6, v = \langle -1, 5,$ and $w = \langle -3, 1 \rangle$. Find:

$$u \cdot v = 2(-1)+6(5) = 28$$

$$u \cdot w = 2(-3)+6(1) = 0$$

$$v \cdot w = -1(-3)+5(1) = 8$$

$$u \cdot (v + w) = u \cdot v + u \cdot w = 28+0 = 28$$

$$v \cdot (u + w) = v \cdot u + v \cdot w = 28+8 = 36$$

$$w \cdot (v + u) = w \cdot v + w \cdot u = 8+0 = 8$$

Alternate Dot Product

Included Angle

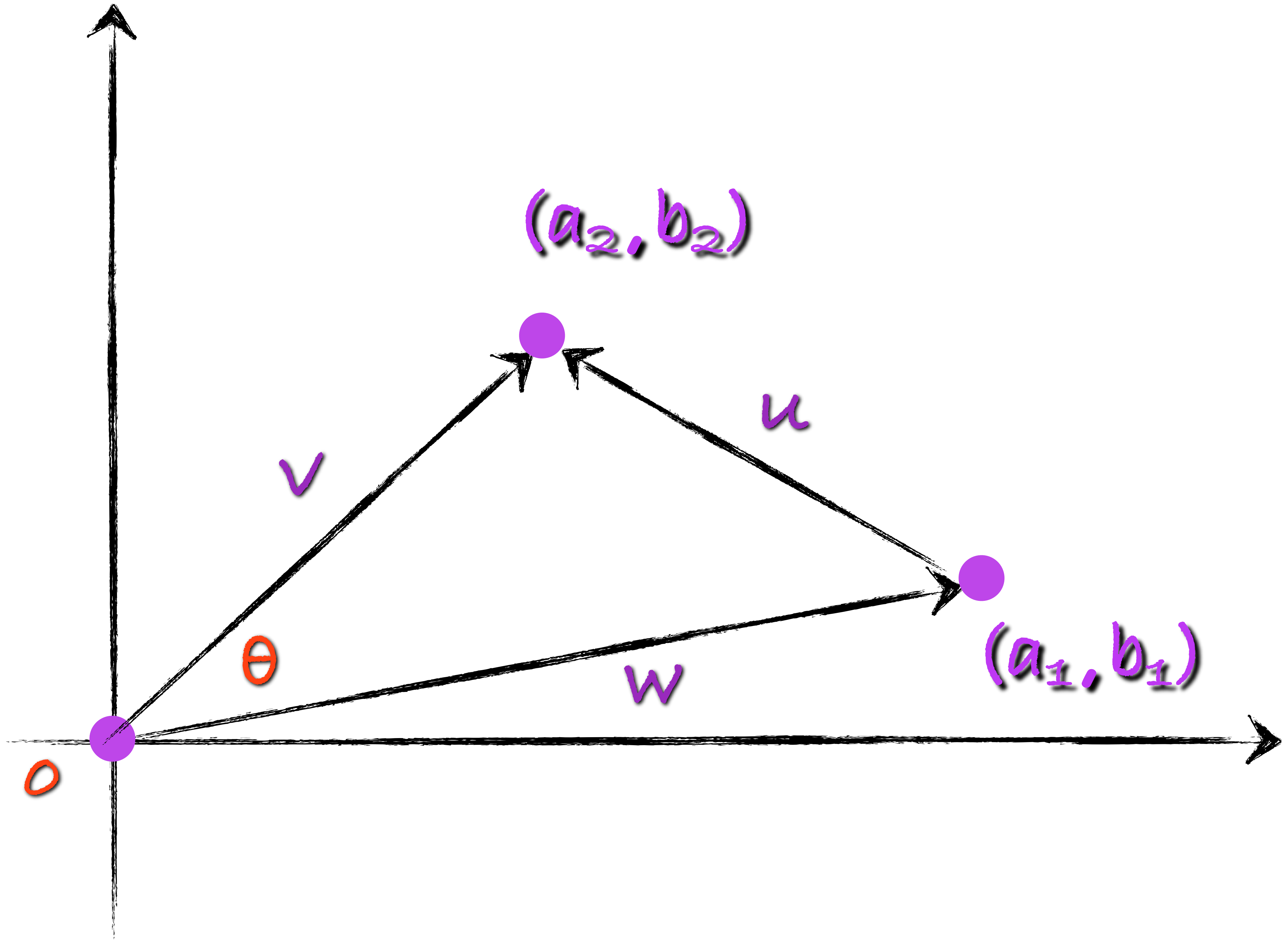
Dot Product and Included Angle

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

If v and w are vectors, and θ is a the smallest, non-negative angle between the vectors in standard position ($0 \le \theta \le \pi$), then:

$$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

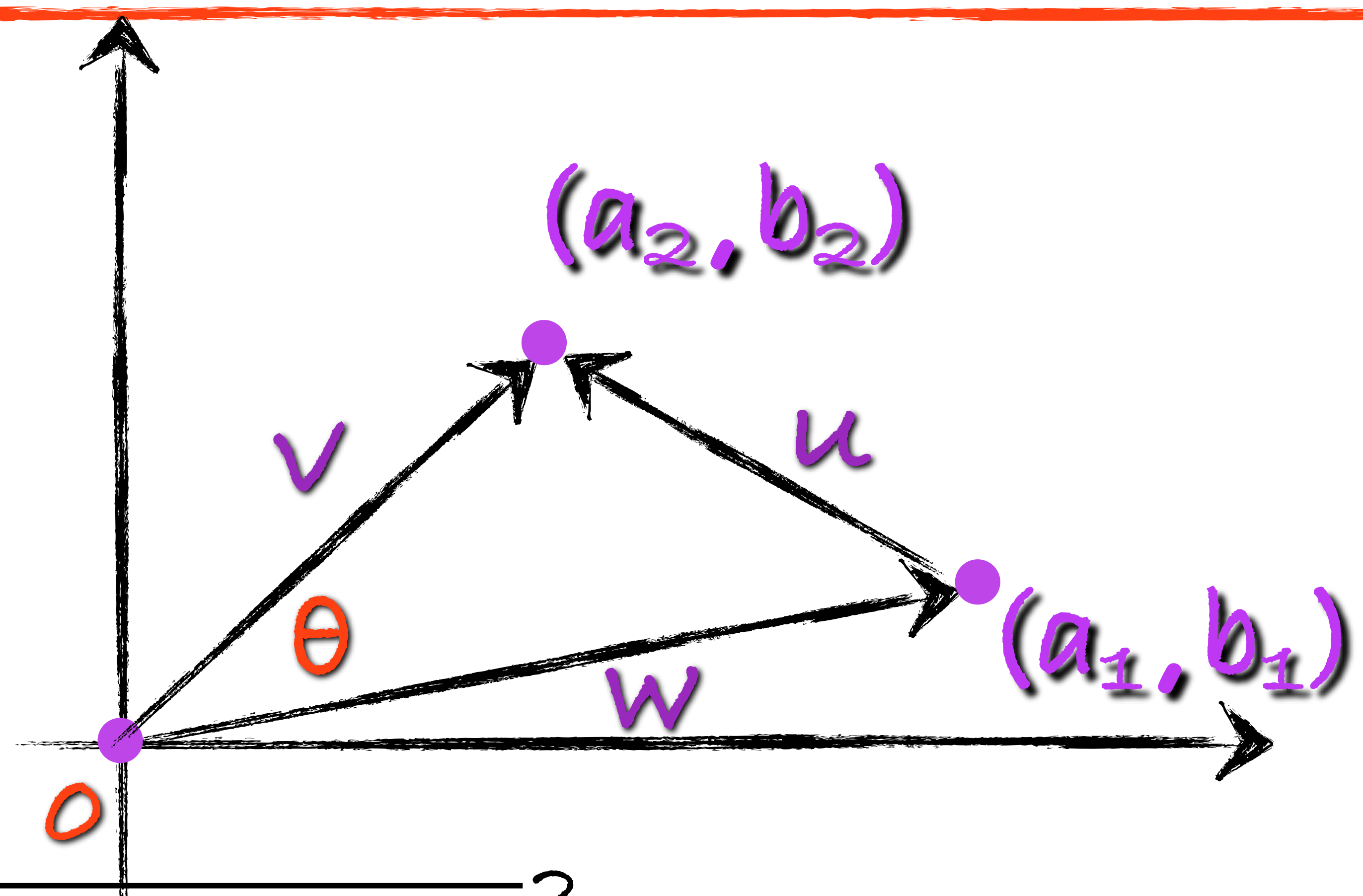
$$\cos\theta = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$



Formula for the Angle between Two Vectors

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

$$v \bullet w = \|v\|\|w\|\cos\theta$$



$$\|u\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta$$

$$\sqrt{(a_2-a_1)^2+(b_2-b_1)^2}^{\,2} = \sqrt{(a_2-0)^2+(b_2-0)^2}^{\,2} + \sqrt{(a_1-0)^2+(b_1-0)^2}^{\,2} - 2\|v\|\|w\|\cos\theta$$

$$(a_2-a_1)^2+(b_2-b_1)^2 = (a_2)^2+(b_2)^2+(a_1)^2+(b_1)^2 - 2\|v\|\|w\|\cos\theta$$

$$(\cancel{a_2^2} - 2a_2a_1 + \cancel{a_1^2}) + (\cancel{b_2^2} - 2b_2b_1 + \cancel{b_1^2}) = \cancel{(a_2)^2} + \cancel{(b_2)^2} + \cancel{(a_1)^2} + \cancel{(b_1)^2} - 2\|v\|\|w\|\cos\theta$$

$$-2a_2a_1 + -2b_2b_1 = -2\|v\|\|w\|\cos\theta$$

$$a_2a_1 + b_2b_1 = \|v\|\|w\|\cos\theta \qquad a_2a_1 + b_2b_1 = v \bullet w$$

$$v \bullet w = \|v\|\|w\|\cos\theta$$

Included Angle

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

From your text

Angle Between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

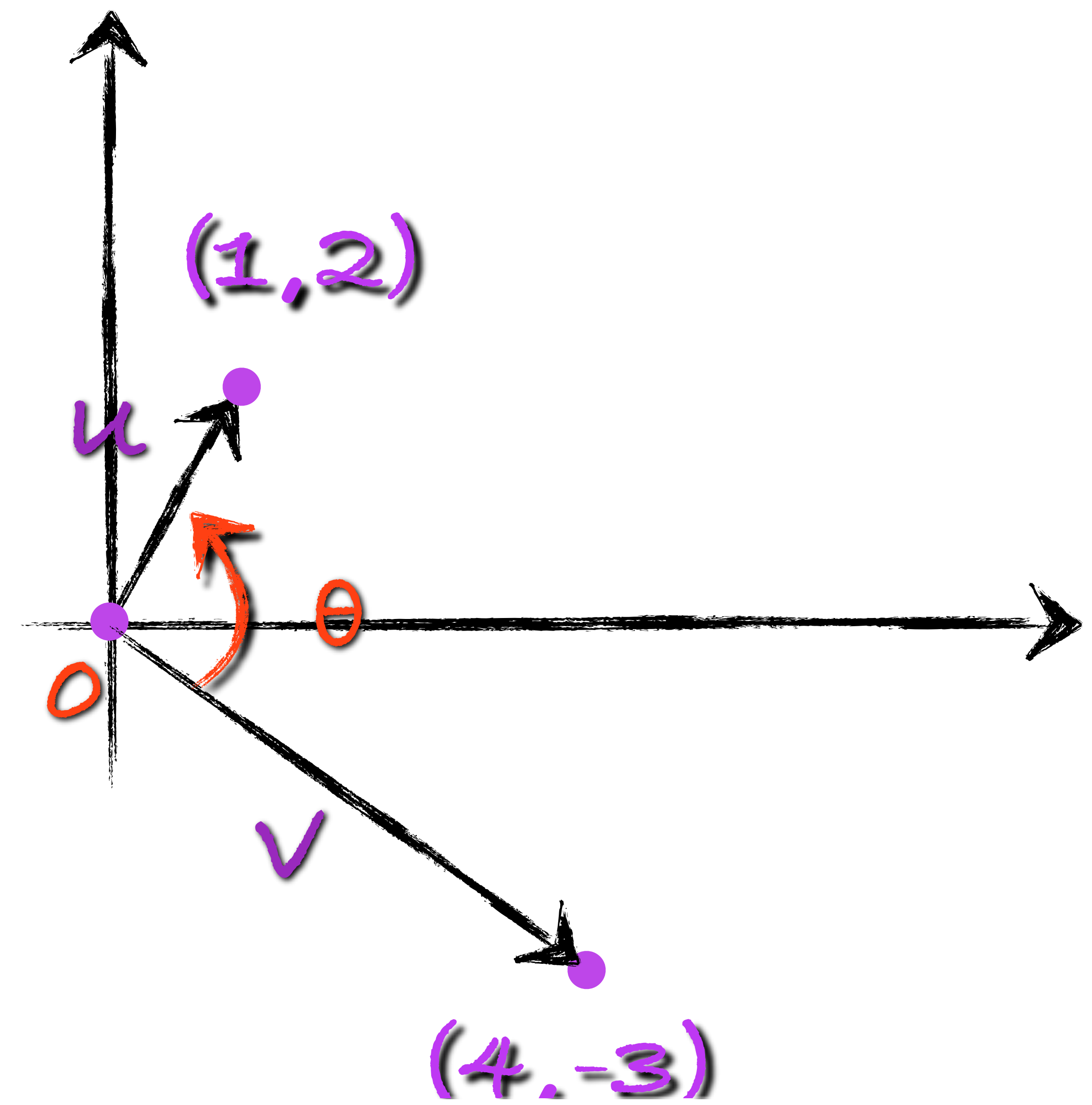
$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Formula for the Angle between Two Vectors

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

Find the angle between the two vectors $v = 4i - 3j$ and $w = i + 2j$. Round to the nearest tenth of a degree.

