Chapter 6

Additional Topics in Trigonometry

6.5 Complex Numbers in Polar Form; DeMoivre's Theorem

Homework

Read Sec 6.5

Complete Notes

Do p478 1-111 Every other odd

Complete worksheets

Objectives

- Plot complex numbers in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find and graph roots of complex numbers in polar form.

Remember

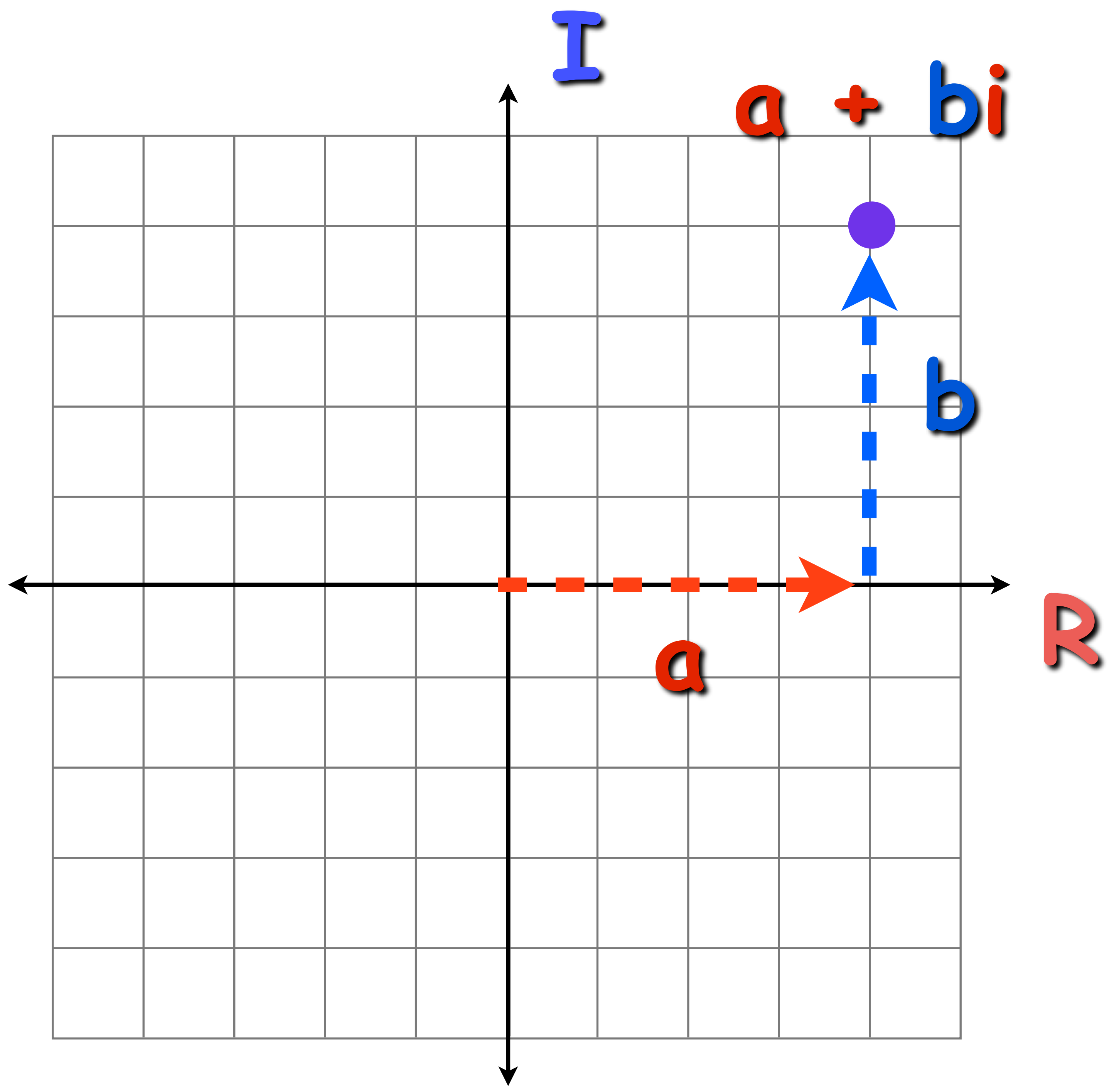
Draw a Picture

The Complex Plane

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

A complex number **z = a + bi** is represented as a point **(a, b)** in a coordinate plane. The horizontal axis of the coordinate plane is called the **real axis**. The vertical axis is the **imaginary axis**. This coordinate system is called the **complex plane**.

When we represent a complex number as a point in the complex plane, we say that we are **plotting the complex number**.



Example: Plotting Complex Numbers

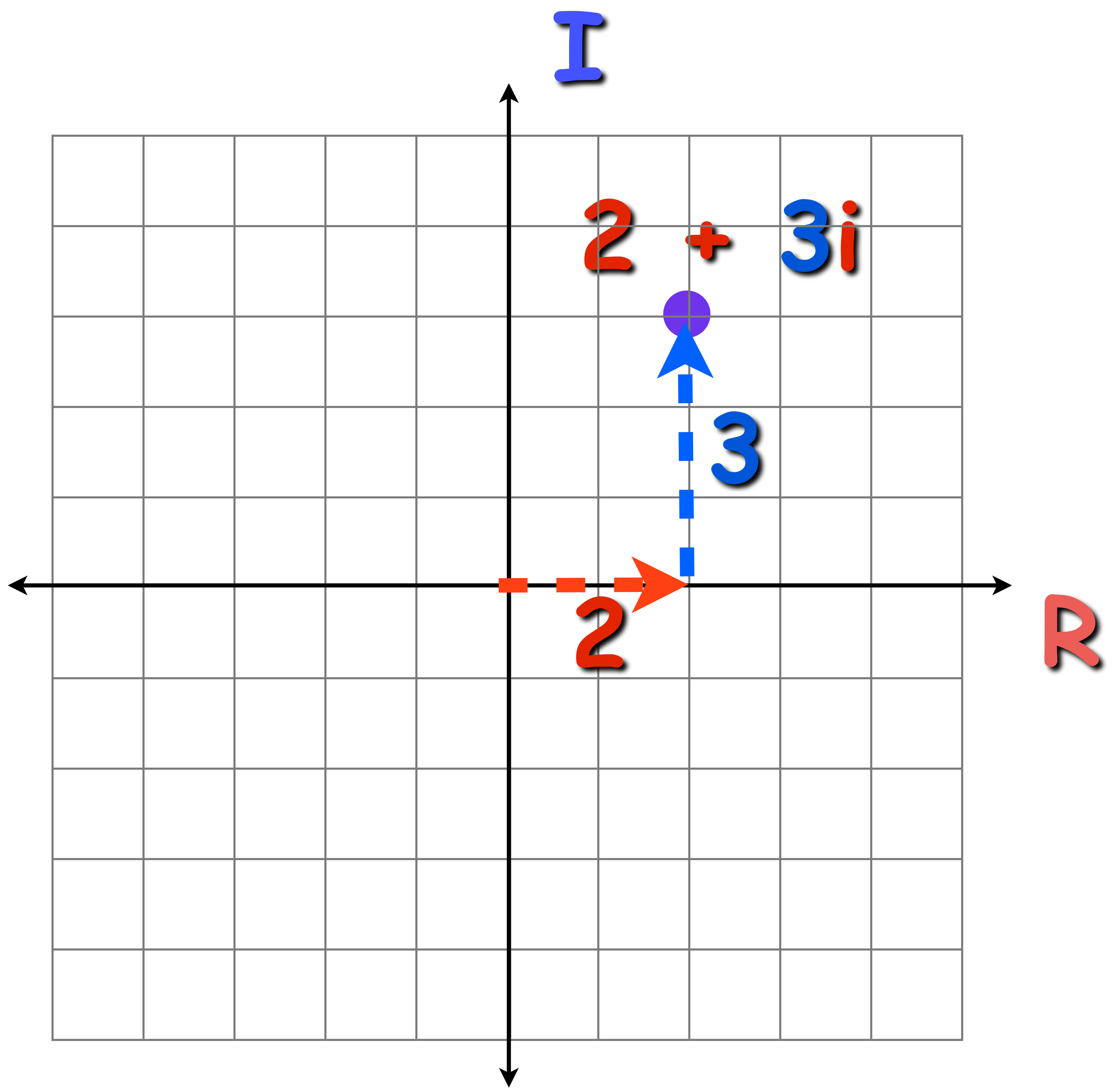
Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Plot the complex number in the complex plane: $z = 2 + 3i$

$z = 2 + 3i$

$a = 2$, real component

$b = 3$, imaginary component



Example: Plotting Complex Numbers

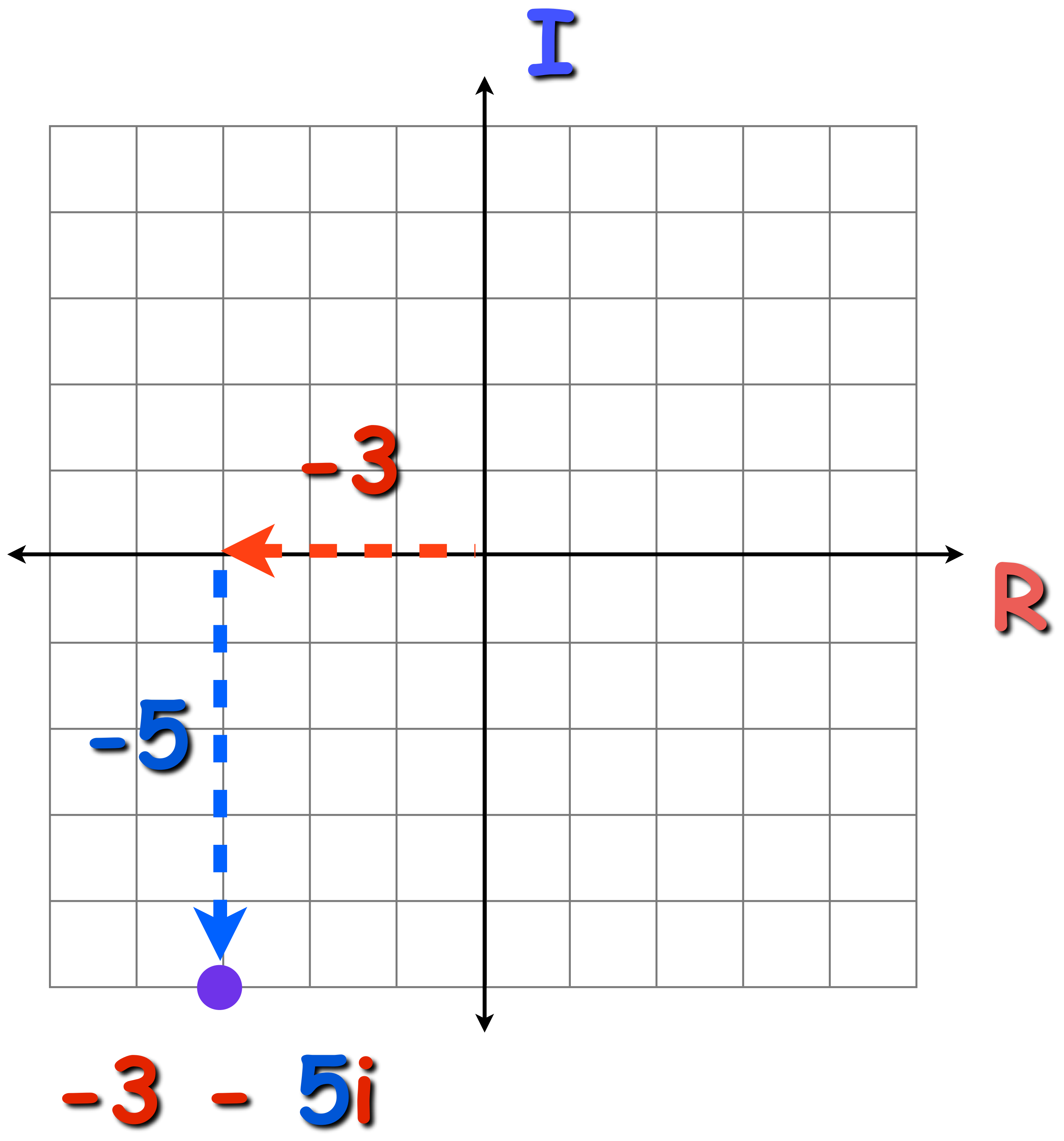
Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Plot the complex number in the complex plane: $z = -3 - 5i$