$$\cos\theta = \frac{v \bullet w}{\|v\|\|w\|}$$



$$\cos\theta = \frac{4(1)+(-3)2}{\sqrt{4^2+(-3)^2}\sqrt{1^2+2^2}}$$

$$\cos\theta = \frac{-2}{\sqrt{25}\sqrt{5}} = \frac{-2}{5\sqrt{5}}$$

$$\cos^{-1}\left(-\frac{2}{5\sqrt{5}}\right) \approx 100.30^\circ$$

Formula for the Angle between Two Vectors

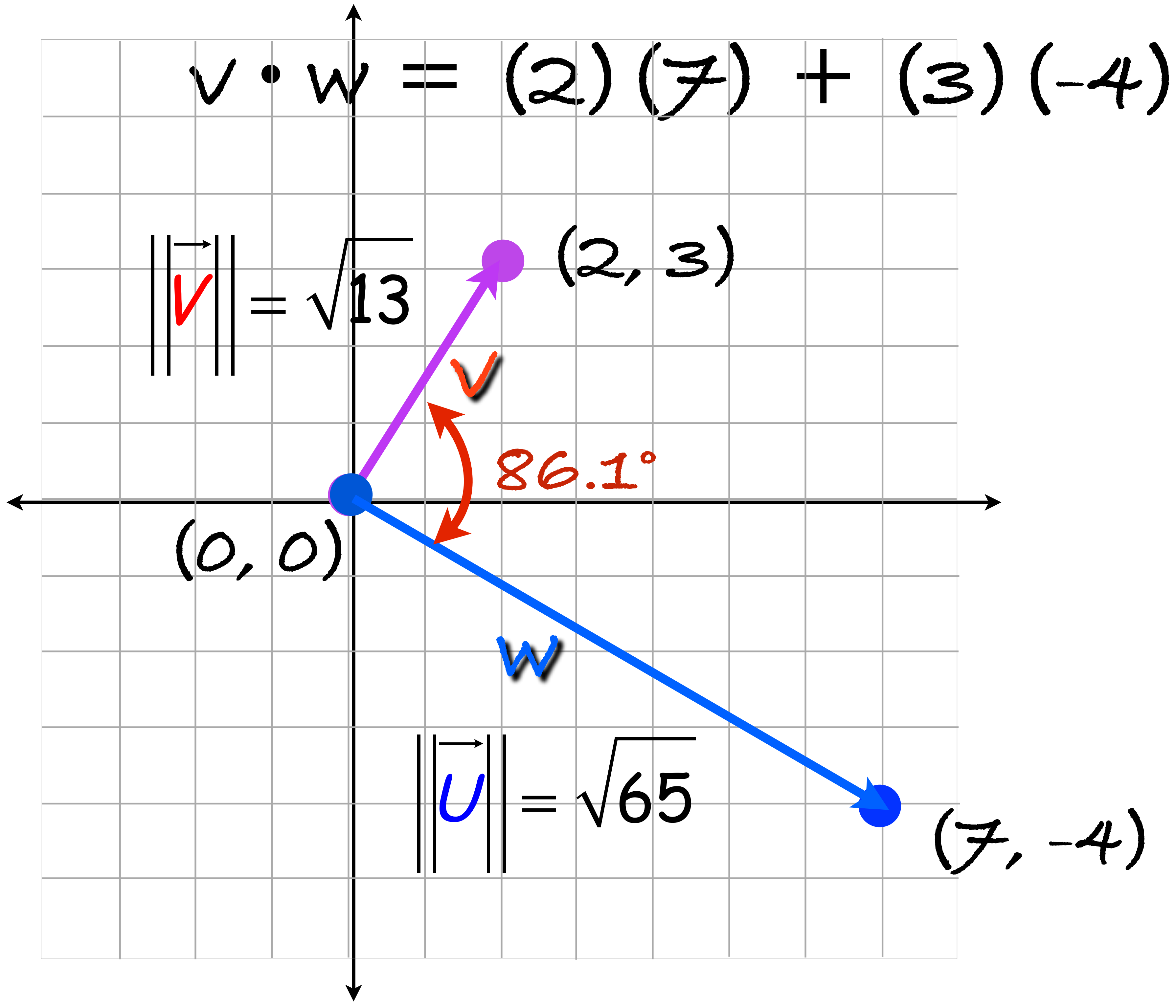
Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

If $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 7\mathbf{i} - 4\mathbf{j}$, find each of the dot product and the angle between them.

$$\mathbf{v} \cdot \mathbf{w} = a_1 a_2 + b_1 b_2$$

$$\cos\theta = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

$$v \cdot w = (2)(7) + (3)(-4) = 2$$



$$\cos\theta = \frac{2}{\sqrt{2^2 + (3)^2}\sqrt{7^2 + (-4)^2}}$$

$$\cos\theta = \frac{2}{\sqrt{13}\sqrt{65}} = \frac{2}{\sqrt{845}} \approx .0688$$

$$\cos^{-1}\left(\frac{2}{\sqrt{845}}\right) \approx 86.1^\circ$$

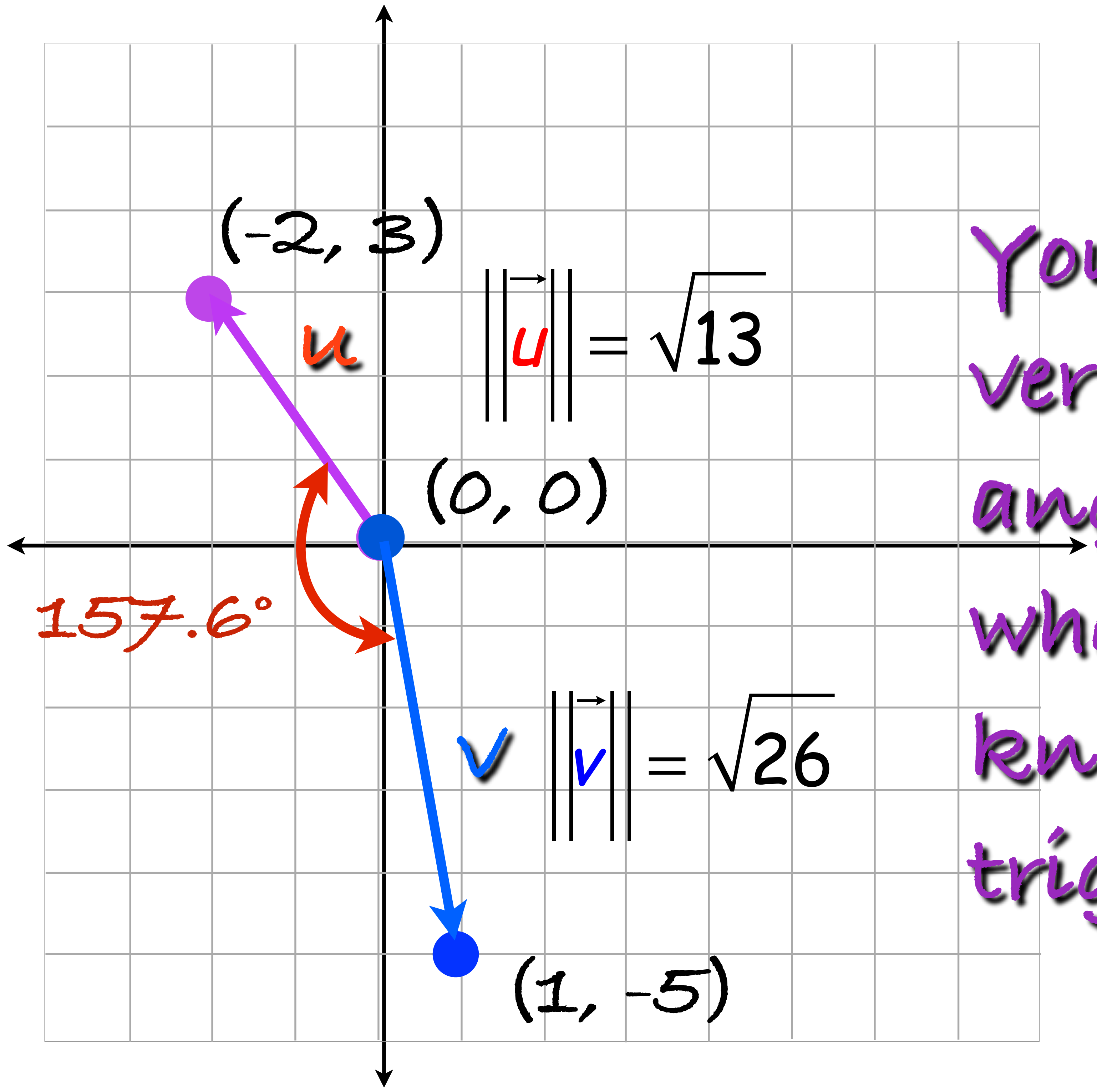
Example

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

Find the angle between $u = \langle -2, 3 \rangle$ and $v = \langle 1, -5 \rangle$

$$u \cdot v = a_1 a_2 + b_1 b_2$$

$$v \cdot w = (-2)(1) + (3)(-5) = -17$$



You can verify the angle using what you know about trigonometry

$$\cos\theta = \frac{v \bullet w}{\|v\|\|w\|}$$

$$\cos\theta = \frac{-17}{\sqrt{(-2)^2 + (3)^2}\sqrt{1^2 + (-5)^2}}$$

$$\cos\theta = \frac{-17}{\sqrt{13}\sqrt{26}} = \frac{-17}{\sqrt{338}}$$

$$\cos^{-1}\left(\frac{-17}{\sqrt{338}}\right) \approx 157.6199^\circ$$

Parallel and Orthogonal Vectors

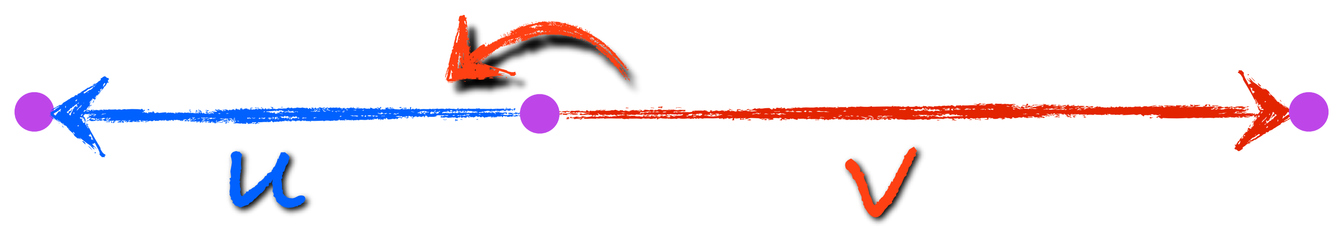
Parallel and Orthogonal Vectors

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

Two vectors are **parallel** when the angle θ between the vectors is 0° or 180°. If $\theta = 0°$, the vectors point in the same direction. If $\theta = 180°$, the vectors point in opposite directions.

vectors point in opposite directions

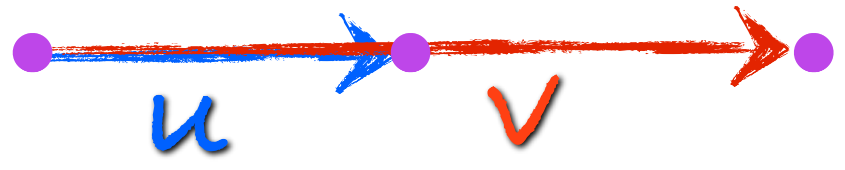
$\theta = 180°$



$\cos\theta = -1$

vectors point in the same direction

$\theta = 0°$



$\cos\theta = 1$

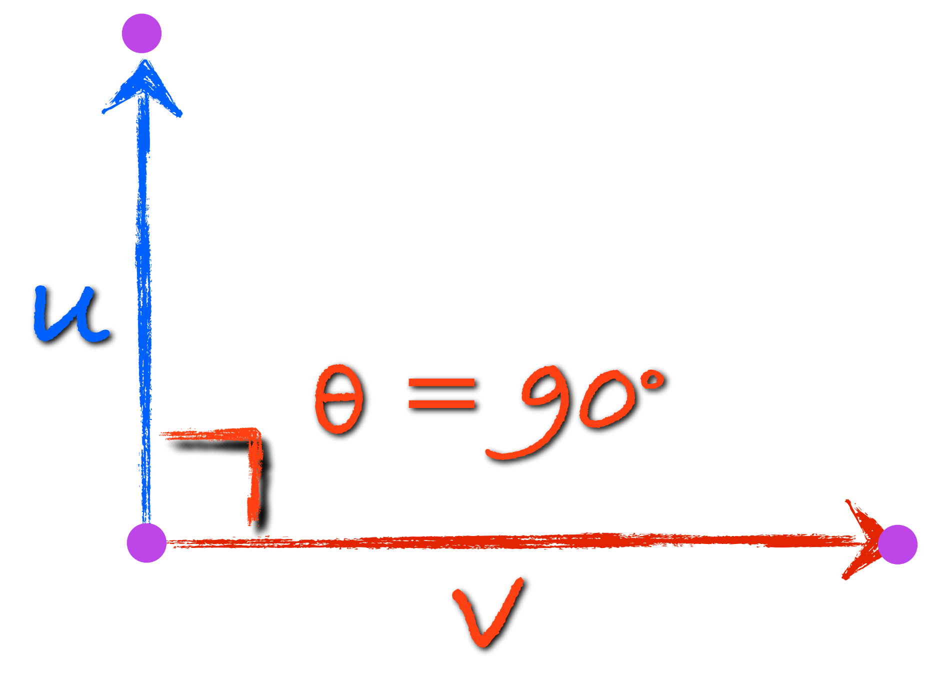
$$v \bullet u = \|v\|\|u\|\cos\theta$$

Parallel and Orthogonal Vectors

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

Two vectors are orthogonal when the angle θ between the vectors is $90°$.

Orthogonal vectors



$$v \bullet u = \|v\|\|u\|\cos\theta$$

$$v \bullet u = 0$$

Two nonzero vectors v and u are orthogonal, iff $v \cdot w = 0$.

Lines are perpendicular, vectors are orthogonal.

The zero vector is orthogonal to every vector.

Determining Whether Vectors are Orthogonal

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

Are the vectors $v = 2i + 3j$ and $w = 6i - 4j$ orthogonal?

$$v \cdot w = a_1a_2 + b_1b_2$$

$$v \cdot w = 2(6) + 3(-4)$$

$$= 12 + -12$$

Since $v \cdot w = 0$. v and w are orthogonal.

Orthogonal

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

Are the two vectors $u = \langle -4, 2 \rangle$ and $v = \langle 1, 2 \rangle$ orthogonal?

$$u \cdot v = a_1 a_2 + b_1 b_2$$

$$u \cdot v = -4(1) + 2(2)$$

$$= -4 + 4$$

Since $u \cdot v = 0$. u and v are orthogonal.

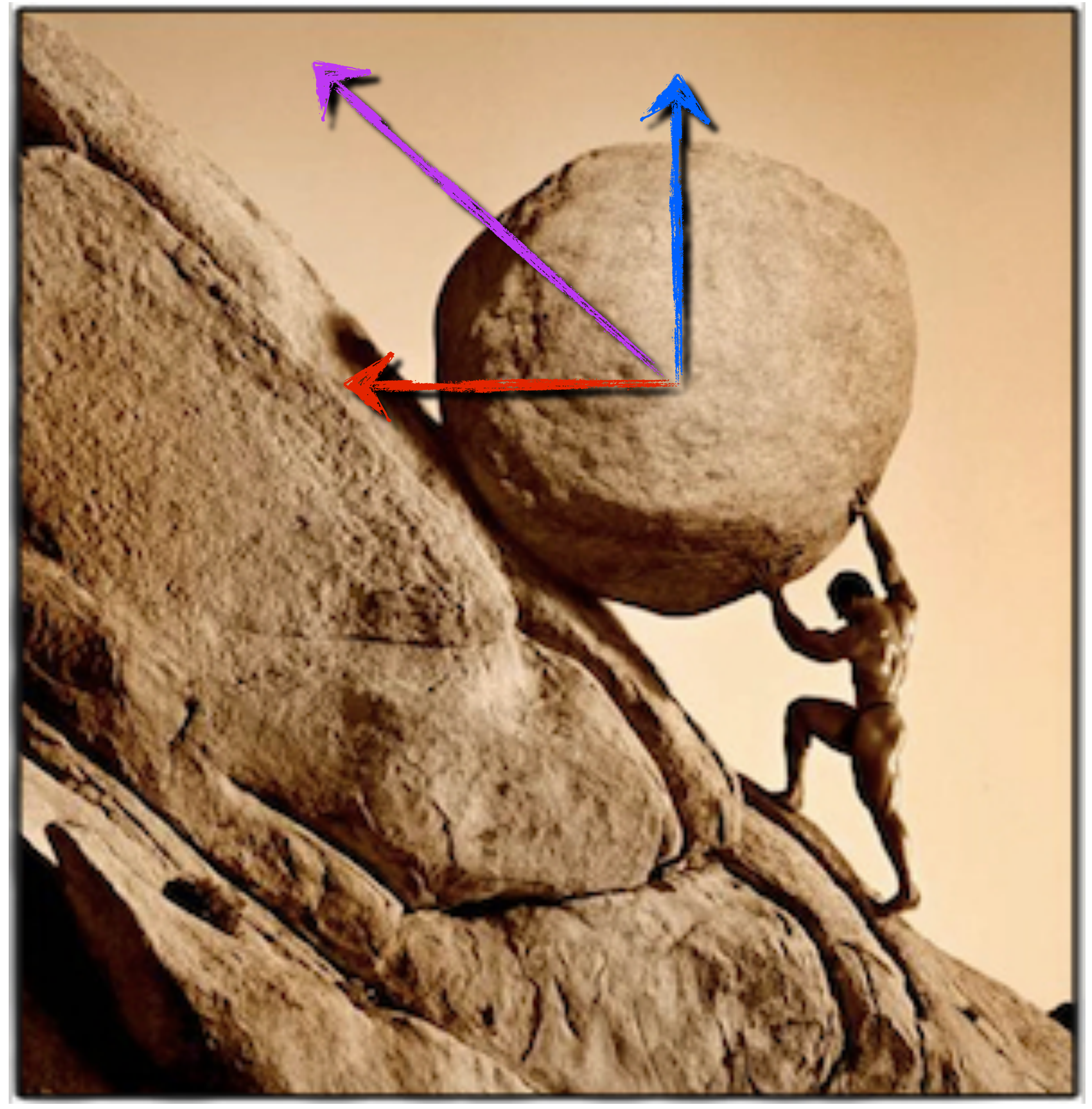
Decomposition of a Vector

Decomposing a Vector into Two Orthogonal Vectors

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

When Sisyphus is pushing his stone up the hill for all eternity, he is applying a force to the stone in two directions. He is pushing the stone forward and up. The force on the stone is the resultant vector of two component vectors that are orthogonal to each other.

We can deconstruct the resultant vector into orthogonal component vectors with a bit of work.



Decomposing a Vector into Two Orthogonal Vectors

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

We now know how to add two vectors to get a resultant vector. But we have yet to discuss how to determine component vectors from what is a resultant vector.

Obviously there are an infinite number of possibilities, so we choose to only find orthogonal component vectors. To do that, we reverse the process of adding two orthogonal vectors.

In other words, we find orthogonal vectors whose sum is the original vector.

If u is a nonzero vector such that $u = w_1 + w_2$, where w_1 and w_2 are orthogonal, then w_1 and w_2 are called **vector components of u**.

The Vector Projection of v onto w

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

We must first define a new process called the projection of v onto w.

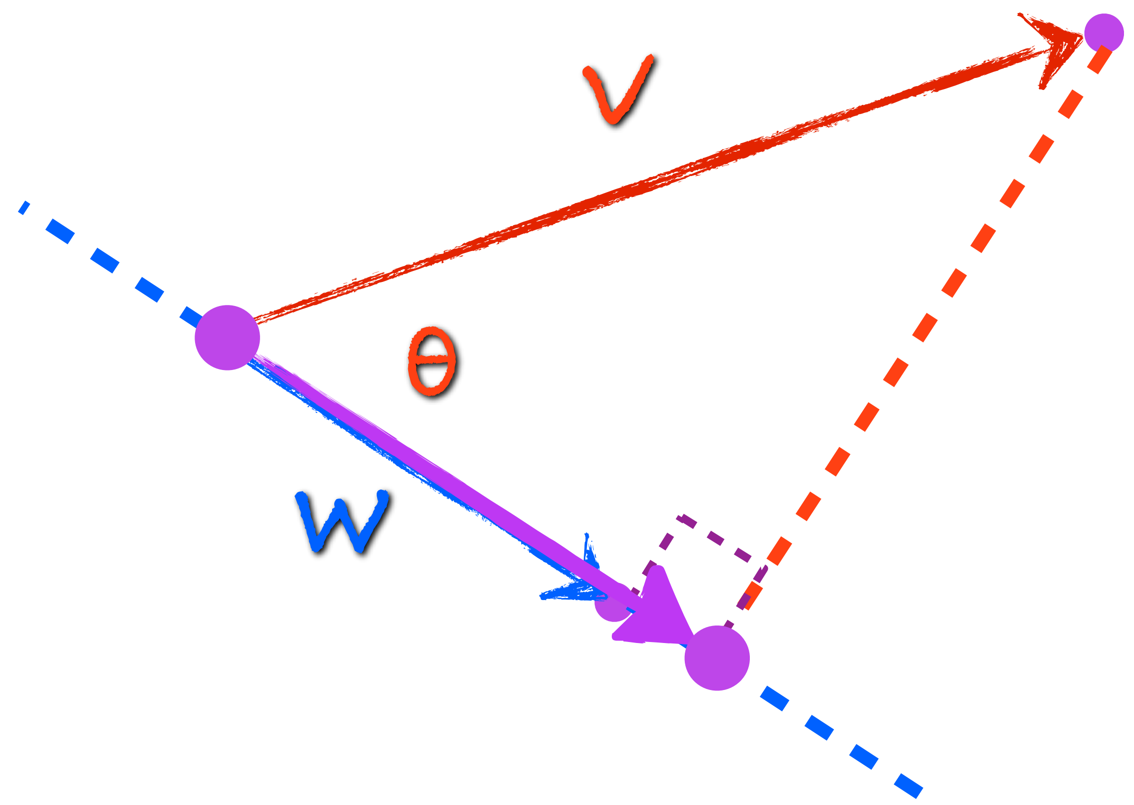
Vector Projection

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

the projection of v onto w

Start with vectors forming an acute angle at an initial point.

Draw a line segment containing w and extending past the initial and terminal points.



From the terminal point of v drop a segment perpendicular to the line containing w.

The new vector, denoted $\text{proj}_w v$ has initial point at the vertex and terminal point at the intersection of our two new segments.

The Vector Projection of v onto w

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

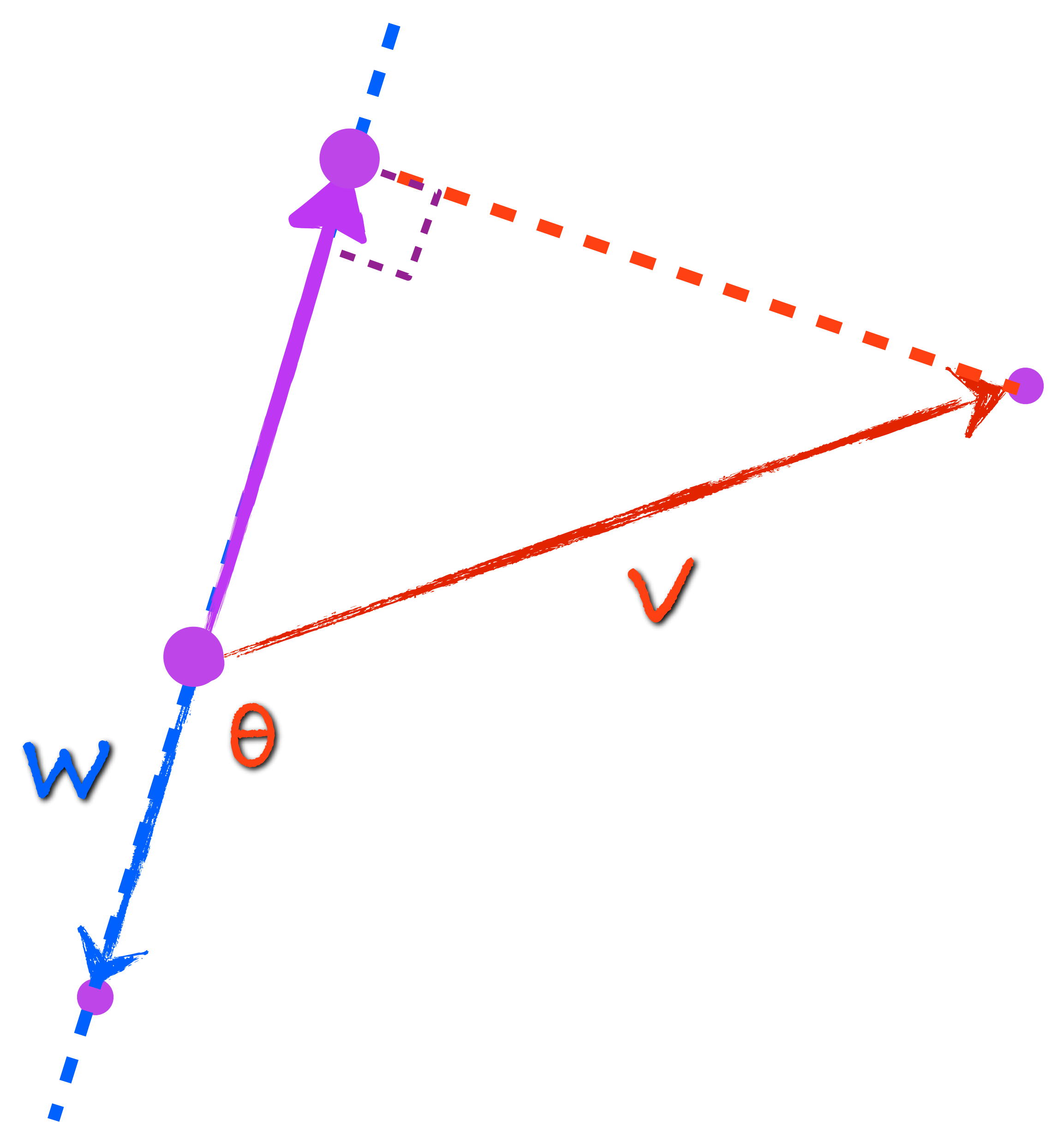
the projection of v onto w

Start with vectors forming an obtuse angle at an initial point.

Draw a line segment containing w and extending past the initial and terminal points.

From the terminal point of v drop a segment perpendicular to the line containing w.

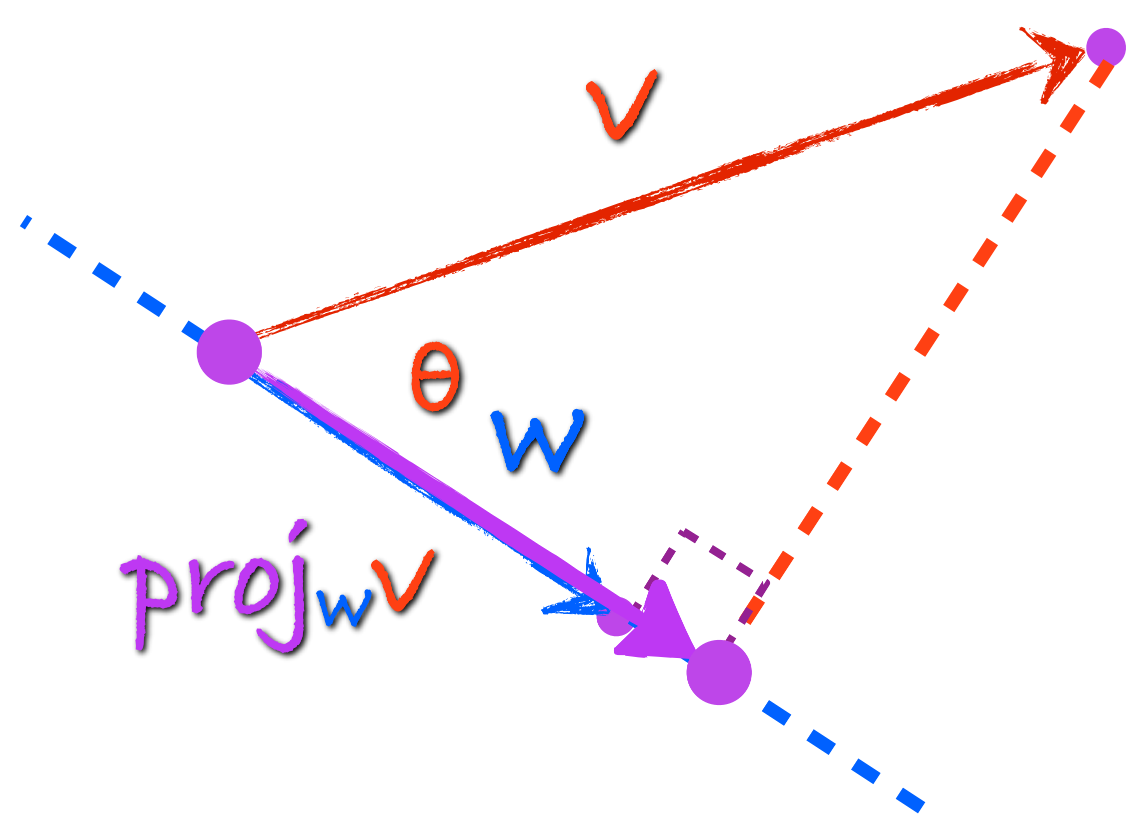
The new vector, denoted $\text{proj}_w v$ has initial point at the vertex and terminal point at the intersection of our two new segments.