$z = -3 + -5i$

$a = -3$, real component

$b = -5$, imaginary component



Example: Plotting Complex Numbers

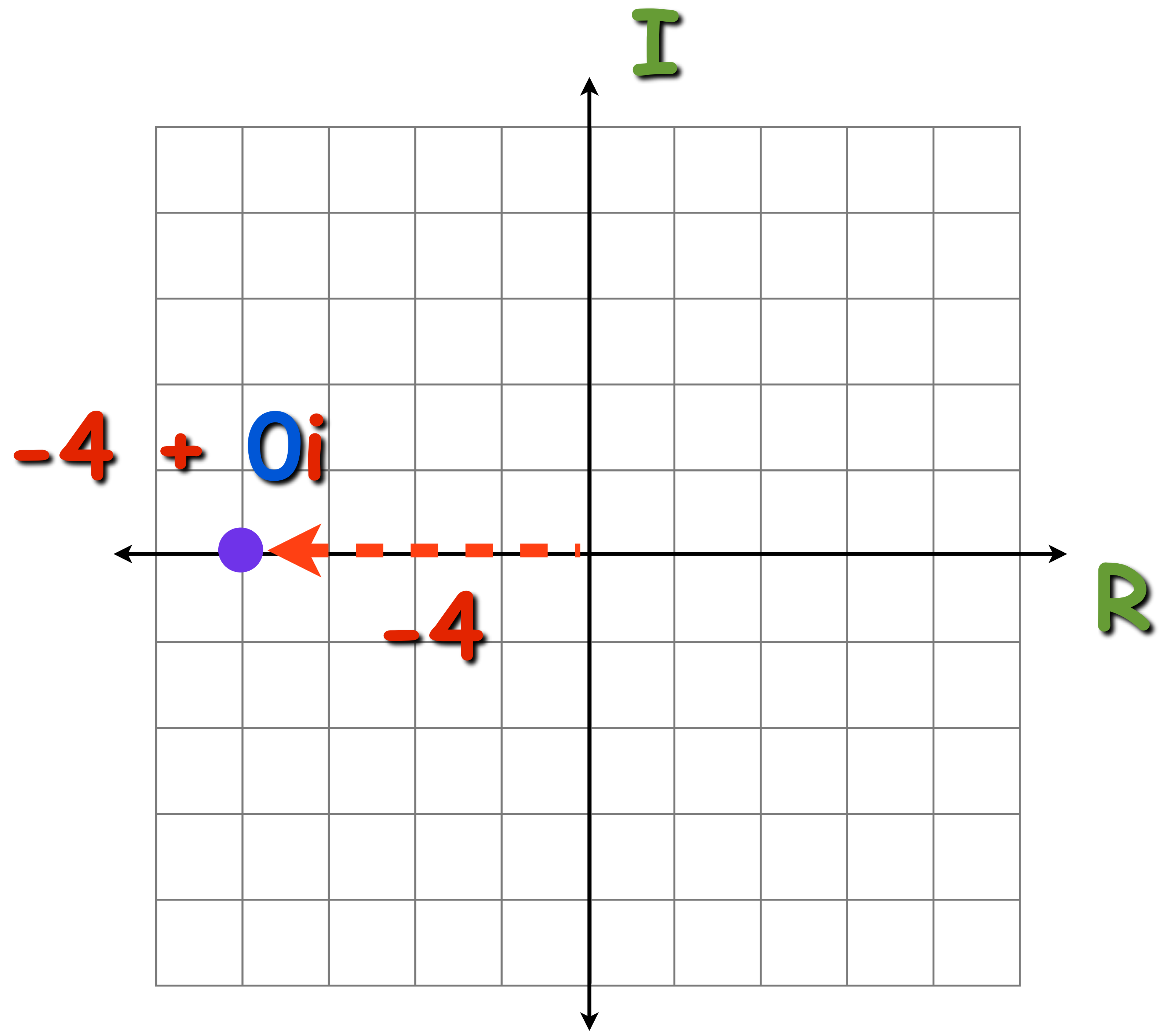
Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Plot the complex number in the complex plane: **z = -4**

z = -4 + 0i

a = -4, real component

b = 0, imaginary component



Example: Plotting Complex Numbers

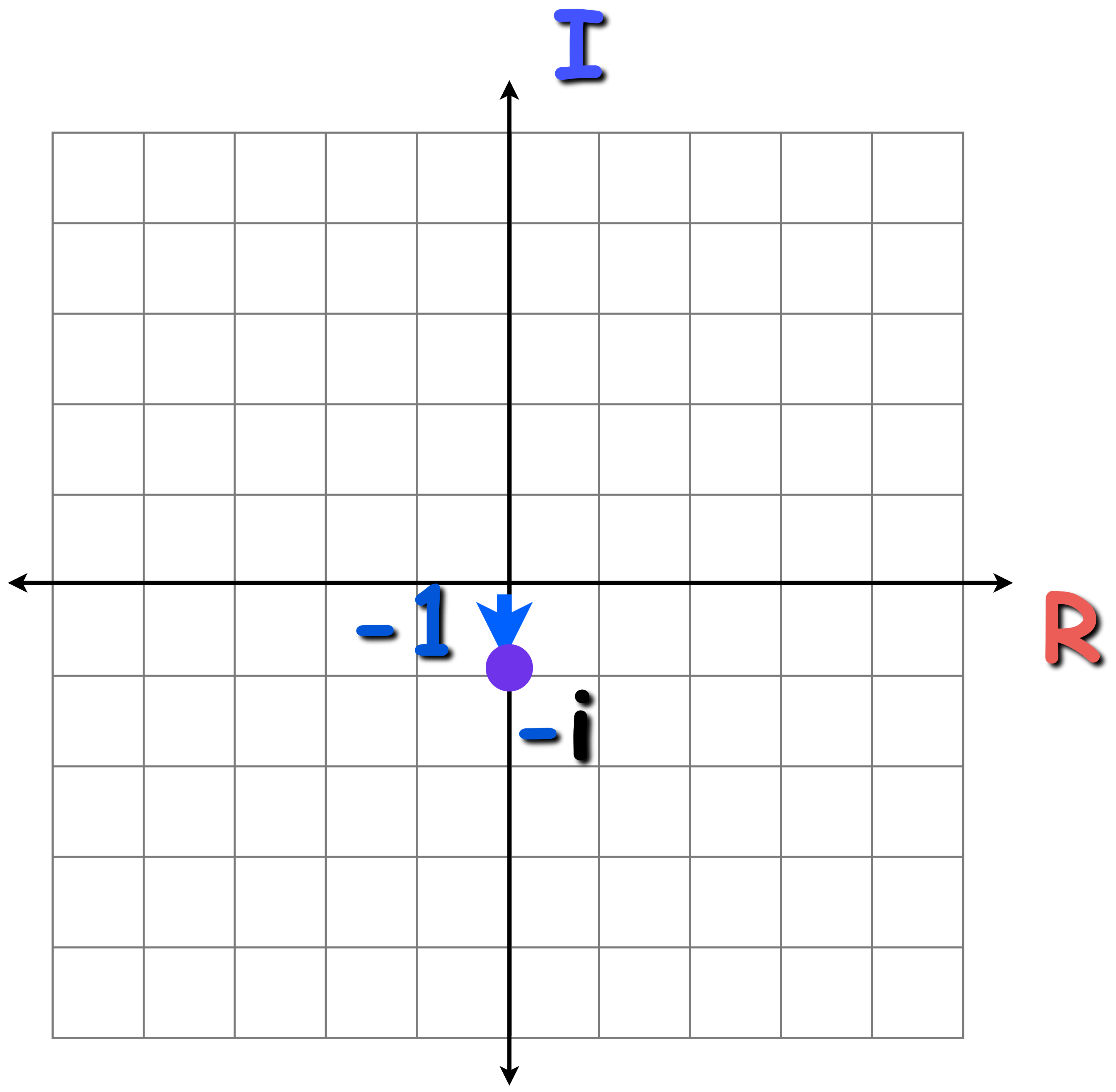
Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Plot the complex number in the complex plane: $z = -i$

$z = 0 + -1i$

$a = 0$, real component

$b = -1$, imaginary component



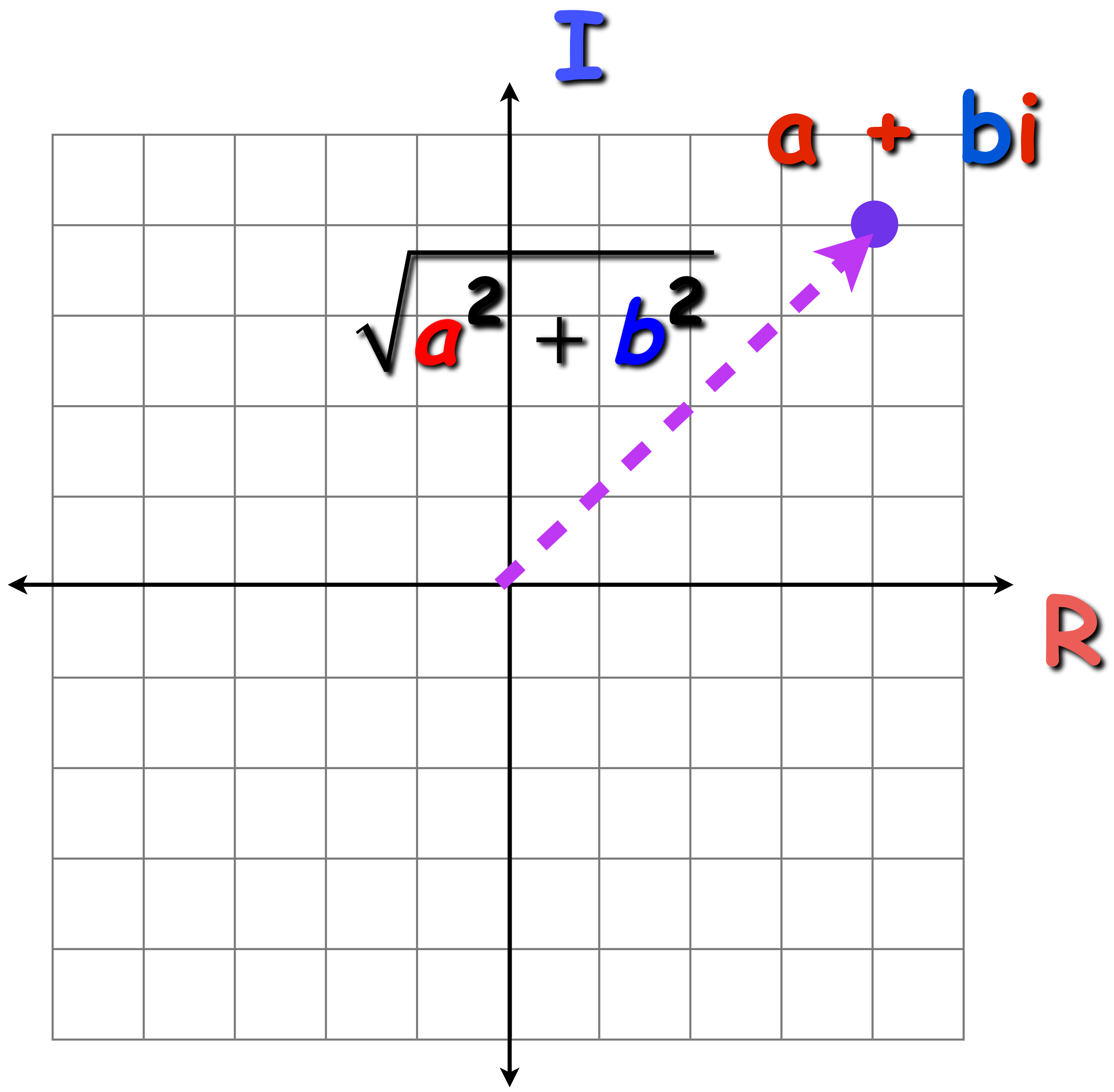
The Absolute Value of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

The absolute value of the complex number **z = a + bi** is the distance from the origin to the point **z** in the complex plane.

The absolute value of the complex number a + bi is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$



Example: Finding the Absolute Value of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Determine the absolute value of the following complex number:

$$z = 5 + 12i$$

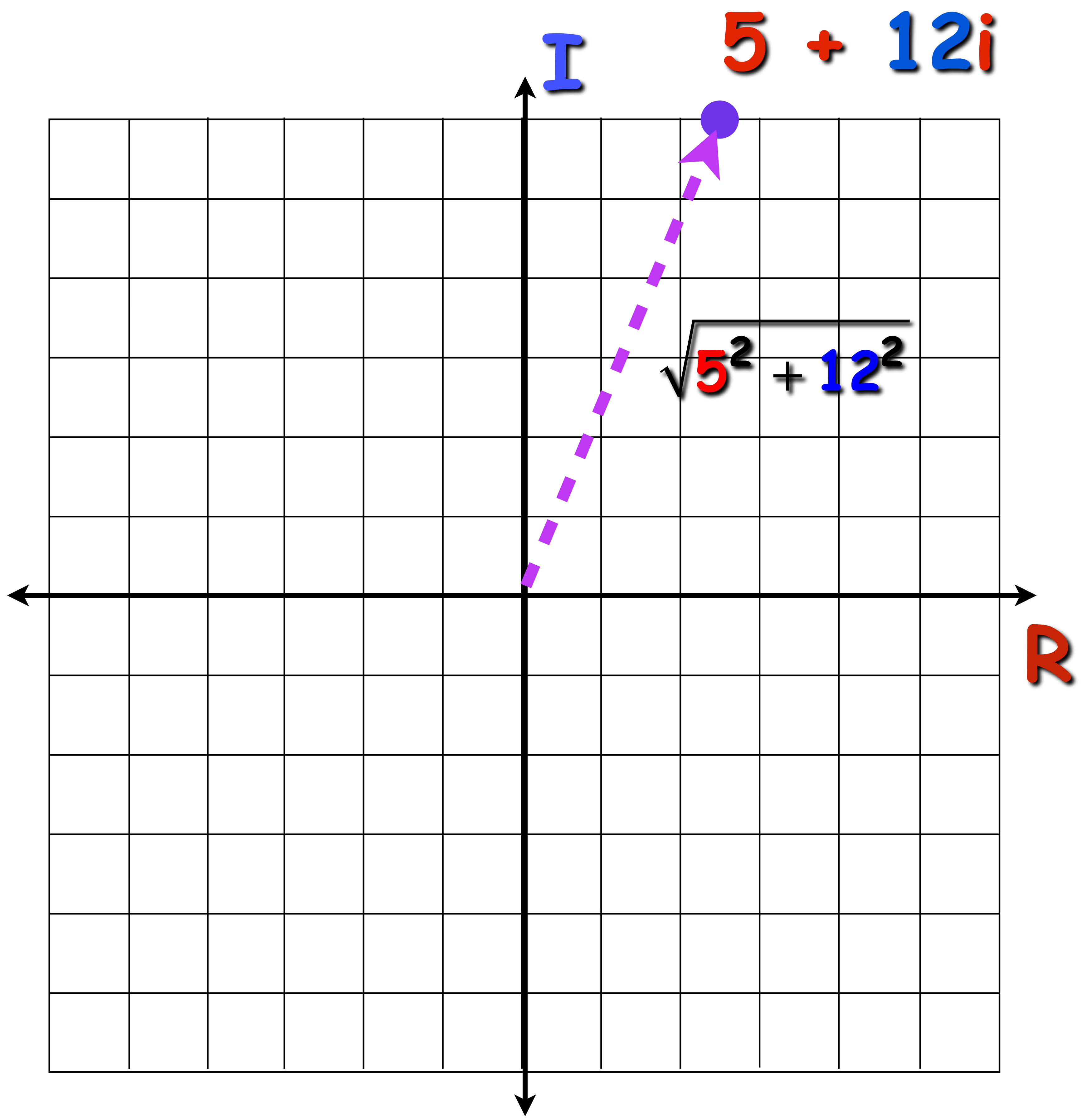
$$|\,z\,| = |\,a + bi\,| = \sqrt{a^2 + b^2}$$

$$|\,z\,| = |\,5 + 12i\,| = \sqrt{5^2 + 12^2}$$

$$= \sqrt{25 + 144}$$

$$= \sqrt{169} = 13$$

$$|\,z\,| = |\,5 + 12i\,| = 13$$



Example: Finding the Absolute Value of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Determine the absolute value of the following complex number:

$$z = 2 - 3i$$

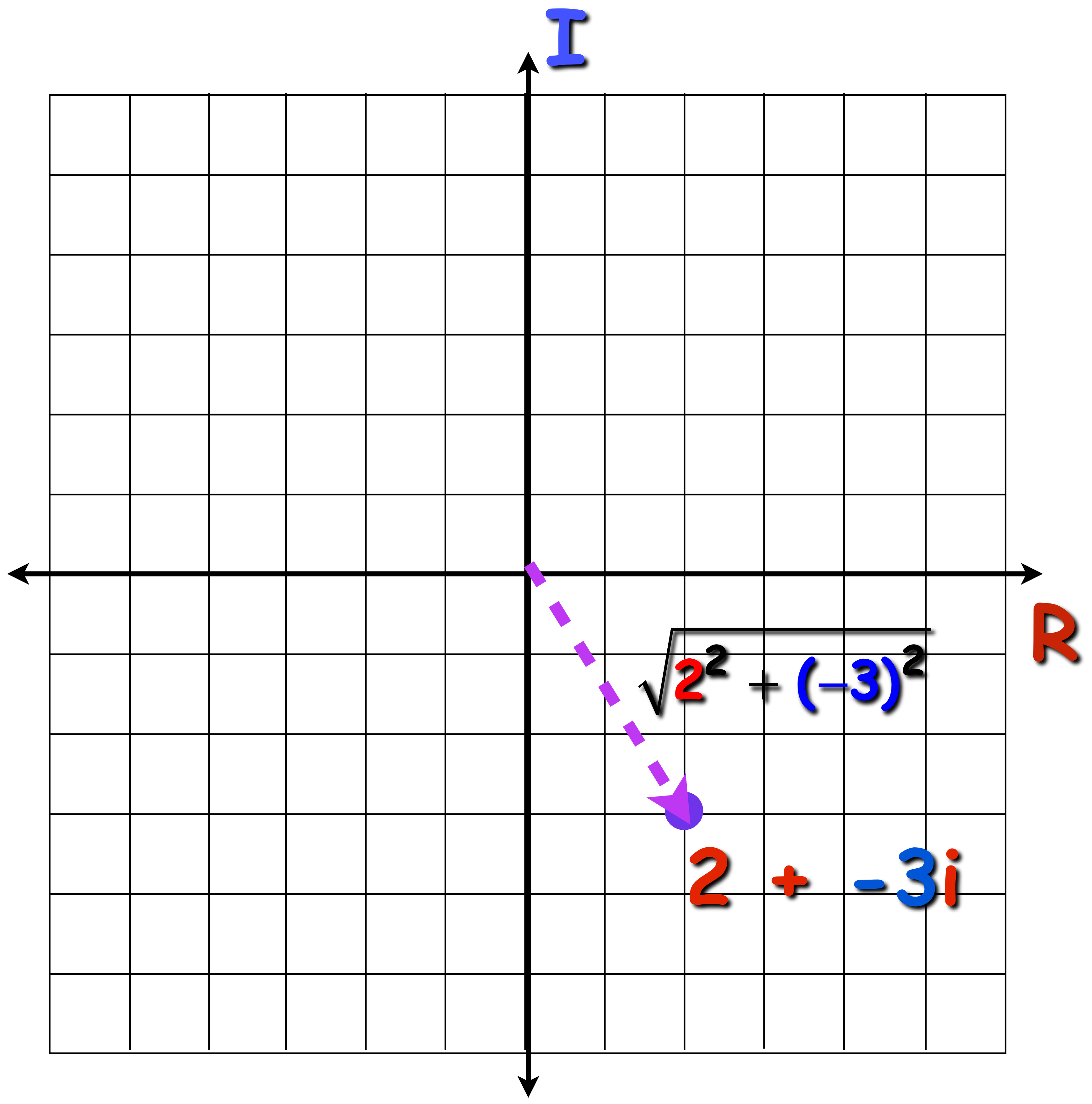
$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