The Vector Projection of v onto w

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

To define $\text{proj}_w v$ we must determine magnitude and direction.



magnitude

From the drawing $\cos\theta = \dfrac{\|\text{proj}_w v\|}{\|v\|}$

Thus $\|\text{proj}_w v\| = \|v\|\cos\theta$

From the dot product $v \bullet w = \|v\|\|w\|\cos\theta$

Thus $\dfrac{v \bullet w}{\|w\|} = \|v\|\cos\theta$

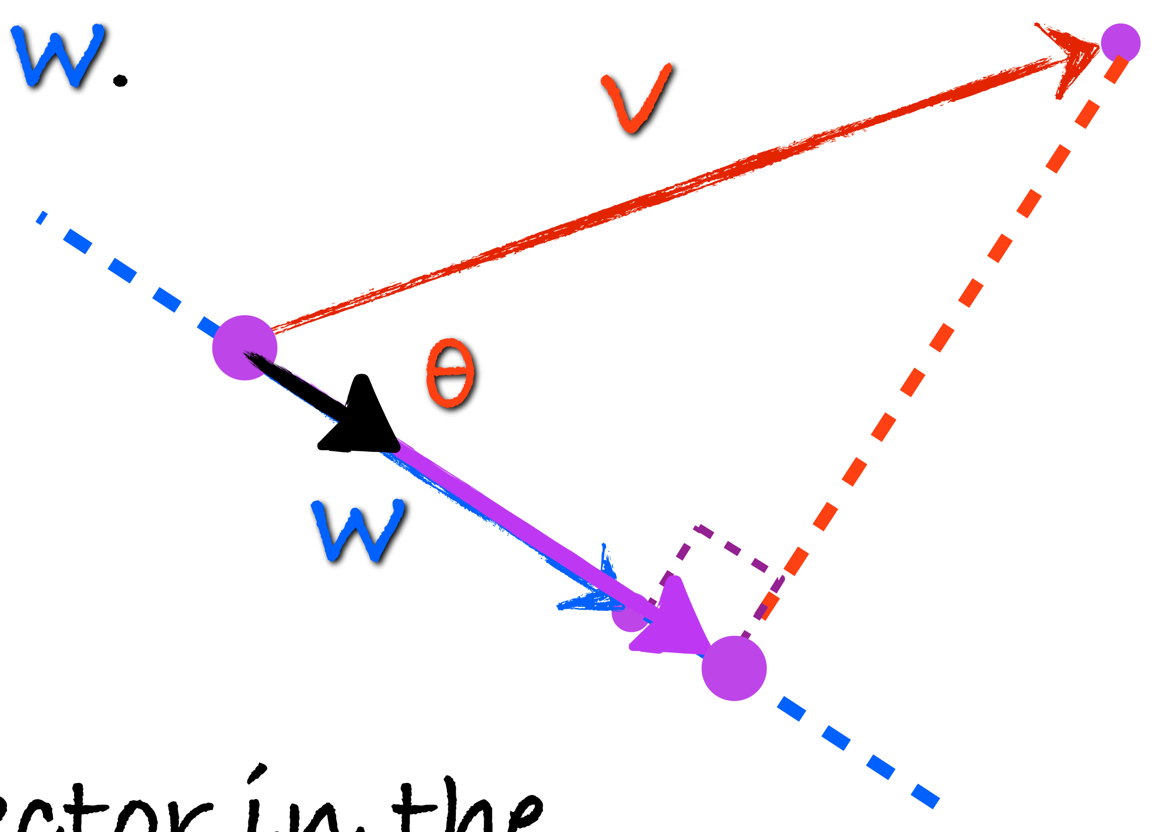
and $\|\text{proj}_w v\| = \|v\|\cos\theta = \dfrac{v \bullet w}{\|w\|}$ This is the magnitude of $\text{proj}_w v$

The Vector Projection of v onto w

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

Now we can fully define $\text{proj}_w v$.

The vector $\text{proj}_w v$ is on the same line segment as w. So we can define $\text{proj}_w v$ by using the unit vector in the same direction as w.



The unit vector with same direction as w is: $\dfrac{w}{\|w\|}$

If we multiply the magnitude $\|\text{proj}_w v\|$ (a scalar) by the unit vector in the direction of w, we will have the vector $\text{proj}_w v$ itself.

$$\text{proj}_w v = \left(\frac{v \bullet w}{\|w\|}\right)\left(\frac{w}{\|w\|}\right) = \frac{v \bullet w}{\|w\|^2} w$$

Component Vectors

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

Definition of Vector Components

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors such that

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal and $\mathbf{w}_1$ is parallel to (or a scalar multiple of) $\mathbf{v}$, as shown in Figure 6.38. The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are called **vector components of u**. The vector $\mathbf{w}_1$ is the **projection** of $\mathbf{u}$ onto $\mathbf{v}$ and is denoted by

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}.$$

The vector $\mathbf{w}_2$ is given by $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

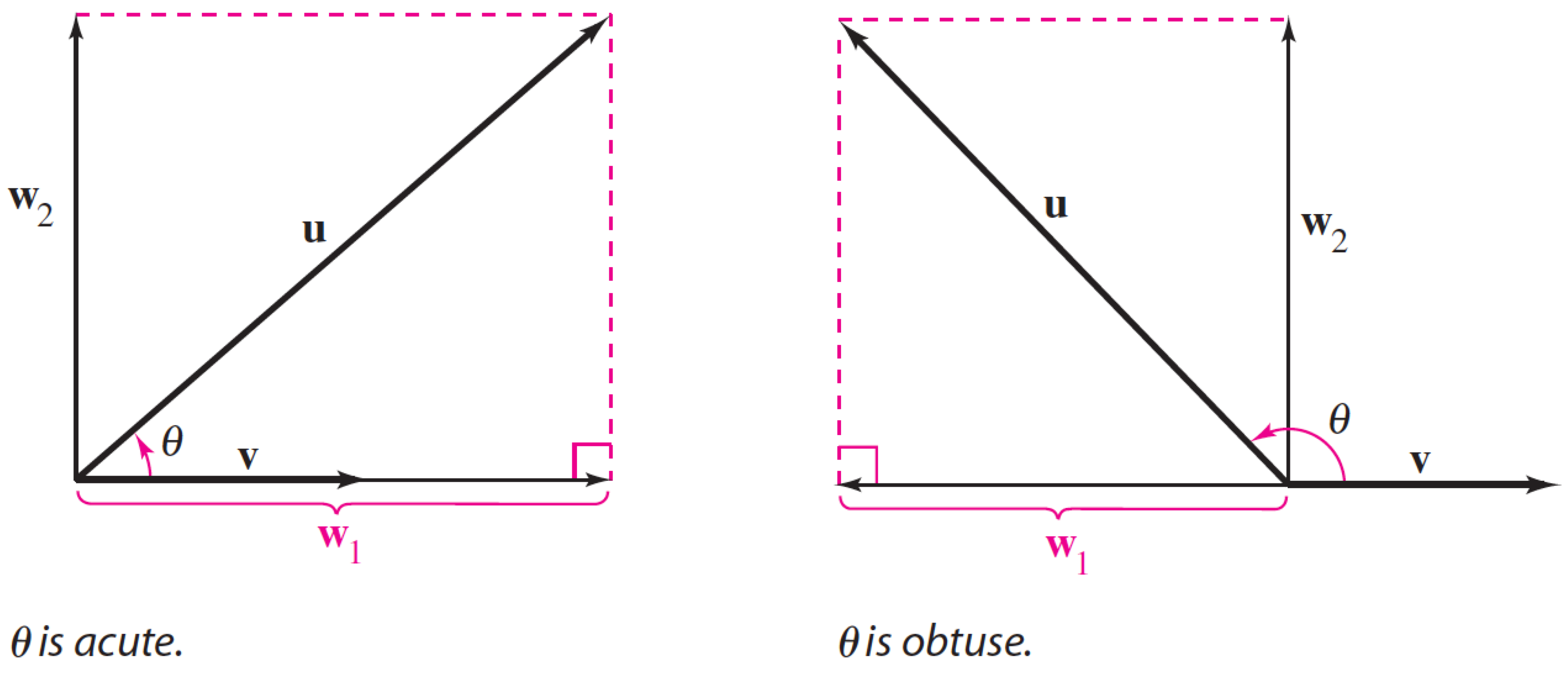


FIGURE 6.38

It may help to think of the projection of u as the shadow cast by u from light rays perpendicular to v.

Example: Finding the Vector Projection

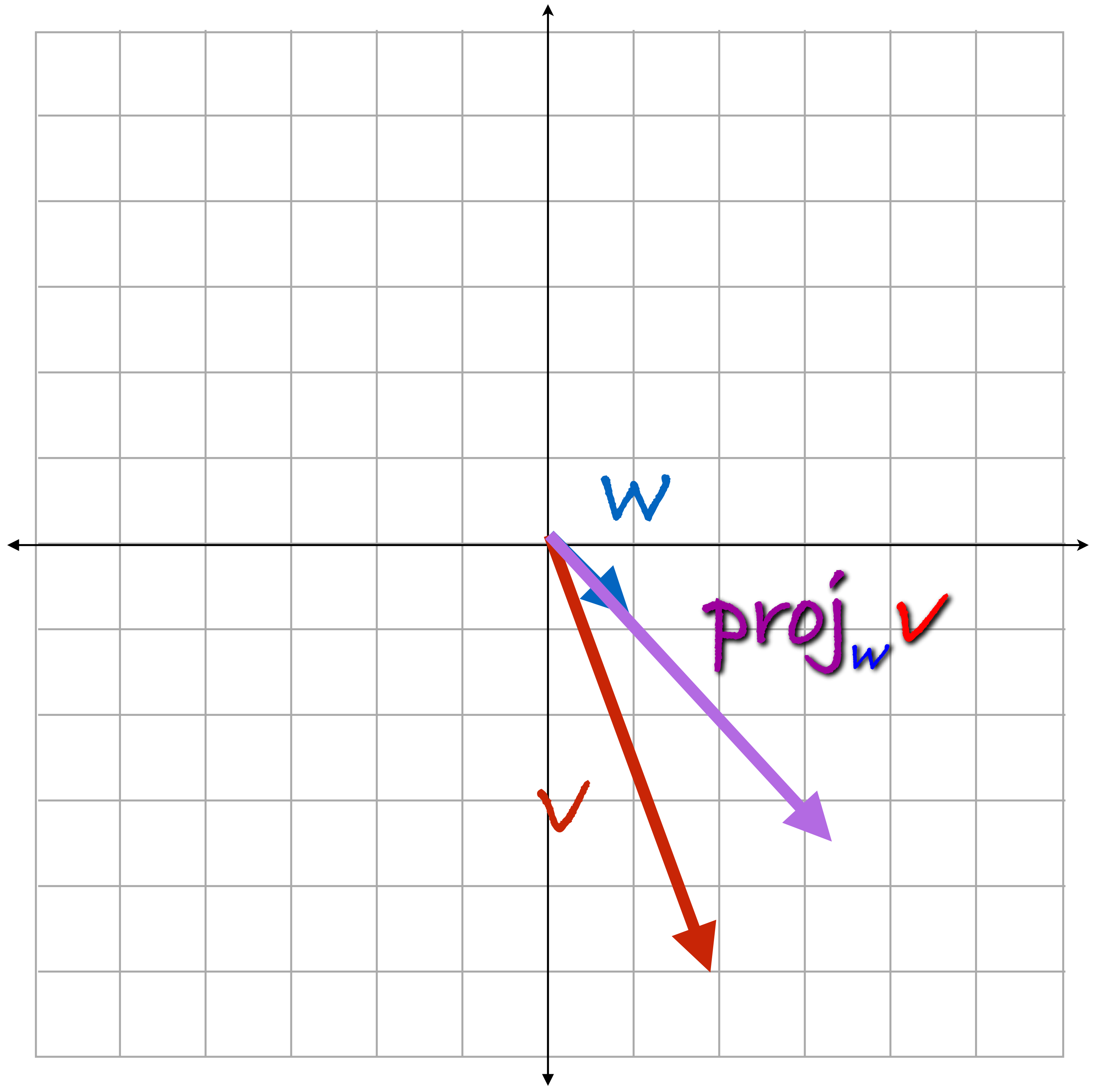
Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

If $v = 2i - 5j$ and $w = i - j$, find the vector projection of v onto w.