$$|z| = |2 + -3i| = \sqrt{2^2 + (-3)^2}$$

$$= \sqrt{4 + 9}$$

$$= \sqrt{13}$$

$$|z| = |2 - 3i| = \sqrt{13}$$



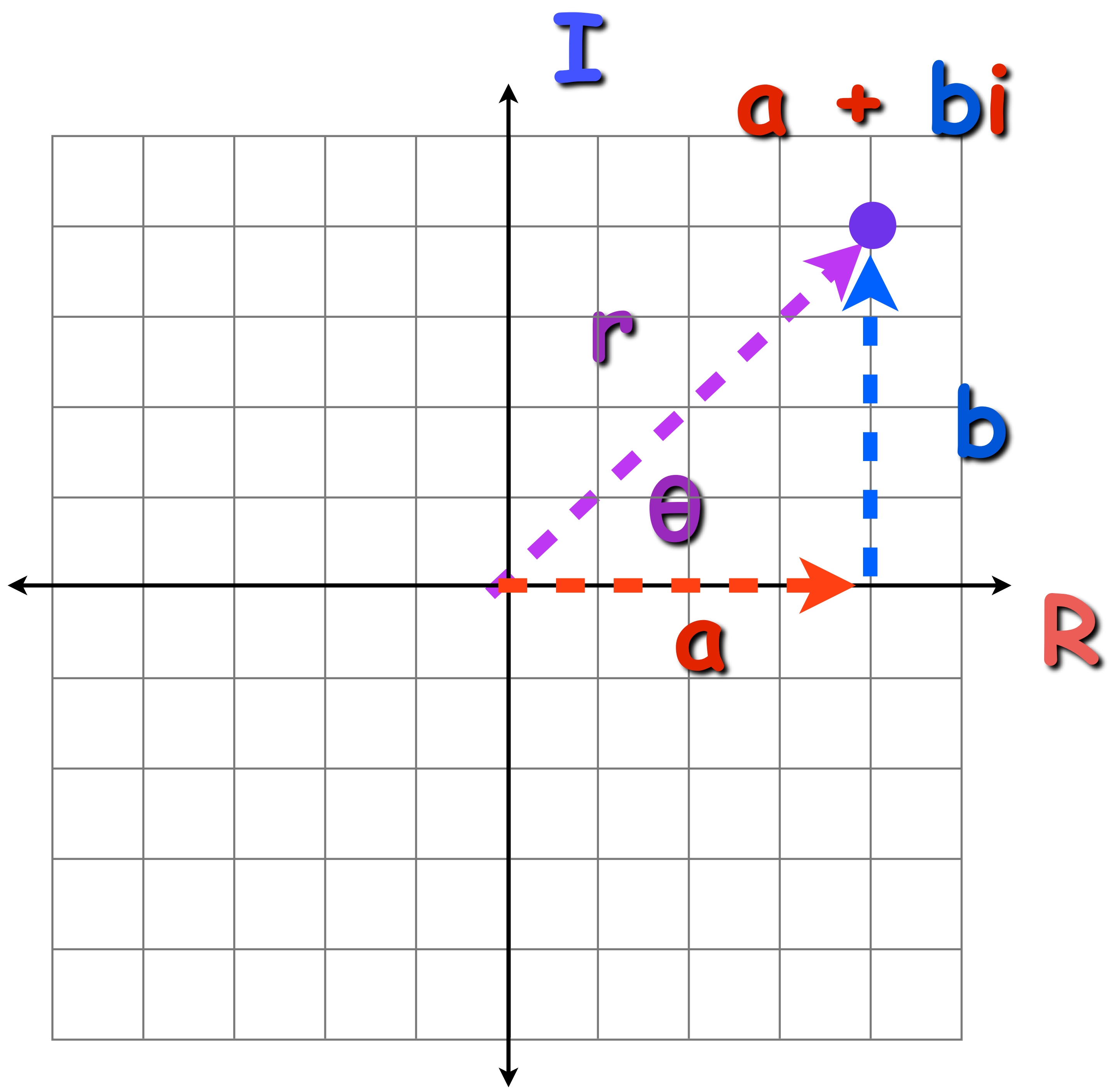
Trigonometric (Polar) Form of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

A complex number **z = a + bi** is said to be in **rectangular form**.

The expression **z = r(cosθ + isinθ)** is called the **trigonometric or polar form of a complex number**. This is often shortened to **r cis θ**

$$\boxed{z = r\left(\cos\theta + i\sin\theta\right)}$$



Where a = rcosθ and b = rsinθ, and

$$r = \sqrt{a^2 + b^2} \qquad \tan\theta = \frac{b}{a}, \text{ or } \theta = \tan^{-1}\frac{b}{a}$$

r is called the **modulus** of **z**, and

θ ($0 \le \theta < 2\pi$) is called the **argument** of **z**.

Trigonometric (Polar) Form of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is called an **argument** of z.

Trigonometric (Polar) Form of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

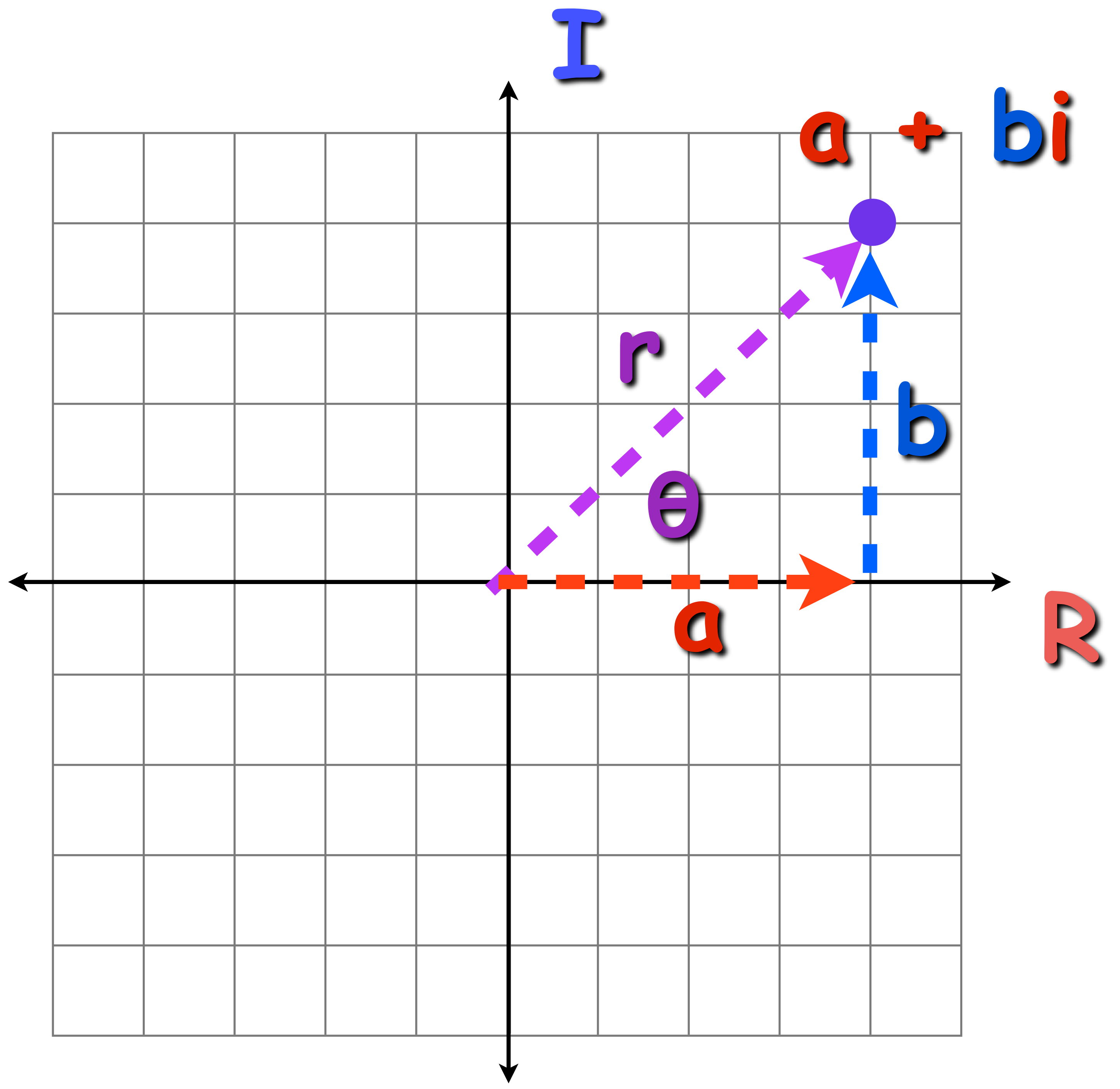
The expression **$z = r(\cos\theta + i\sin\theta)$** (or r cis θ) can be easily derived from

$$\cos\theta = \frac{a}{r} \qquad \sin\theta = \frac{b}{r}$$

$$a = r\cos\theta \qquad b = r\sin\theta.$$

$$a + bi = r\cos\theta + ir\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

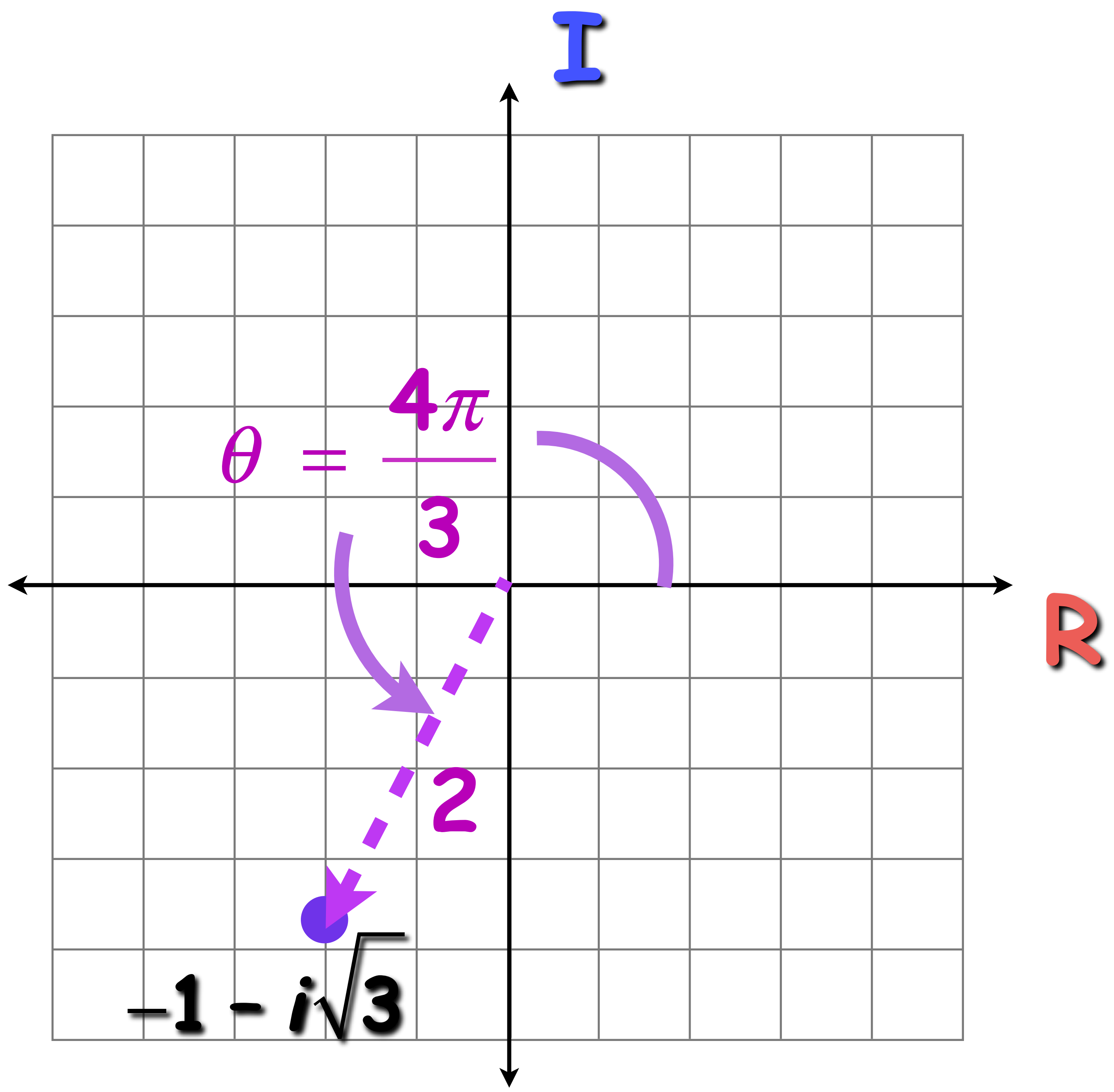


Example: Writing a Complex Number in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Plot the complex number in the complex plane, then write the number in polar form:

$$z = -1 - i\sqrt{3}$$



$$r = \sqrt{a^2 + b^2}$$

$$r = \sqrt{-1^2 + \left(-\sqrt{3}\right)^2}$$

$$r = \sqrt{4} = 2$$

$$\tan\theta = \frac{b}{a}$$

$$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

$$\theta = \frac{4\pi}{3}, QIII$$

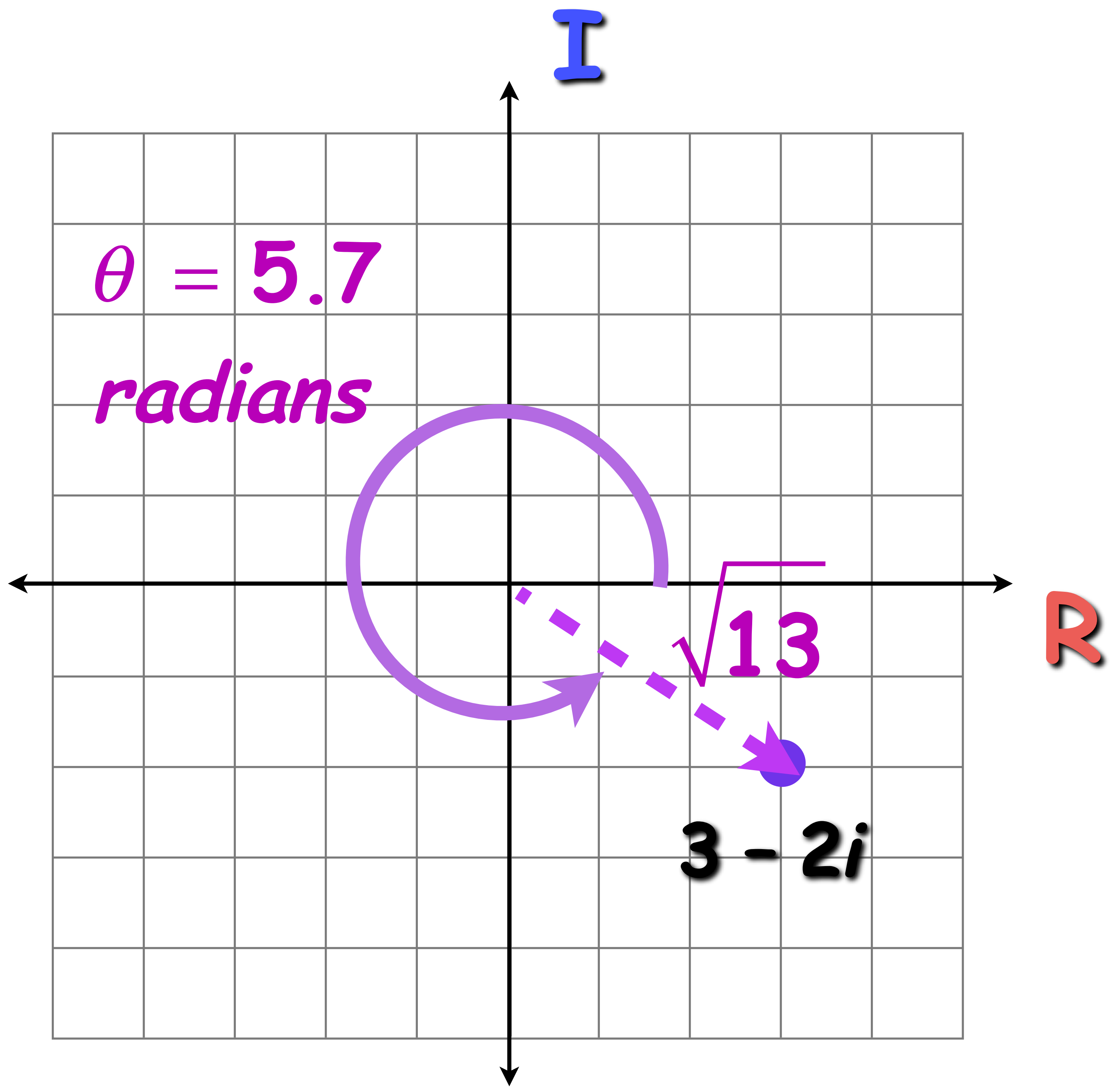
$$\boxed{z = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) \qquad z = 2cis\frac{4\pi}{3}}$$

Example: Writing a Complex Number in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Plot the complex number in the complex plane, then write the number in polar form:

$$z = 3 - 2i$$



$$r = \sqrt{a^2 + b^2}$$

$$r = \sqrt{3^2 + (-2)^2}$$

$$r = \sqrt{13}$$

$$\tan\theta = \frac{b}{a}$$

$$\tan\theta = \frac{-2}{3} \qquad \theta = \tan^{-1}\frac{-2}{3}$$

$$\theta \approx 5.7 \text{ radians, QIV}$$

$$z = \sqrt{13}\left(\cos 5.7 + i\sin 5.7\right)$$

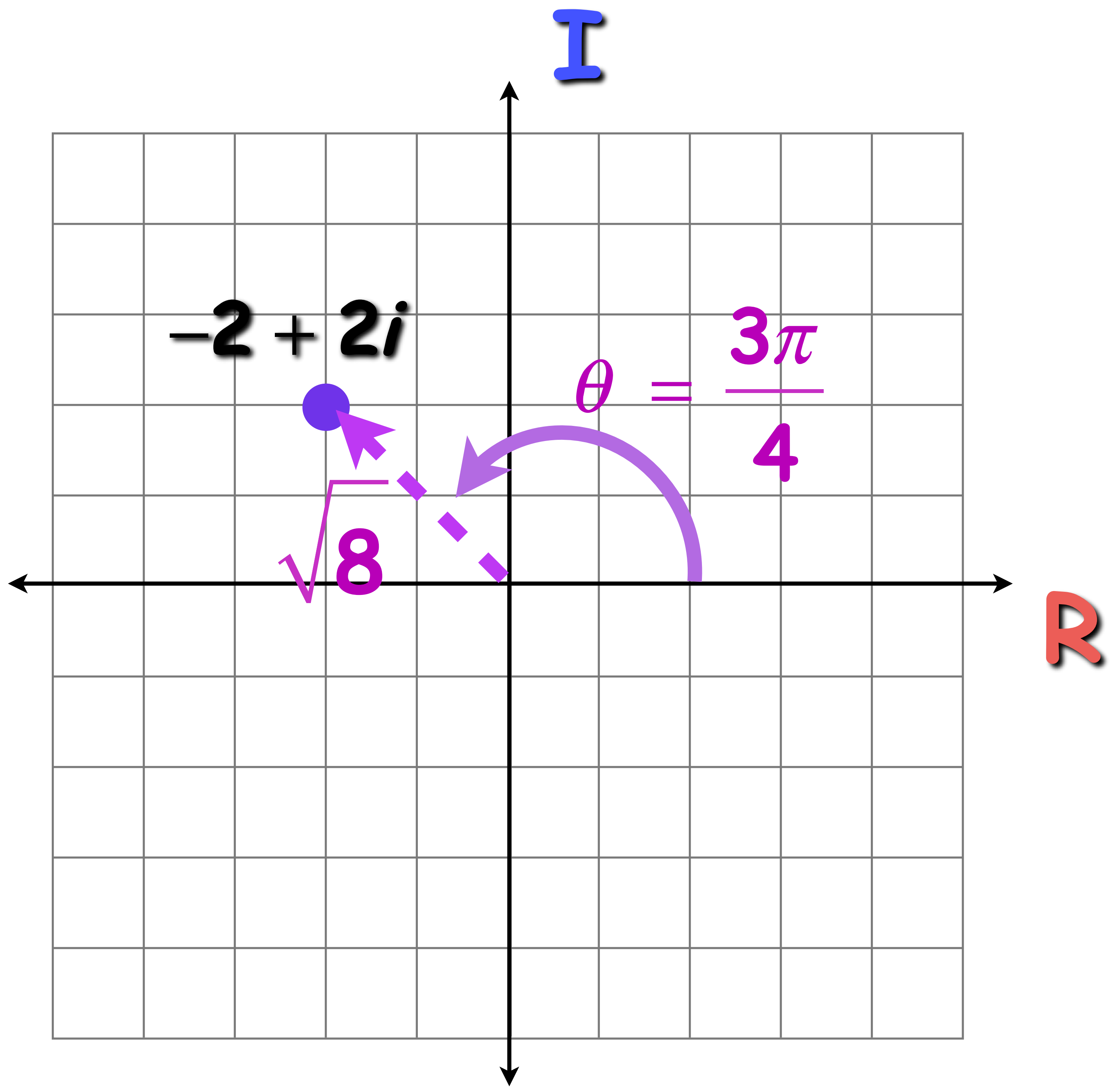
$$z = \sqrt{13}\,cis 5.7$$

Example: Writing a Complex Number in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Plot the complex number in the complex plane, then write the number in polar form:

$$z = -2 + 2i$$



$$r = \sqrt{a^2 + b^2}$$

$$r = \sqrt{(-2)^2 + (2)^2}$$

$$r = \sqrt{8}$$

$$\tan\theta = \frac{b}{a}$$

$$\tan\theta = \frac{2}{-2} = -1$$

$$\theta = \frac{3\pi}{4} \text{ radians, QII}$$

$$\boxed{\begin{aligned} z &= \sqrt{8}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \\ z &= \sqrt{8}\,\mathrm{cis}\,\frac{3\pi}{4} \end{aligned}}$$