$$\text{proj}_w v = \frac{v \bullet w}{\|w\|^2} w \qquad v \cdot w = a_1 a_2 + b_1 b_2 \qquad \|\vec{u}\| = \sqrt{a^2 + b^2}$$

$$\text{proj}_w v = \frac{v \bullet w}{\|w\|^2} w = \frac{2(1)+(-5)(-1)}{\sqrt{1^2+(-1)^2}^{\,2}}(i-j)$$

$$= \frac{7}{2}(i-j) \quad = \frac{7}{2}i - \frac{7}{2}j$$



Example: Finding the Vector Projection

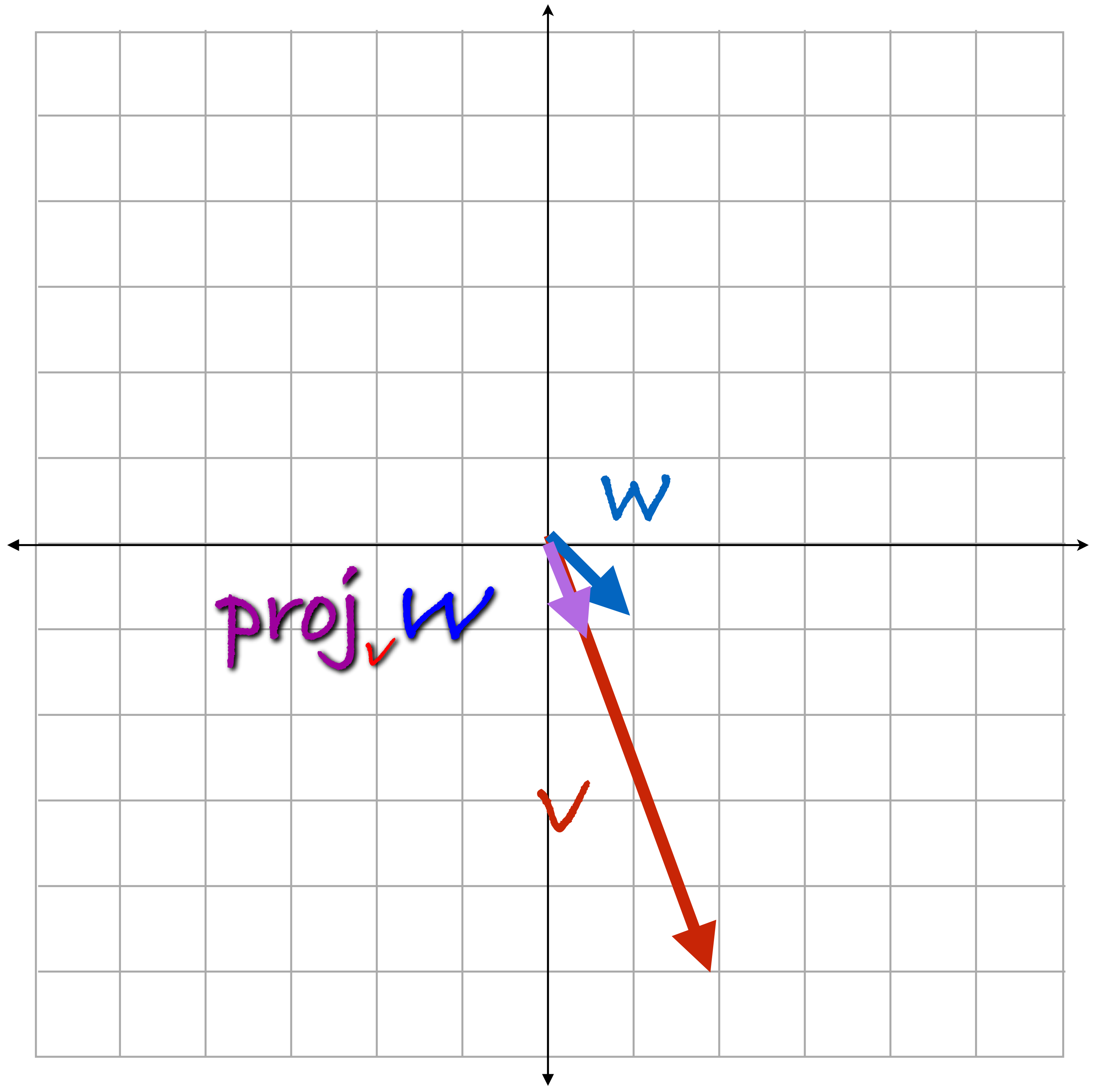
Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

If $v = 2i - 5j$ and $w = i - j$, find the vector projection of w onto v.

$$\text{proj}_v w = \frac{w \bullet v}{\|v\|^2} v \qquad v \cdot w = a_1 a_2 + b_1 b_2 \qquad \|\vec{u}\| = \sqrt{a^2 + b^2}$$

$$\text{proj}_v w = \frac{w \bullet v}{\|v\|^2} v = \frac{2(1) + (-5)(-1)}{\sqrt{2^2 + (-5)^2}^{\,2}} (2i - 5j)$$

$$= \frac{7}{29}(2i - 5j) \quad = \frac{14}{29} i - \frac{35}{29} j$$



Example: Finding the Vector Projection

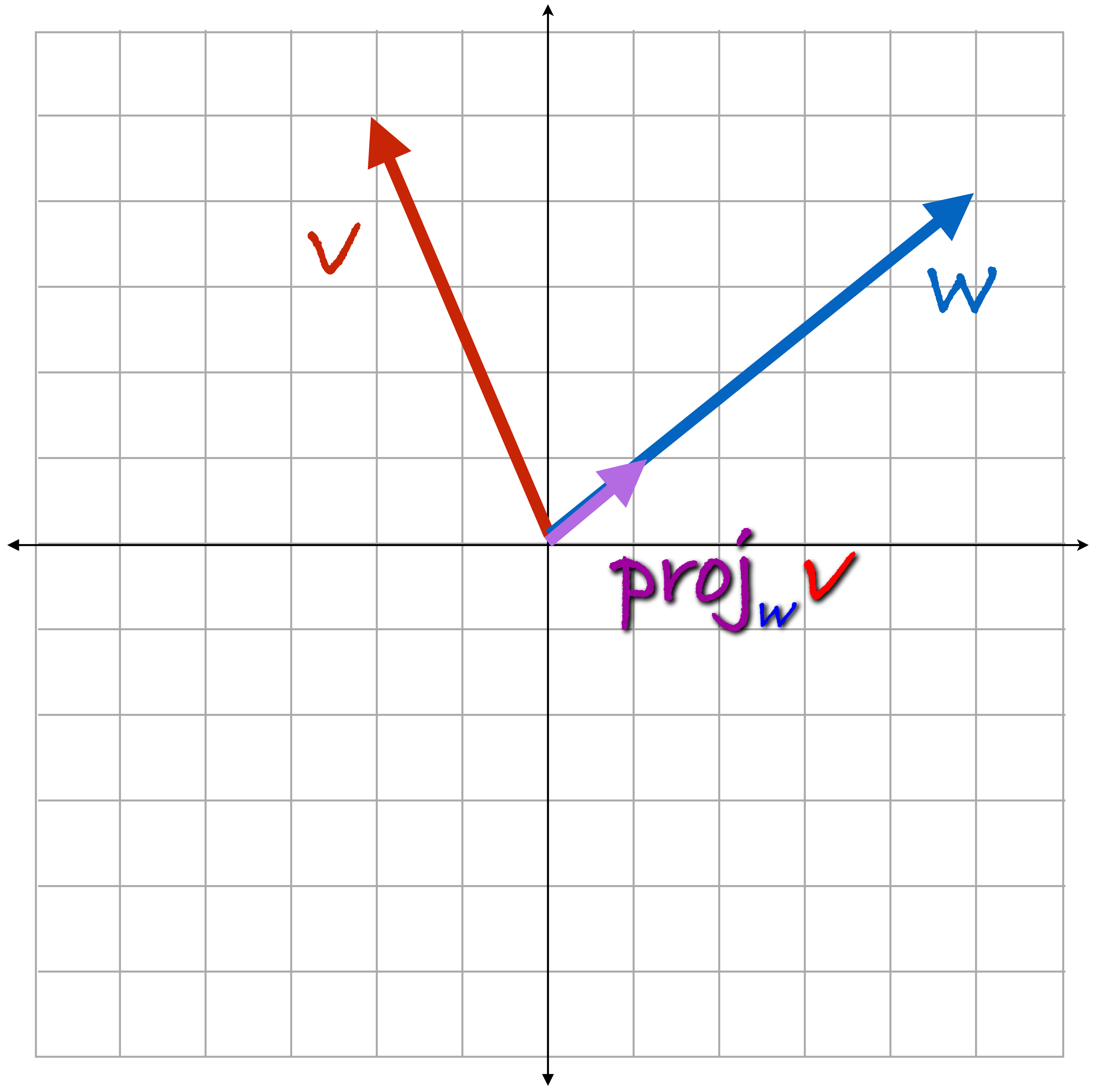
Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

If v = -2i + 5j and w = 5i + 4j, find the vector projection of v onto w.

$$\text{proj}_w v = \frac{v \bullet w}{\|w\|^2} w \qquad v \cdot w = a_1a_2 + b_1b_2 \qquad \|\vec{u}\| = \sqrt{a^2 + b^2}$$

$$= \frac{(-2)(5)+(5)(4)}{\sqrt{5^2+(4)^2}^{\,2}}(5i+4j)$$

$$= \frac{10}{41}(5i+4j) = \frac{50}{41}i + \frac{40}{41}j$$



Example

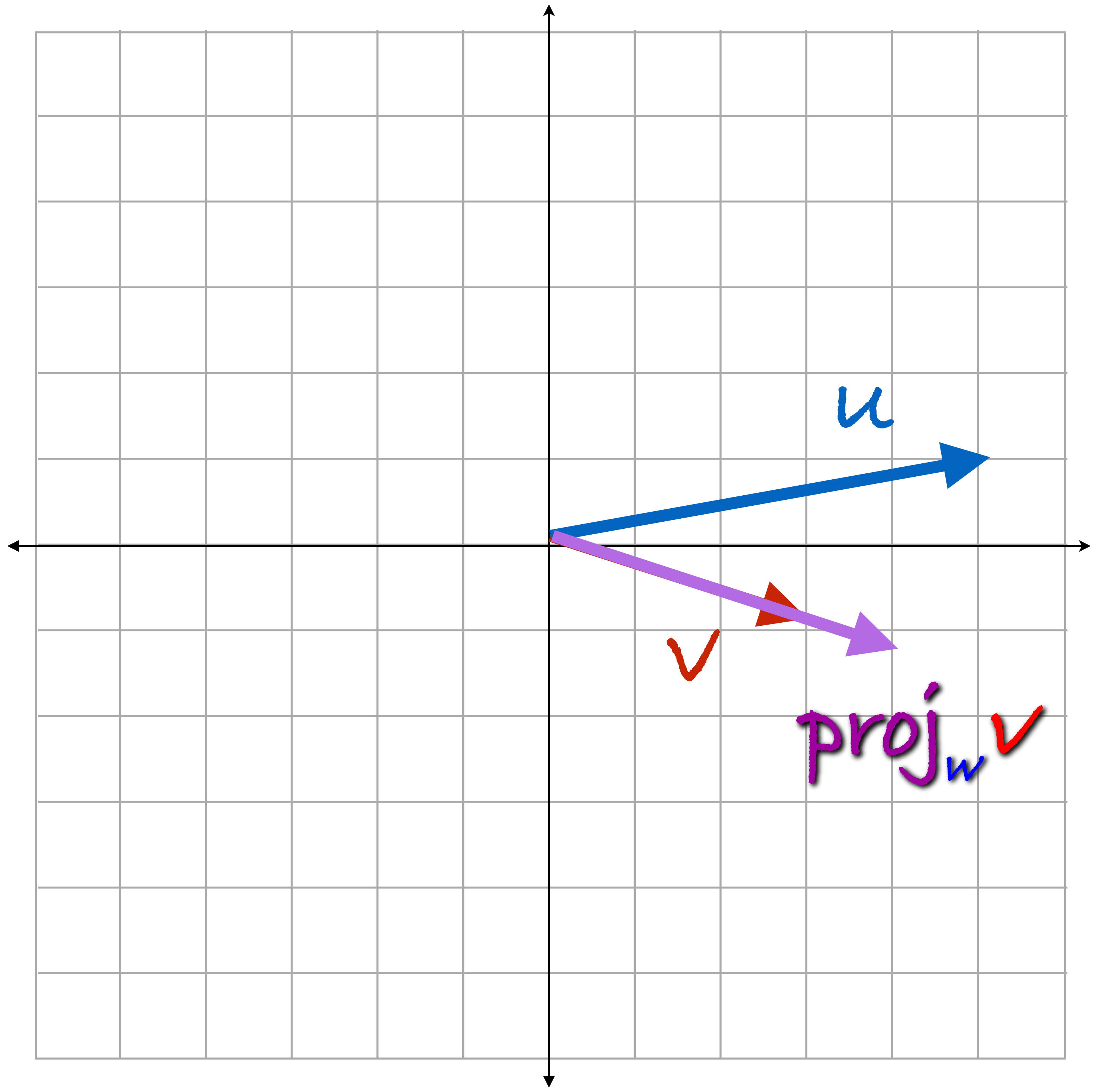
Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

Find the projection of $u = \langle 5, 2 \rangle$ onto $v = \langle 3, -1 \rangle$.

$$\text{proj}_w v = \frac{v \bullet w}{\|w\|^2} w \qquad v \cdot w = a_1 a_2 + b_1 b_2 \qquad \|\vec{u}\| = \sqrt{a^2 + b^2}$$

$$= \frac{(5)(3)+(2)(-1)}{\sqrt{3^2+(-1)^2}^{\,2}}(3,-1)$$

$$= \frac{13}{10}(3,-1) = \left(\frac{39}{10}, -\frac{13}{10}\right)$$



Vector Decomposition

The Vector Components of v

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

Now we are ready to find the vector components of a resultant vector, v.

We start by remembering $v = v_1 + v_2$. where v_1 and v_2 are orthogonal.

Then, to give us a direction we need a vector w to establish direction for one of the orthogonal components of v.

Let v and w be our vectors. Vector v can be expressed as the sum of two orthogonal vectors, v_1 and v_2, where v_1 is parallel to w, and v_2 is orthogonal to v_1. This is the decomposition of v into v_1 and v_2.

$$v_1 = \text{proj}_w v \text{ and } v_2 = v - v_1$$

The Vector Components of v

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

Let $v = 2i - 5j$ and $w = i - j$. Decompose v into two vectors v_1 and v_2, where v_1 is parallel to w and v_2 is orthogonal to w.

$$v_1 = \text{proj}_w v \text{ and } v_2 = v - v_1$$

$$v_1 = \text{proj}_w v = \frac{v \bullet w}{\|w\|^2} w = \frac{2(1) + (-5)(-1)}{\sqrt{1^2 + (-1)^2}^{\,2}} (i - j) = \frac{7}{2}(i - j) = \frac{7}{2} i - \frac{7}{2} j$$

$$v_2 = v - v_1 = (2i - 5j) - \left(\frac{7}{2} i - \frac{7}{2} j\right) \qquad v_2 = -\frac{3}{2} i - \frac{3}{2} j$$

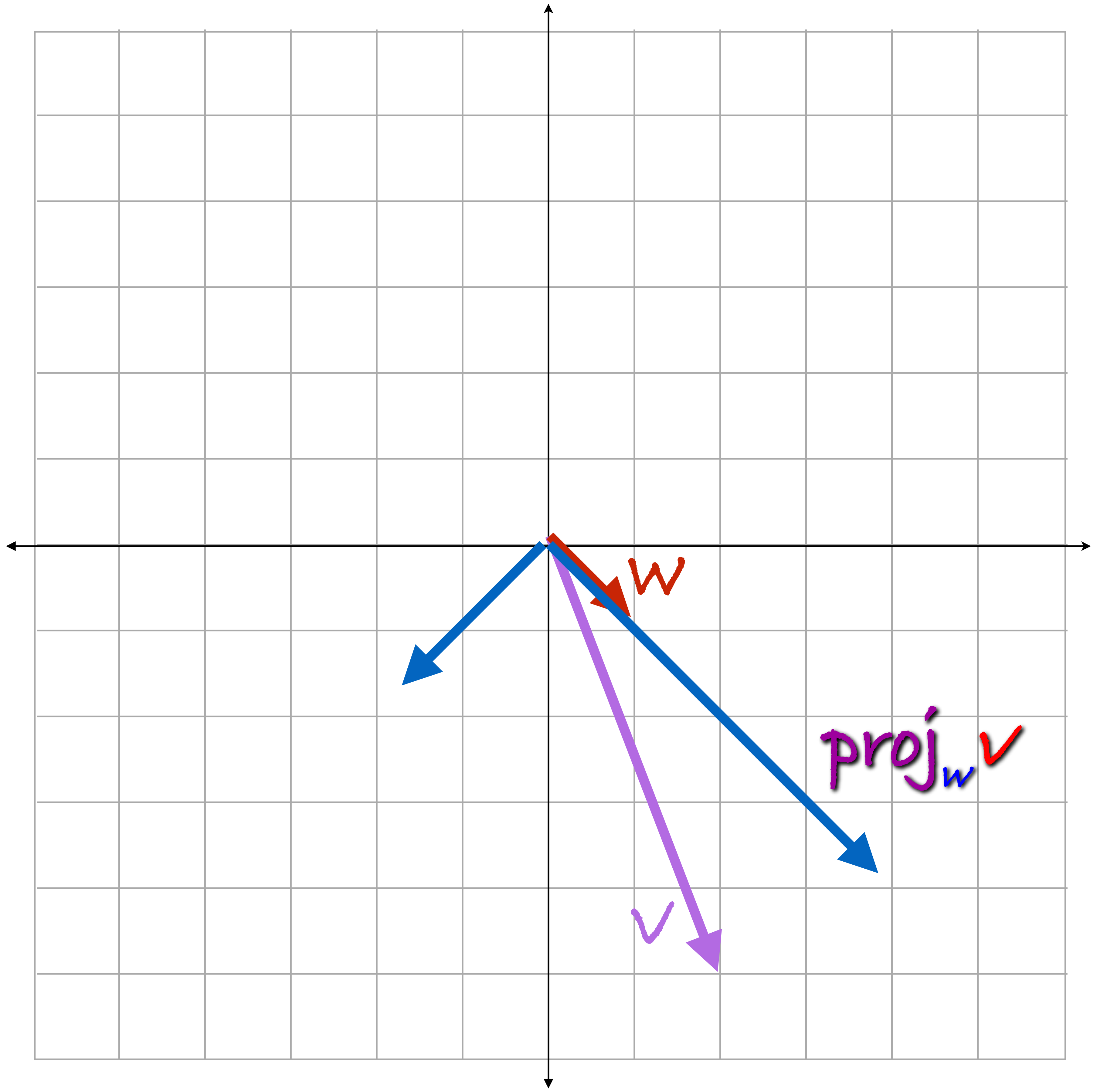
$$v = \left(\frac{7}{2} i - \frac{7}{2} j\right) + \left(-\frac{3}{2} i - \frac{3}{2} j\right)$$

Components

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

Let $v = 2i - 5j$ and $w = i - j$. Decompose v into two vectors v_1 and v_2, where v_1 is parallel to w and v_2 is orthogonal to w.

$$v = \left(\frac{7}{2}i - \frac{7}{2}j\right) + \left(-\frac{3}{2}i - \frac{3}{2}j\right)$$



Decomposing a Vector into Two Orthogonal Vectors

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

Let $v = 3i - 2j$ and $w = 2i + j$. Decompose v into two vectors v_1 and v_2, where v_1 is parallel to w and v_2 is orthogonal to w.

$$v_1 = \text{proj}_w v \text{ and } v_2 = v - v_1$$

$$v_1 = \text{proj}_w v = \frac{v \bullet w}{\|w\|^2} w = \frac{3(2)+(-2)(1)}{\sqrt{2^2+(1)^2}^{\,2}}(2i+j) = \frac{4}{5}(2i+j) = \frac{8}{5}i + \frac{4}{5}j$$

$$v_2 = v - v_1 = (3i - 2j) - \left(\frac{8}{5}i + \frac{4}{5}j\right)$$

$$v_2 = \frac{7}{5}i - \frac{14}{5}j$$

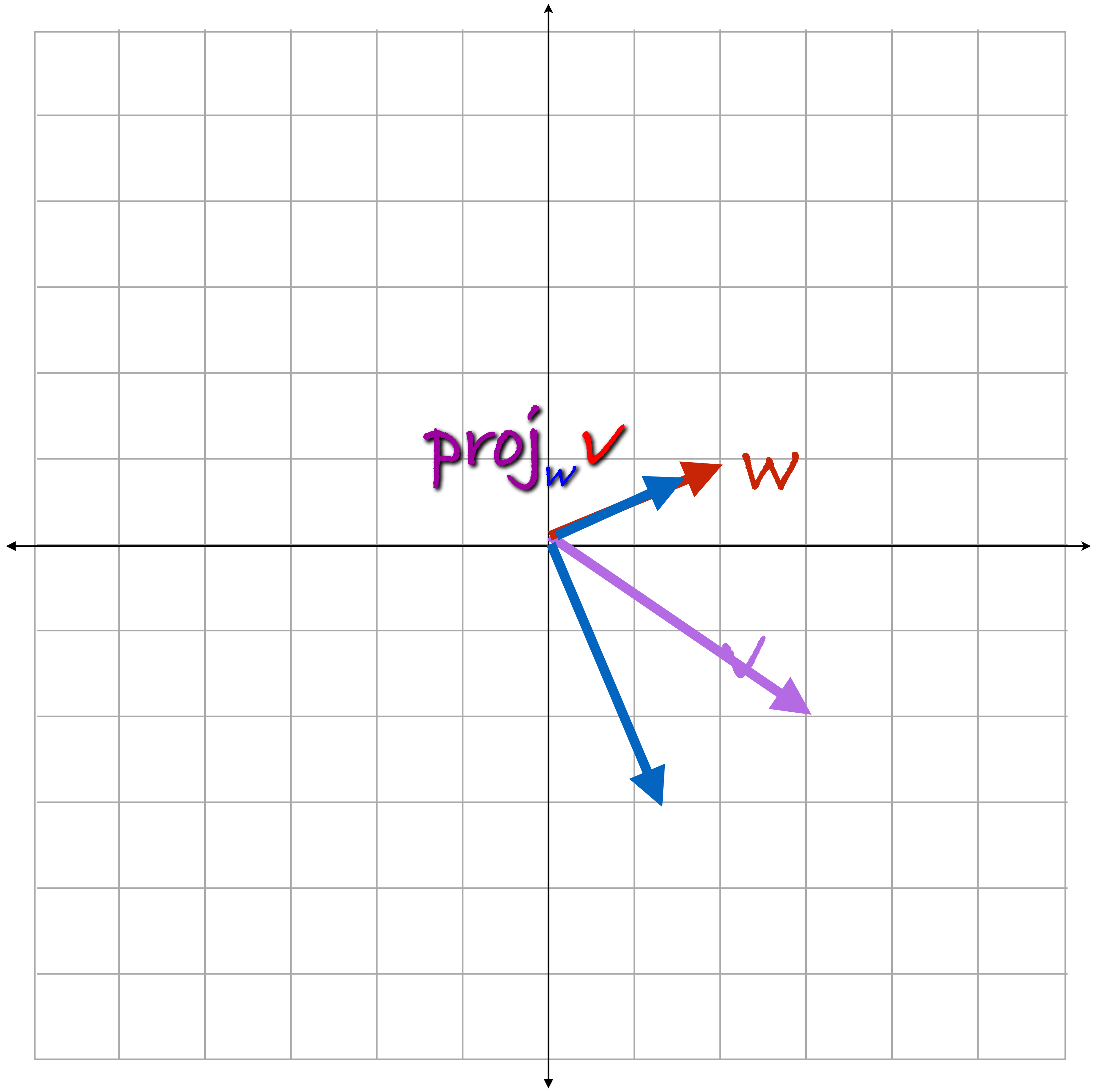
$$v = \left(\frac{8}{5}i + \frac{4}{5}j\right) + \left(\frac{7}{5}i - \frac{14}{5}j\right)$$

Decomposing a Vector into Two Orthogonal Vectors

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

Let $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} + \mathbf{j}$. Decompose $\mathbf{v}$ into two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{w}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.

$$v = \left(\frac{8}{5}i + \frac{4}{5}j\right) + \left(\frac{7}{5}i - \frac{14}{5}j\right)$$



Decomposing a Vector into Two Orthogonal Vectors

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

If $v = -i + 2j$ and $w = 3i - j$, find the vector projection of v onto w, then decompose v into v_1 and v_2, where v_1 is parallel to w and v_2 is orthogonal to v_1.

$$v_1 = \text{proj}_w v = \frac{v \bullet w}{\|w\|^2} w$$

$$v_1 = \frac{(-1)(3)+(2)(-1)}{\sqrt{3^2+(-1)^2}^{\,2}}(3i-j) \quad = \frac{-5}{10}(3i-j) \qquad v_1 = -\frac{3}{2}i + \frac{1}{2}j$$

$$v_2 = v - v_1 \qquad v_2 = (-i+2j) - \left(-\frac{3}{2}i + \frac{1}{2}j\right) \qquad v_2 = \frac{1}{2}i + \frac{3}{2}j$$

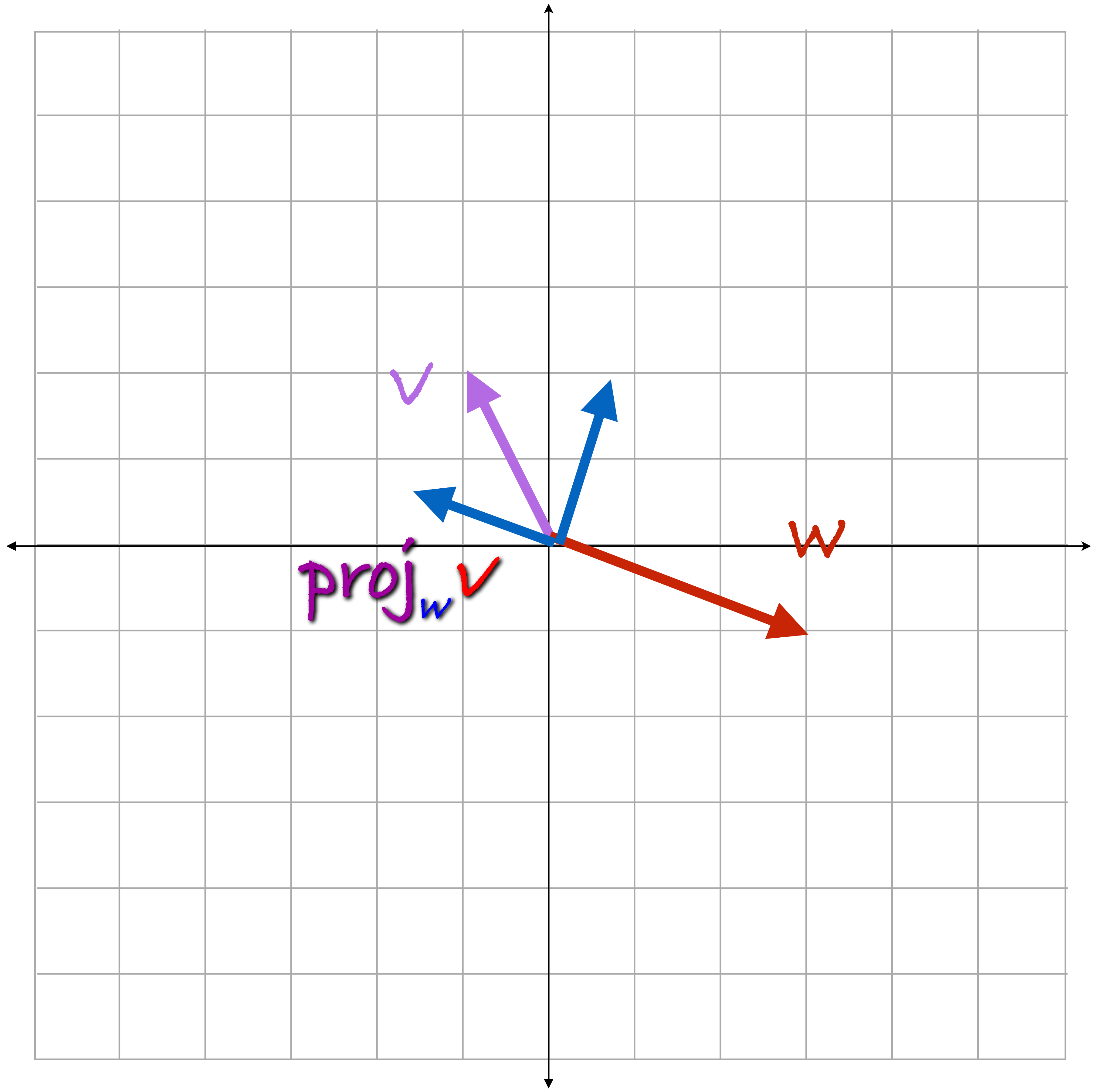
$$v = \left(-\frac{3}{2}i + \frac{1}{2}j\right) + \left(\frac{1}{2}i + \frac{3}{2}j\right)$$