Example: Writing a Complex Number in Rectangular Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Write the complex number in the rectangular form, then plot the number in the complex plane:

$$z = 3\left(\cos 300^\circ + i \sin 300^\circ\right)$$

z = r(cosθ + isinθ)

a = rcosθ and **b = rsinθ**

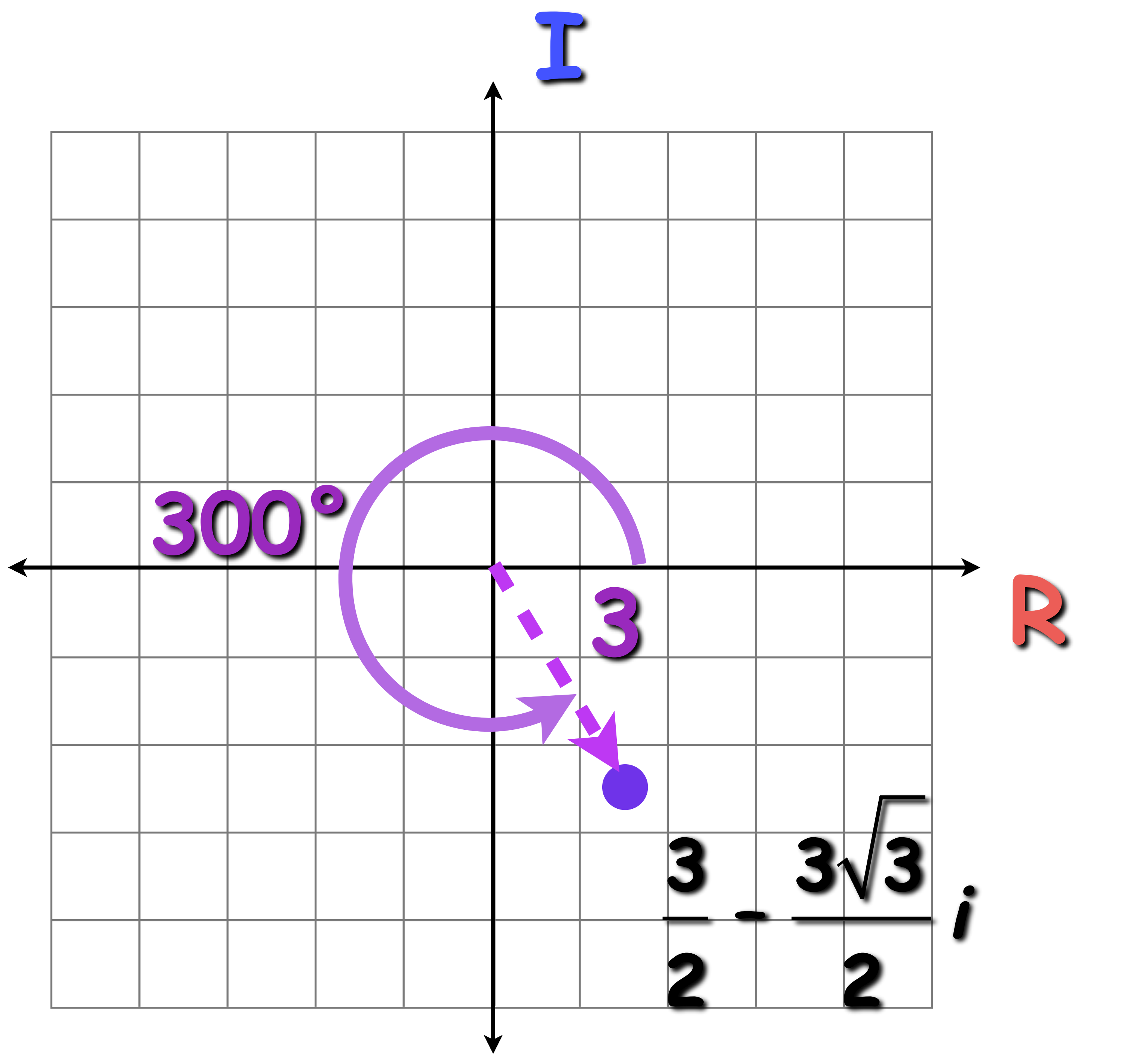
$$a = 3\cos 300^\circ$$

$$a = 3 \bullet \frac{1}{2} = \frac{3}{2}$$

$$b = 3\sin 300^\circ$$

$$b = 3 \bullet -\frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$$

$$\boxed{z = \frac{3}{2} + \left(-\frac{3\sqrt{3}}{2}\right)i}$$



Example: Writing a Complex Number in Rectangular Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Write the complex number in the rectangular form, then plot the number in the complex plane:

$$z = 4\left(\cos 30^\circ + i\sin 30^\circ\right)$$

z = r(cosθ + isinθ)

a = rcosθ and **b = rsinθ**

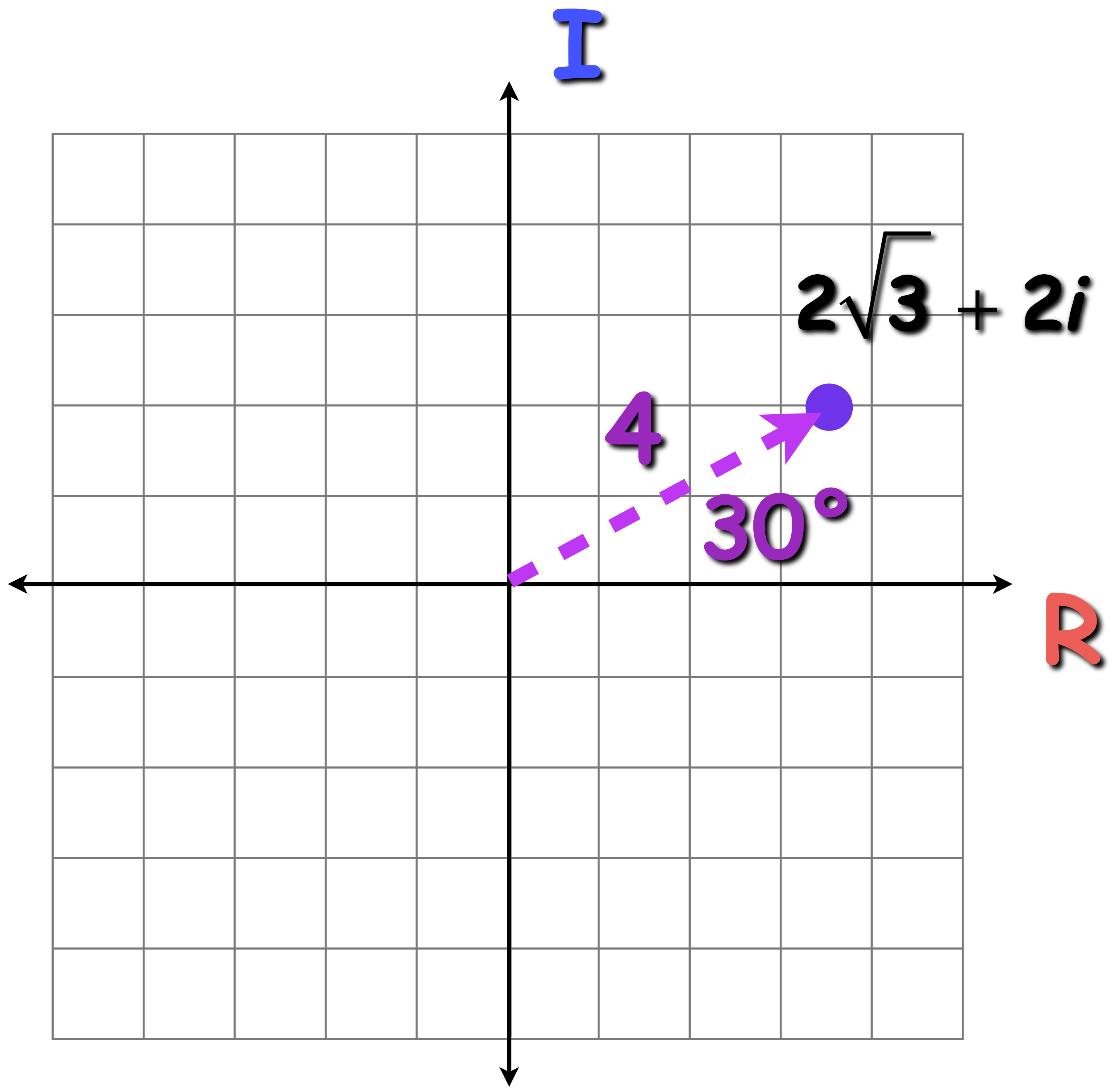
$$a = 4\cos 30^\circ$$

$$a = 4 \bullet \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$$b = 4\sin 30^\circ$$

$$b = 4 \bullet \frac{1}{2} = 2$$

$$\boxed{z = 2\sqrt{3} + 2i}$$



Example: Writing a Complex Number in Rectangular Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Write the complex number in the rectangular form, then plot the number in the complex plane:

$$z = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

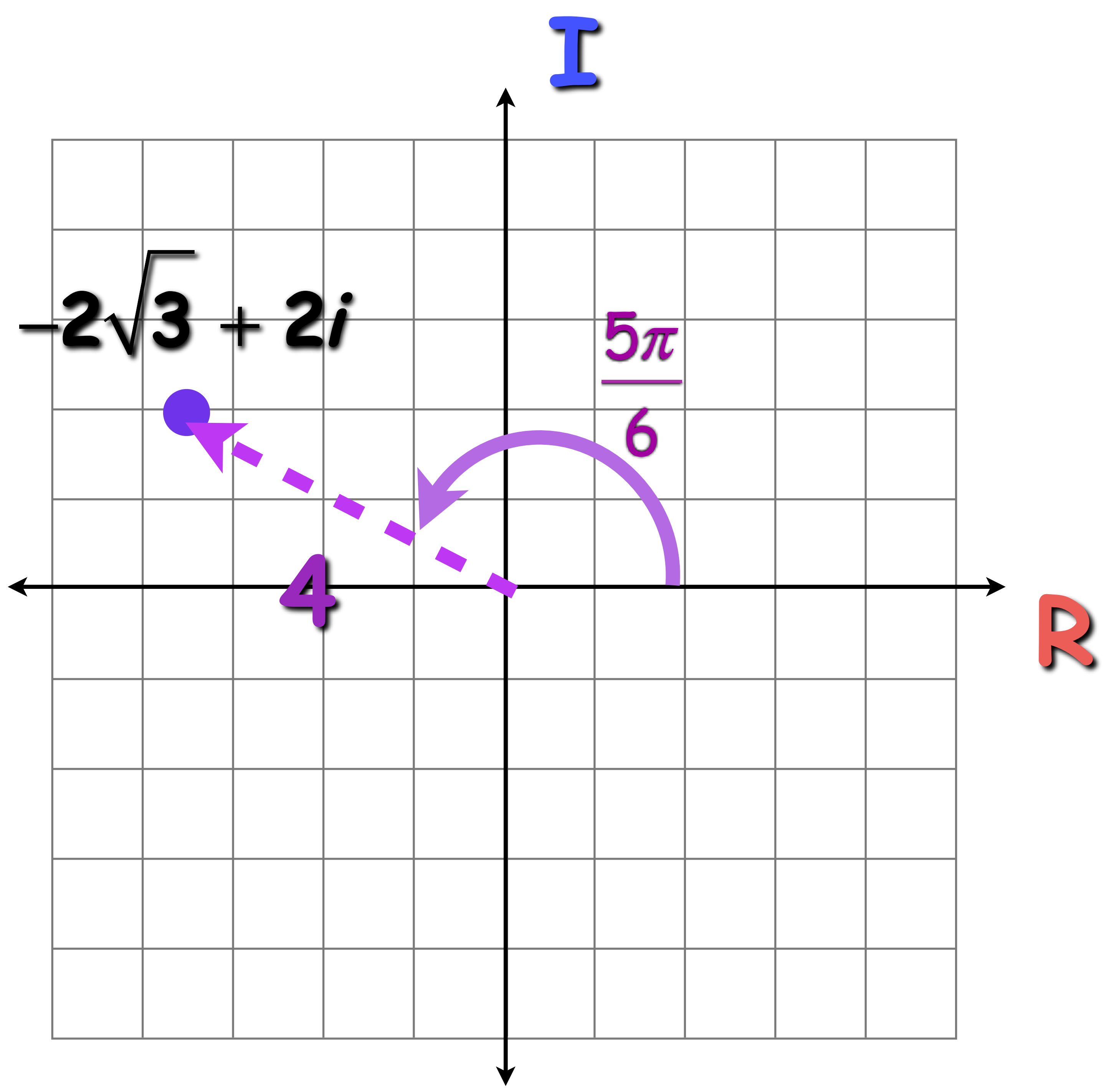
z = r(cosθ + isinθ)