Decomposing a Vector into Two Orthogonal Vectors

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

If $v = -i + 2j$ and $w = 3i - j$, find the vector projection of v onto w, then decompose v into v_1 and v_2, where v_1 is parallel to w and v_2 is orthogonal to v_1.

$$v = \left(-\frac{3}{2}i + \frac{1}{2}j\right) + \left(\frac{1}{2}i + \frac{3}{2}j\right)$$



Decomposing a Vector into Two Orthogonal Vectors

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

Find the projection of $u = \langle 5, 2 \rangle$ onto $v = \langle 3, -1 \rangle$. Then write u as the sum of two orthogonal vectors, one of which is $\text{proj}_v u$.

$$\text{proj}_v u = \frac{u \bullet v}{\|v\|^2} v$$

$$u \cdot v = a_1 a_2 + b_1 b_2$$

$$\|\vec{v}\| = \sqrt{a^2 + b^2}$$

$$v_1 = \frac{(5)(3)+(2)(-1)}{\sqrt{3^2+(-1)^2}^{\,2}}(3,-1)$$

$$= \frac{13}{10}(3,-1) = \left(\frac{39}{10}, -\frac{13}{10}\right)$$

$$u = v_1 + v_2$$

$$v_2 = u - v_1$$

$$v_2 = (5,2) - \left(\frac{39}{10}, -\frac{13}{10}\right)$$

$$v_2 = \left(\frac{11}{10}, \frac{33}{10}\right)$$

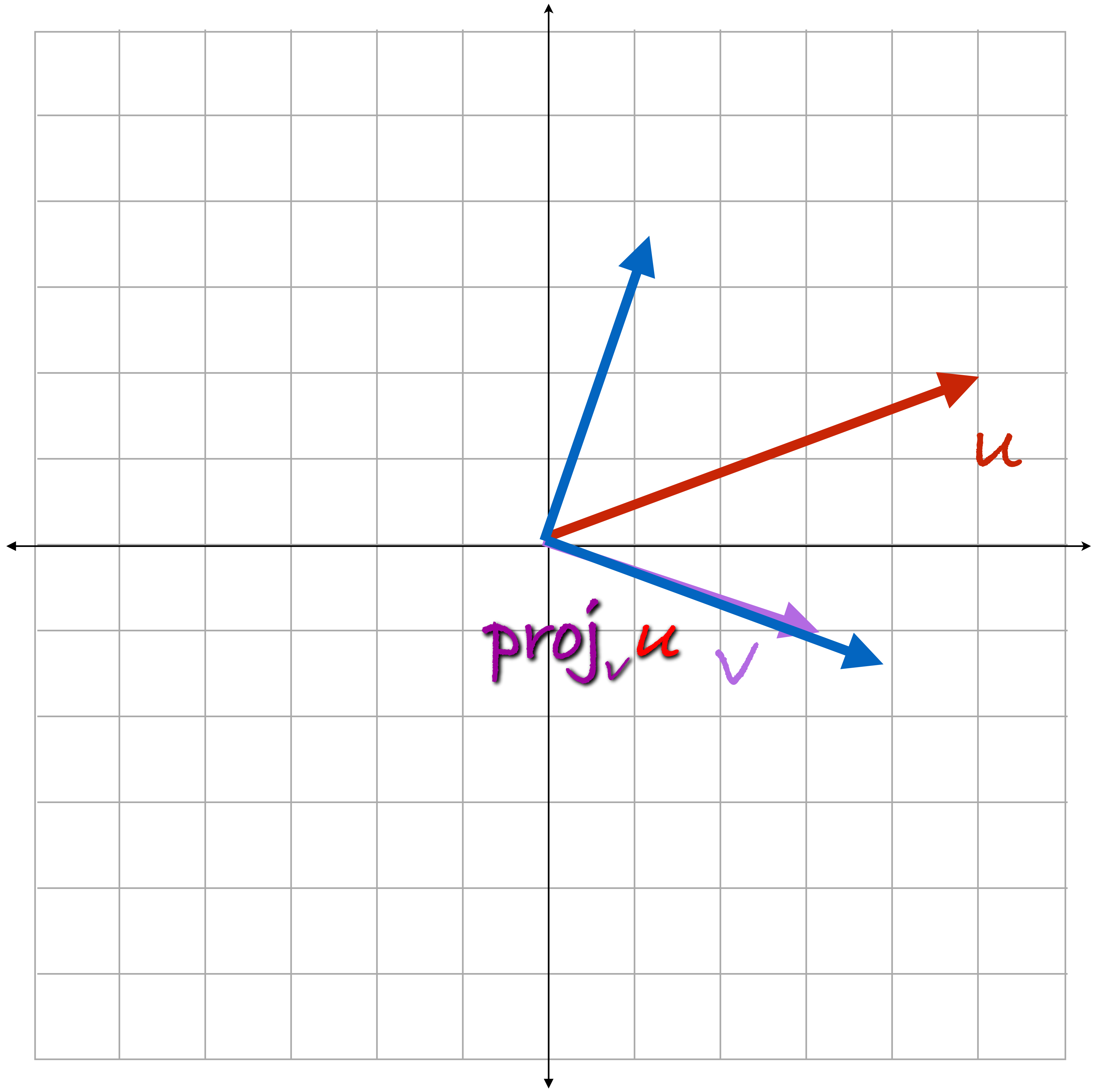
$$u = \left(\frac{39}{10}, -\frac{13}{10}\right) + \left(\frac{11}{10}, \frac{33}{10}\right)$$

Decomposing a Vector into Two Orthogonal Vectors

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

Find the projection of $u = \langle 5, 2 \rangle$ onto $v = \langle 3, -1 \rangle$. Then write u as the sum of two orthogonal vectors, one of which is $\text{proj}_v u$.

$$u = \left(\frac{39}{10}, -\frac{13}{10}\right) + \left(\frac{11}{10}, \frac{33}{10}\right)$$

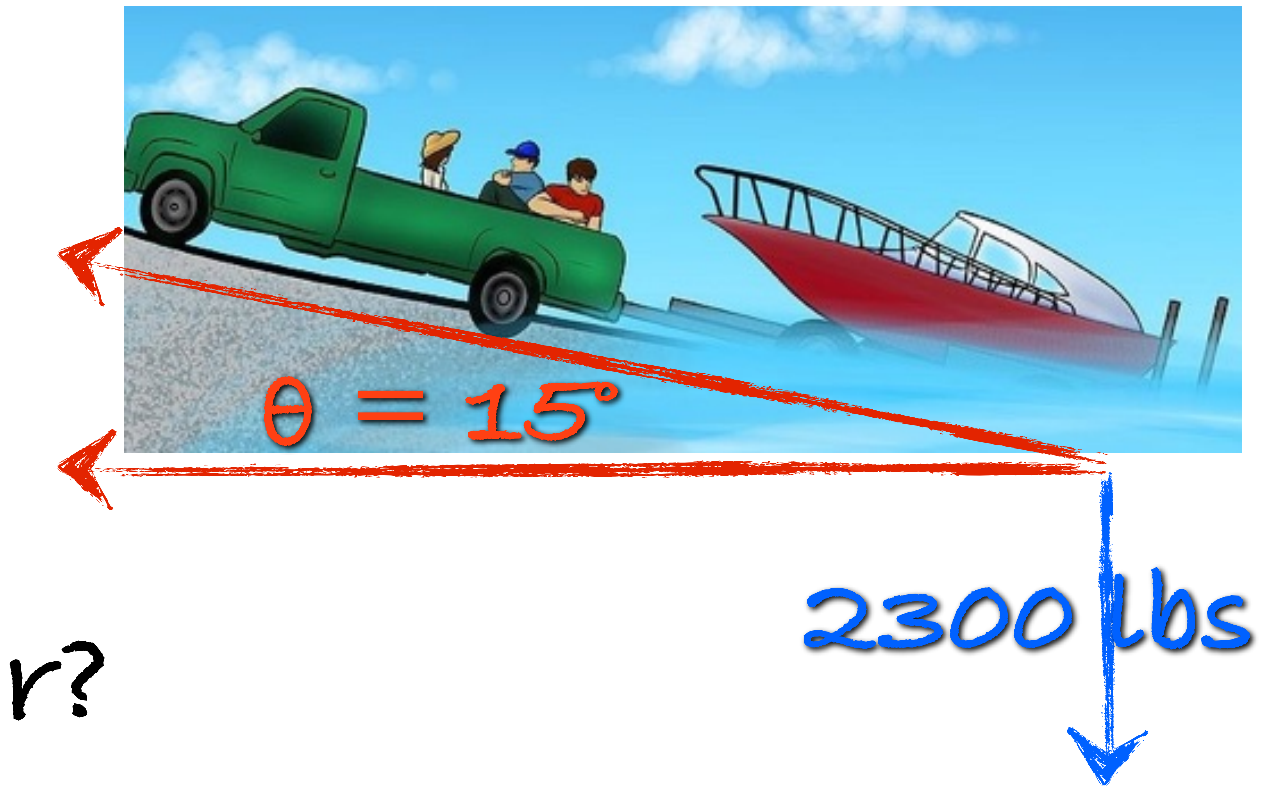


Work

Force

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

We want to pull a boat and trailer weighing 2300 lbs out of the water at the bottom of a 15° ramp.



Assuming the wheels roll without friction, what is the minimum force required to pull the boat from the water?

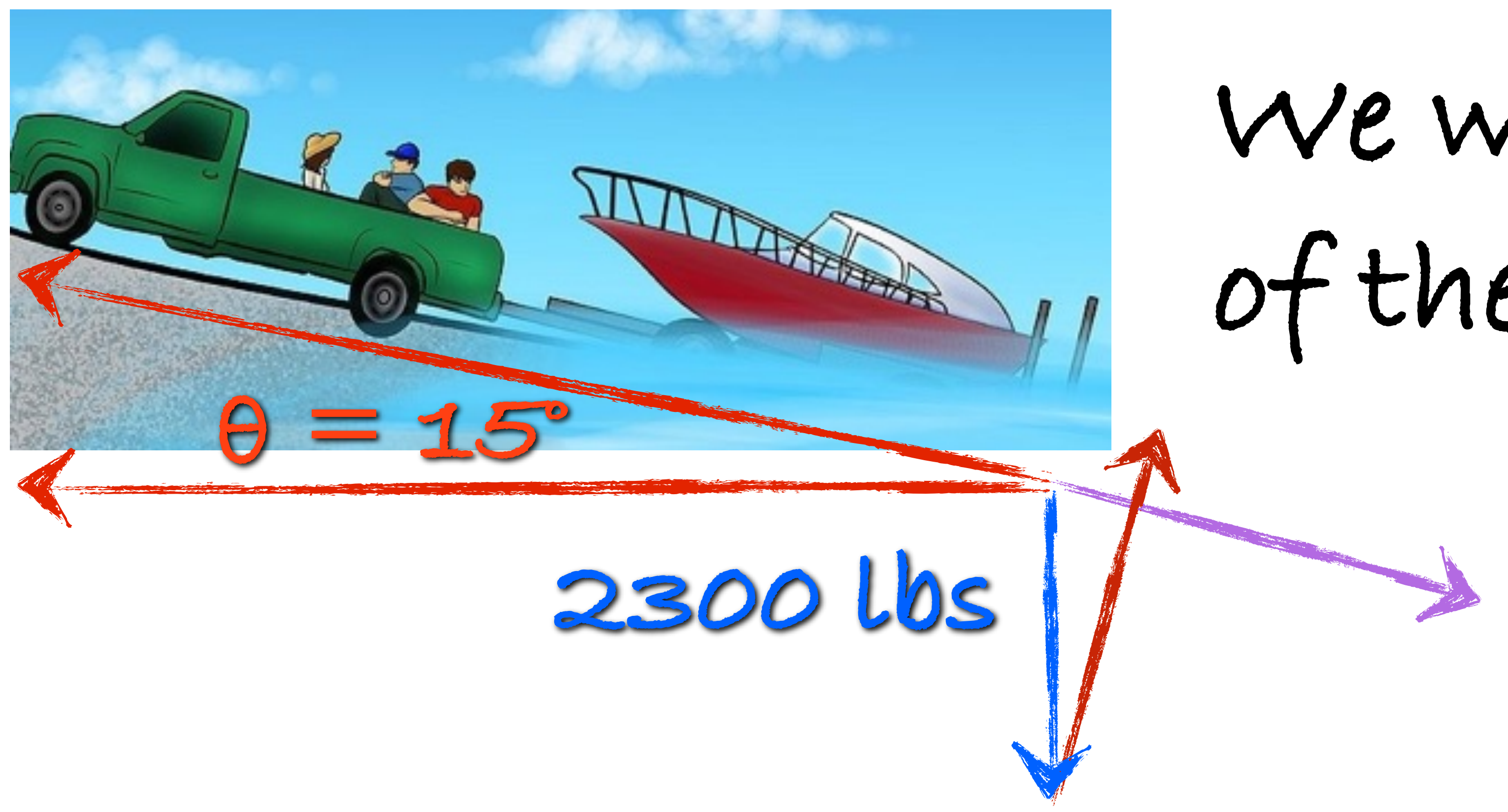
The weight of the boat and trailer is a vector applied downward (-) perpendicular to the ground $w = (0, -2300) = (0i - 2300j)$

The Force required to pull the boat from the water must be greater than the Force of the boat pulling back into the water.

To find the force pulling back we simply find $\text{proj}_r w$, the force of the boat projected onto the ramp.

Force

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.



We want to pull a boat and trailer weighing 2300 lbs out of the water at the bottom of a 15° ramp.

To find the force pulling back we simply find $\text{proj}_r w$, the force of the boat projected onto the ramp.