a = rcosθ and **b = rsinθ**

$$a = 4\cos\frac{5\pi}{6} \qquad b = 4\sin\frac{5\pi}{6}$$

$$a = 4 \bullet -\frac{\sqrt{3}}{2} = -2\sqrt{3} \qquad b = 4 \bullet \frac{1}{2} = 2$$

$$\boxed{z = -2\sqrt{3} + 2i}$$



The Product of Two Complex Numbers in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

The product is:
$$\boxed{z_1 \cdot z_2 = r_1 \cdot r_2\left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right]}$$

Multiply the moduli and add the arguments.

The Product of Two Complex Numbers in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$z_1 \cdot z_2 = r_1[\cos\theta_1 + i\sin\theta_1] \cdot r_2[\cos\theta_2 + i\sin\theta_2]$$

$$z_1 \cdot z_2 = r_1 \cdot r_2\,[\cos\theta_1 + i\sin\theta_1][\cos\theta_2 + i\sin\theta_2]$$

$$= r_1 \cdot r_2\,[\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2]$$

$$= r_1 \cdot r_2\,[\cos\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2]$$

$$= r_1 \cdot r_2\,[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$$

$$\boxed{z_1 \cdot z_2 = r_1 \cdot r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]}$$

Example: The Product of Two Complex Numbers in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find the product of the complex numbers. Write the answer in polar form and rectangular form.

$$z_1 = 6(\cos 40^\circ + i \sin 40^\circ) \qquad z_2 = 5(\cos 20^\circ + i \sin 20^\circ)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$z_1 z_2 = 6 \bullet 5[(\cos 40^\circ + 20^\circ) + i \sin 40^\circ + 20^\circ)]$$

$$z_1 z_2 = 30(\cos 60^\circ + i \sin 60^\circ)$$

Change to rectangular form

$$z_1 z_2 = 30\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 15 + 15i\sqrt{3}$$

Example: The Product of Two Complex Numbers in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find the product of the complex numbers. Write the answer in polar form and rectangular form.

$$z_1 = 8(\cos 120^\circ + i \sin 120^\circ) \quad z_2 = 6(\cos 150^\circ + i \sin 150^\circ)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$z_1 z_2 = 8 \bullet 6[(\cos 120^\circ + 150^\circ) + i\sin(120^\circ + 150^\circ)]$$

$$z_1 z_2 = 48(\cos 270^\circ + i\sin 270^\circ)$$

Change to rectangular form

$$z_1 z_2 = 48\left(0 + i\left(-1\right)\right) = -48i$$

The Quotient of Two Complex Numbers in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

The quotient is: $$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]}$$

Divide the moduli and subtract the arguments.

The Quotient of Two Complex Numbers in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$\frac{z_1}{z_2} = \frac{r_1\left(\cos\theta_1 + i\sin\theta_1\right)}{r_2\left(\cos\theta_2 + i\sin\theta_2\right)} = \frac{r_1\left(\cos\theta_1 + i\sin\theta_1\right)\left(\cos\theta_2 - i\sin\theta_2\right)}{r_2\left(\cos\theta_2 + i\sin\theta_2\right)\left(\cos\theta_2 - i\sin\theta_2\right)}$$

$$= \frac{r_1\left(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2\right)}{r_2\left(\cos^2\theta_2 - i^2\sin^2\theta_2\right)}$$

$$= \frac{r_1\left(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + \sin\theta_1\sin\theta_2\right)}{r_2\left(\cos^2\theta_2 + \sin^2\theta_2\right)}$$

$$= \frac{r_1\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\right)}{r_2}$$

$$= \frac{r_1}{r_2}\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)$$

Product and Quotient

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Product and Quotient of Two Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex numbers.

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \quad z_2 \neq 0 \qquad \text{Quotient}$$

The Quotient of Two Complex Numbers in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find the quotient of the complex numbers. Leave the answer in polar form.

$$z_1 = 50(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) \qquad z_2 = 5(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

$$\frac{z_1}{z_2} = \frac{50}{5}\left[\cos\left(\frac{4\pi}{3} - \frac{\pi}{3}\right) + i\sin\left(\frac{4\pi}{3} - \frac{\pi}{3}\right)\right]$$

$$\frac{z_1}{z_2} = 10\left(\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}\right) = 10\left(\cos\pi + i\sin\pi\right)$$

Example: Finding Quotients of Complex Numbers in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find the quotient of the complex numbers. Leave the answer in polar form.

$$z_1 = 3(\cos 45^\circ + i \sin 45^\circ) \qquad z_2 = 2(\cos 135^\circ + i \sin 135^\circ)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{3}{2}\left[\cos\left(45^\circ - 135^\circ\right) + i\sin\left(45^\circ - 135^\circ\right)\right]$$

$$\frac{z_1}{z_2} = \frac{3}{2}\left[\cos\left(-90^\circ\right) + i\sin\left(-90^\circ\right)\right]$$

$$\frac{z_1}{z_2} = \frac{3}{2}\left[\cos\left(270^\circ\right) + i\sin\left(270^\circ\right)\right]$$

Example: Finding Quotients of Complex Numbers in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find the quotient of the complex numbers. Write the answer in polar form and rectangular form.

$$z_1 = 8(\cos 120^\circ + i\sin 120^\circ) \quad z_2 = 6(\cos 150^\circ + i\sin 150^\circ)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{8}{6}[(\cos 120^\circ - 150^\circ) + i\sin(120^\circ - 150^\circ)]$$

$$\frac{z_1}{z_2} = \frac{4}{3}\left(\cos(-30)^\circ + i\sin(-30)^\circ\right)$$

Change to rectangular form $\quad z_1 z_2 = \frac{4}{3}\left(\frac{\sqrt{3}}{2} + i\left(\frac{-1}{2}\right)\right) = \frac{2\sqrt{3}}{3} + \left(\frac{-2}{3}\right)i$

Power of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Write a complex number raised to a power 2. $(a+bi)^2 = \left(r(\cos\theta + i\sin\theta)\right)^2$

Expanding $\left(r(\cos\theta + i\sin\theta)\right)^2 = r(\cos\theta + i\sin\theta)\cdot r(\cos\theta + i\sin\theta)$

$z_1 \cdot z_2 = r_1 \cdot r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ $\quad = r^2(\cos 2\theta + i\sin 2\theta)$

Now let's try cubing $(a+bi)^3 = \left(r(\cos\theta + i\sin\theta)\right)^3 = \left(r(\cos\theta + i\sin\theta)\right)^2\left(r(\cos\theta + i\sin\theta)\right)$

$$= r^2(\cos 2\theta + i\sin 2\theta)\left(r(\cos\theta + i\sin\theta)\right)$$

$z_1 \cdot z_2 = r_1 \cdot r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$$= r^2 \cdot r\left(\cos(2\theta+\theta) + i\sin(2\theta+\theta)\right)$$

$$= r^3(\cos 3\theta + i\sin 3\theta)$$

We could do this all day, but you should see the pattern emerging …

DeMoivre's Theorem

If z is a complex number in polar form ($z = \cos\theta + i\sin\theta$) and n is a positive integer, then:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

Raise the moduli to the power and multiply the arguments by the power.

Note that the argument must be the same angle.

De Moivre's Theorem

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n$$
$$= r^n(\cos n\theta + i\sin n\theta).$$

Example: Powers of Complex Numbers in Polar Form

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find $[2(\cos 30° + i \sin 30°)]^5$. Write the answer in rectangular form.

$$\boxed{z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i \sin n\theta)}$$

$$[2(\cos 30° + i \sin 30°)]^5 = 2^5(\cos 5\cdot 30° + i \sin 5\cdot 30°)$$

$$= 32(\cos 150° + i \sin 150°)$$

$$= 32\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -16\sqrt{3} + 16i$$

Example: Finding the Power of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find $\left(\frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^2$. Write the answer in rectangular form.

$$\boxed{z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)}$$

$$\left[\frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^2 = \left(\frac{1}{2}\right)^2\left[\cos\left(2\cdot\frac{\pi}{4}\right) + i\sin\left(2\cdot\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{4}\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$= \frac{1}{4}\left[0 + i\right] = 0 + \frac{1}{4}i$$

Example: Finding the Power of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find $\left(-2\sqrt{3}-2i\right)^5$. Write the answer in rectangular form, **a + bi**.

First we need to change to polar form.

$$r=\sqrt{a^2+b^2} \qquad \tan\theta=\frac{b}{a} \qquad r=\sqrt{\left(-2\sqrt{3}\right)^2+\left(-2\right)^2}=4$$

$$\tan\theta=\frac{-2}{-2\sqrt{3}}=\frac{1}{\sqrt{3}} \qquad \theta=\frac{7\pi}{6},\ QIII$$

$$\left(-2\sqrt{3}-2i\right)^5=\left(4\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)\right)^5$$

Example: Finding the Power of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

$$\left(4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\right)^5 = (4)^5\left[\cos\left(5\cdot\frac{7\pi}{6}\right) + i\sin\left(5\cdot\frac{7\pi}{6}\right)\right]$$

$$= 1024\left[\cos\frac{35\pi}{6} + i\sin\frac{35\pi}{6}\right]$$

$$= 1024\left[\cos\frac{11\pi}{6} + i\left(\sin\frac{11\pi}{6}\right)\right]$$

Back to rectangular form: $= 1024\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = 512\sqrt{3} - 512i$

nth Roots of Complex Numbers

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

An nth root of a number is the number that when raised to the nth power the result is our original number. That is:

If $z^n = a$, then z is an nth root of a.