$w = (0, -2300)$ let r be a unit vector in the direction of pull.

$$r = (\cos 15)i + (\sin 15)j$$

$$\text{proj}_r w = \frac{w \bullet r}{1^2} r = (w \bullet r) r = ((0i + -2300j) \bullet (.9659i + .2588j)) r$$

$$= (-2300 \bullet .2588) r = (-595.2838) r$$

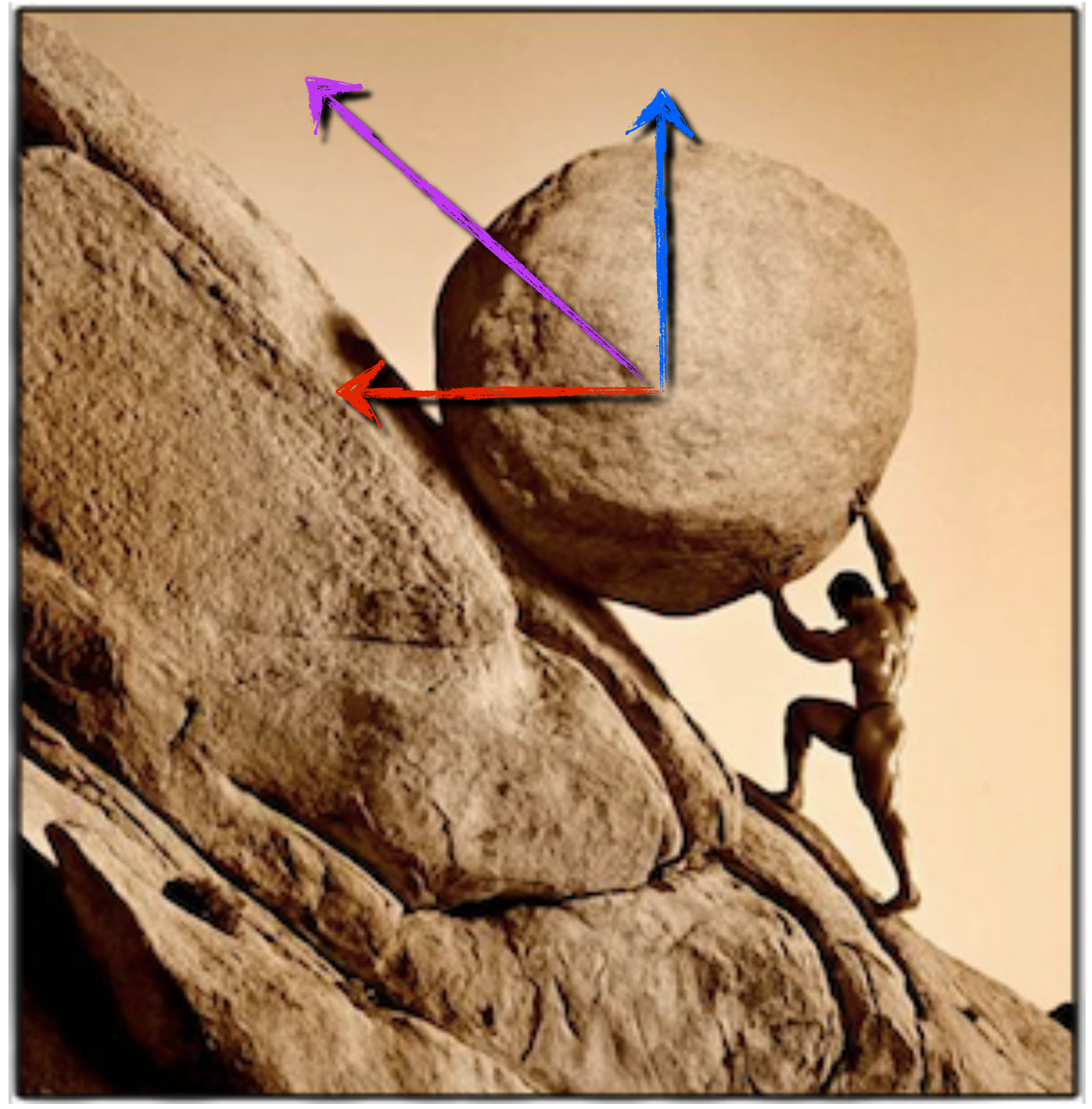
$$= -595.2838(.9659i + .2588j)$$

The magnitude of r = 1, so the force necessary to pull the boat out of the water must exceed 595.2538 foot pounds.

Definition of Work

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

Now we can figure out how much work poor Sisyphus is doing pushing that stone up the hill.



The work, W, done by force $\mathbf{F}$ moving that big pebble from point A to point B is:

$$W = \mathbf{F} \cdot \overrightarrow{AB}$$

Using alternate form of Dot Product

Amount of Work

$$W = \|\mathbf{F}\| \|\overrightarrow{AB}\| \cos\theta$$

Magnitude of Force

Angle force is being applied

Distance force is applied

Alternate Definition

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

An alternate definition of work, W, done by force $\mathbf{F}$ moving from A to B is:

$$W = \left\| \mathrm{proj}_{\overrightarrow{AB}} F \right\| \left\| \overrightarrow{AB} \right\|$$

Amount of Force applied in direction of AB.

Distance force applied in direction of AB.

Combining the two we find:

$$\left\| \mathrm{proj}_{\overrightarrow{AB}} F \right\| \left\| \overrightarrow{AB} \right\| = \left\| F \right\| \left\| \overrightarrow{AB} \right\| \cos\theta \qquad \frac{\left\| \mathrm{proj}_{\overrightarrow{AB}} F \right\|}{\cos\theta} = \left\| F \right\|$$

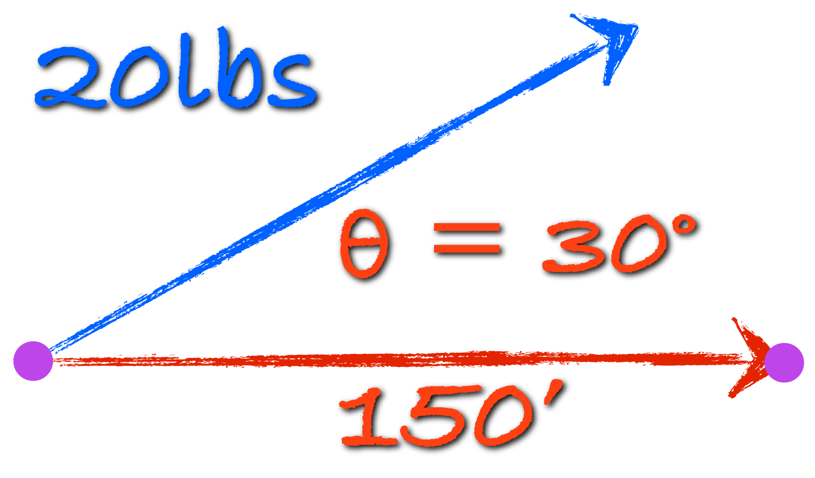
Given that finding the projection requires more work, we will primarily use the alternate dot product form.

Example: Computing Work

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

A child pulls a wagon along level ground by exerting a force of 20 pounds on a handle that makes an angle of 30° with the horizontal. How much work is done pulling the wagon 150 feet.

$$W = \|F\|\|\overrightarrow{AB}\|\cos\theta$$



$$W = 20(150)\cos 30^\circ$$

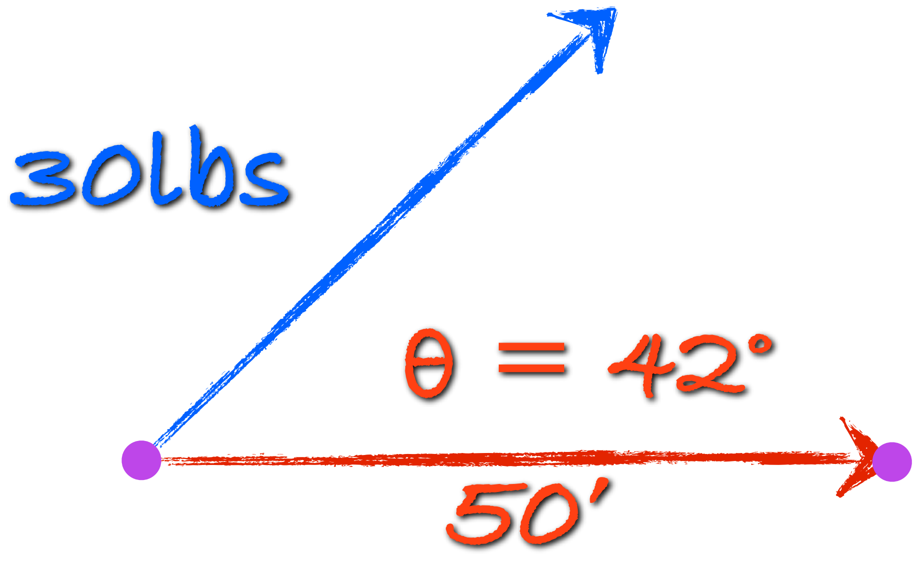
$$W = (3000)\frac{\sqrt{3}}{2} \approx 2598 \text{ foot pounds of work}$$

Example: Computing Work

Objective: *Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.*

A heavy crate is dragged 50 feet along a level floor. Find the work done if a force of 30 pounds at an angle of 42° is used.

$$W = \|F\| \|\overrightarrow{AB}\| \cos\theta$$



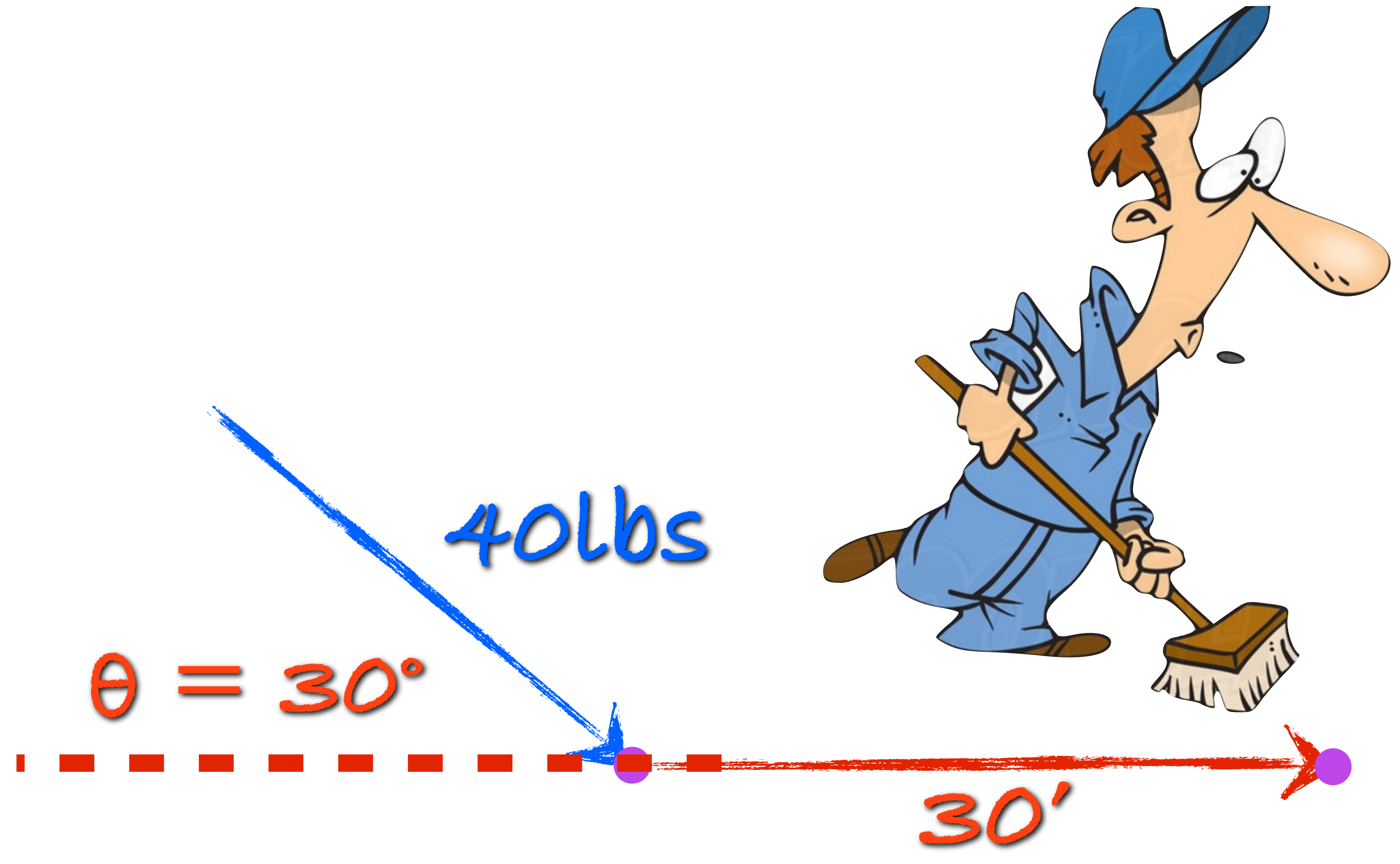
$$W = 30(50)\cos 42^\circ$$

$$W = (1500)(.7431) \approx 1114.7 \text{ foot pounds of work}$$

Example

Objective: Find the dot product of two vectors, the angle between two vectors, determine whether two vectors are orthogonal, write a vector as the sum of two vector components, and calculate work.

A man pushes a broom with a constant force of 40 pounds at an angle of 30°. How much work is done pushing the broom 30 feet?



The force vector of 40 lbs is applied at 30°.

$$F = 40\left((\cos 30^\circ)i + (\sin 30^\circ)j\right)$$

$$= 40\left(\frac{\sqrt{3}}{2}i + \frac{1}{2}j\right) = 20\sqrt{3}i + 20j$$

The distance traveled vector is simply 30i. $\overrightarrow{AB} = 30i + 0j$

$$W = F \cdot \overrightarrow{AB} \quad = \left(20\sqrt{3}i + 20j\right) \bullet \left(30i + 0j\right) \quad = \left(20\sqrt{3} \cdot 30\right) + 20 \cdot 0$$

$$= \left(600\sqrt{3}\right) \approx 1039.23 \text{ foot pounds}$$