Let a(cosα + isinα) be an nth root of r(cosθ + isinθ)

Then $\left(a\left(\cos\alpha + i\sin\alpha\right)\right)^n = r\left(\cos\theta + i\sin\theta\right)$

and $a^n\left(\cos n\alpha + i\sin n\alpha\right) = r\left(\cos\theta + i\sin\theta\right)$

so $a^n = r \qquad \cos n\alpha = \cos\theta \qquad \sin n\alpha = \sin\theta$

Finally $a = \sqrt[n]{r} \qquad n\alpha = \theta + k2\pi \quad \alpha = \dfrac{\theta + k2\pi}{n}$

cosine and sine are periodic functions with period 2π

DeMoivre's Theorem for Finding Complex Roots

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

If **w** is a complex number in polar form (**w = cosθ + isinθ**) then **w** has **n** distinct nth roots given by:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360k^\circ}{n}\right) + i\sin\left(\frac{\theta + 360k^\circ}{n}\right)\right]$$

$k = 0, 1, 2, 3, \ldots, n-1$

In other words, **w** has **n** roots spread evenly around the circle. For square root **w** has two roots, for cube root **w** has 3 roots, etc.

Finding Complex Roots

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Definition of the *n*th Root of a Complex Number

The complex number $u = a + bi$ is an ***n*th root** of the complex number z if

$$z = u^n = (a + bi)^n.$$

Finding *n*th Roots of a Complex Number

For a positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots given by

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

where $k = 0, 1, 2, \ldots, n - 1$.

Example: DeMoivre's Theorem for Finding Complex Roots

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find all the complex cube roots of **8(cos210° + *i*sin210°)**.
Write roots in polar form, with θ in degrees.

$$\boxed{z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360k^\circ}{n}\right) + i\sin\left(\frac{\theta + 360k^\circ}{n}\right)\right]}$$

$$z_0 = \sqrt[3]{8}\left[\cos\left(\frac{210^\circ + 360 \bullet 0^\circ}{3}\right) + i\sin\left(\frac{210^\circ + 360 \bullet 0^\circ}{3}\right)\right] = 2\left[\cos 70^\circ + i\sin 70^\circ\right]$$

$$z_1 = \sqrt[3]{8}\left[\cos\left(\frac{210^\circ + 360 \bullet 1^\circ}{3}\right) + i\sin\left(\frac{210^\circ + 360 \bullet 1^\circ}{3}\right)\right] = 2\left[\cos 190^\circ + i\sin 190^\circ\right]$$

$$z_2 = \sqrt[3]{8}\left[\cos\left(\frac{210^\circ + 360 \bullet 2^\circ}{3}\right) + i\sin\left(\frac{210^\circ + 360 \bullet 2^\circ}{3}\right)\right] = 2\left[\cos 310^\circ + i\sin 310^\circ\right]$$

Example: Finding the Roots of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find all the complex cube roots of **8(cos210° + *i*sin210°)**.
Write roots in polar form, with θ in degrees.

And, of course, we can convert to rectangular form

$$= 2\left[\cos 70^\circ + i\sin 70^\circ\right] = 2\left[.3420 + .9397i\right] = .6840 + 1.8794i$$

$$= 2\left[\cos 190^\circ + i\sin 190^\circ\right] = 2\left[-.9848 + -.1736i\right] = -1.9696 - .3473i$$

$$= 2\left[\cos 310^\circ + i\sin 310^\circ\right] = 2\left[.6427 + -.7660i\right] = 1.2856 - 1.5321i$$

Example: Finding the Roots of a Complex Number

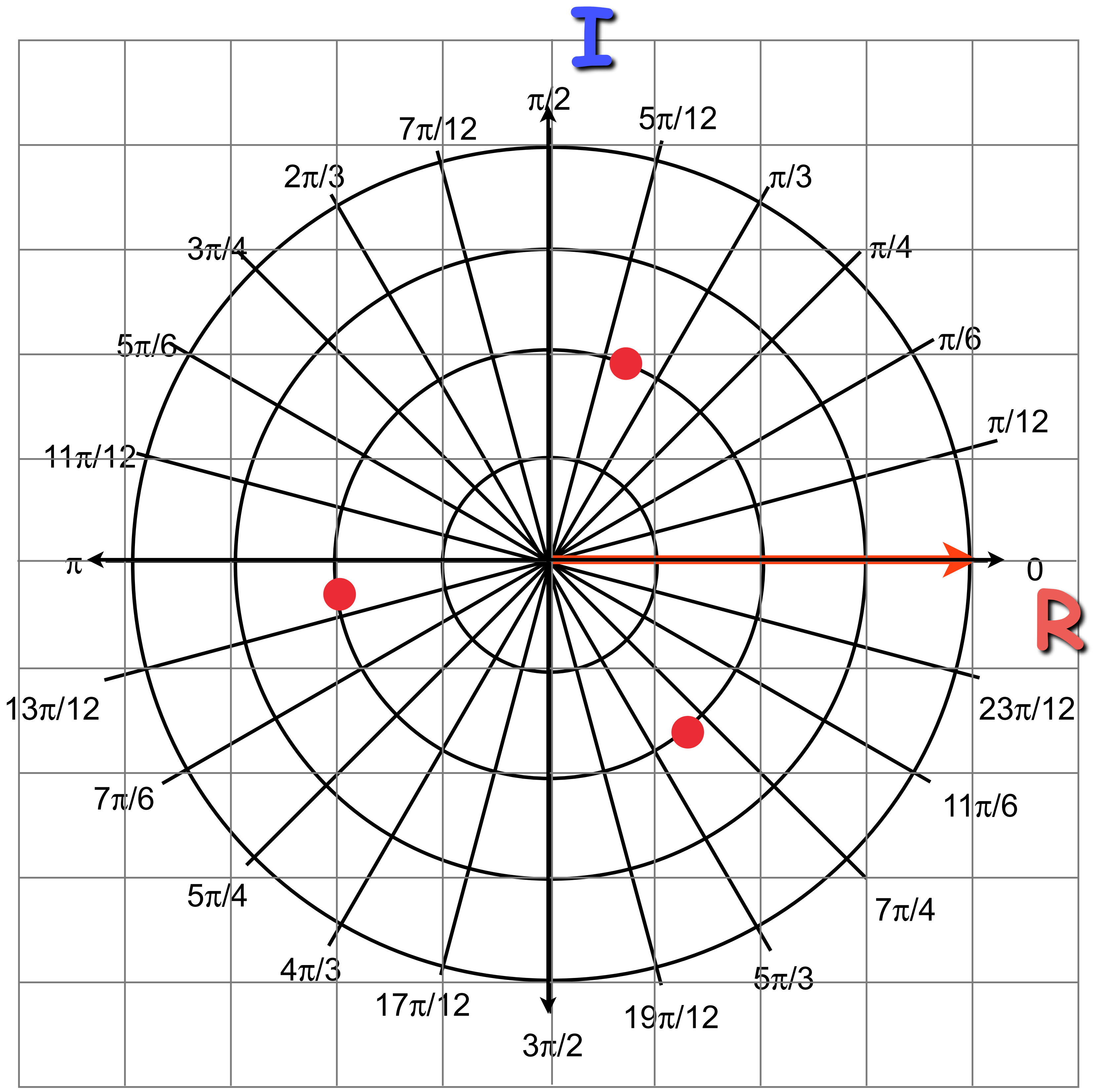
Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

These roots can be plotted on the imaginary plane, or on what we will soon call the **polar coordinate plane**.

$$= 2\left[\cos 70^\circ + i\sin 70^\circ\right] = .6840 + 1.8794i$$

$$= 2\left[\cos 190^\circ + i\sin 190^\circ\right] = -1.9696 - .3473i$$

$$= 2\left[\cos 310^\circ + i\sin 310^\circ\right] = 1.2856 - 1.5321i$$



Example: Finding the Roots of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find all the complex fourth roots of $\omega = -8 + 8i\sqrt{3}$

Write roots in polar form, with θ in degrees.

First we must write $\omega = -8 + 8i\sqrt{3}$ in polar form.

$$r = \sqrt{a^2 + b^2}$$

$$r = \sqrt{(-8)^2 + \left(8\sqrt{3}\right)^2} = 16$$

$$\tan\theta = \frac{b}{a}$$

$$\theta = \tan^{-1}\frac{8\sqrt{3}}{-8} = -\sqrt{3}$$

$$\theta = 120^\circ$$

$$\omega = 16(\cos 120^\circ + i\sin 120^\circ)$$

Example: Finding the Roots of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find all the complex fourth roots of $\omega = -8 + 8i\sqrt{3}$

Write roots in polar form, with θ in degrees.

$$\omega = 16(\cos 60^\circ + i\sin 60^\circ)$$

$$\boxed{z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360k^\circ}{n}\right) + i\sin\left(\frac{\theta + 360k^\circ}{n}\right)\right]}$$

$$z_0 = \sqrt[4]{16}\left[\cos\left(\frac{120^\circ + (360 \bullet 0)^\circ}{4}\right) + i\sin\left(\frac{120^\circ + (360 \bullet 0)^\circ}{4}\right)\right] = 2\left[\cos 30^\circ + i\sin 30^\circ\right]$$

$$z_1 = \sqrt[4]{16}\left[\cos\left(\frac{120^\circ + (360 \bullet 1)^\circ}{4}\right) + i\sin\left(\frac{120^\circ + (360 \bullet 1)^\circ}{4}\right)\right] = 2\left[\cos 120^\circ + i\sin 120^\circ\right]$$

Example: Finding the Roots of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find all the complex fourth roots of $\omega = -8 + 8i\sqrt{3}$

Write roots in polar form, with θ in degrees.

$$\omega = 16(\cos 60^\circ + i \sin 60^\circ)$$

$$\boxed{z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360k^\circ}{n}\right) + i\sin\left(\frac{\theta + 360k^\circ}{n}\right)\right]}$$

$$z_2 = \sqrt[4]{16}\left[\cos\left(\frac{120^\circ + (360 \bullet 2)^\circ}{4}\right) + i\sin\left(\frac{120^\circ + (360 \bullet 2)^\circ}{4}\right)\right] = 2\left[\cos 210^\circ + i \sin 210^\circ\right]$$

$$z_3 = \sqrt[4]{16}\left[\cos\left(\frac{120^\circ + (360 \bullet 3)^\circ}{4}\right) + i\sin\left(\frac{120^\circ + (360 \bullet 3)^\circ}{4}\right)\right] = 2\left[\cos 300^\circ + i \sin 300^\circ\right]$$

Example: Finding the Roots of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find all the complex fourth roots of $\omega = -8 + 8i\sqrt{3}$
Write roots in polar form, with θ in degrees.

$$\omega = 16(\cos 60^\circ + i \sin 60^\circ)$$

Converting to rectangular form:

$$\sqrt[4]{\omega} = 2\left[\cos 30^\circ + i \sin 30^\circ\right] = 2\left[\frac{\sqrt{3}}{2} + \frac{1}{2}i\right] = \sqrt{3} + i$$

$$= 2\left[\cos 120^\circ + i \sin 120^\circ\right] = 2\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right] = -1 + i\sqrt{3}$$

$$= 2\left[\cos 210^\circ + i \sin 210^\circ\right] = 2\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = -\sqrt{3} - i$$

$$= 2\left[\cos 300^\circ + i \sin 300^\circ\right] = 2\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] = 1 - i\sqrt{3}$$

Graphing Roots

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

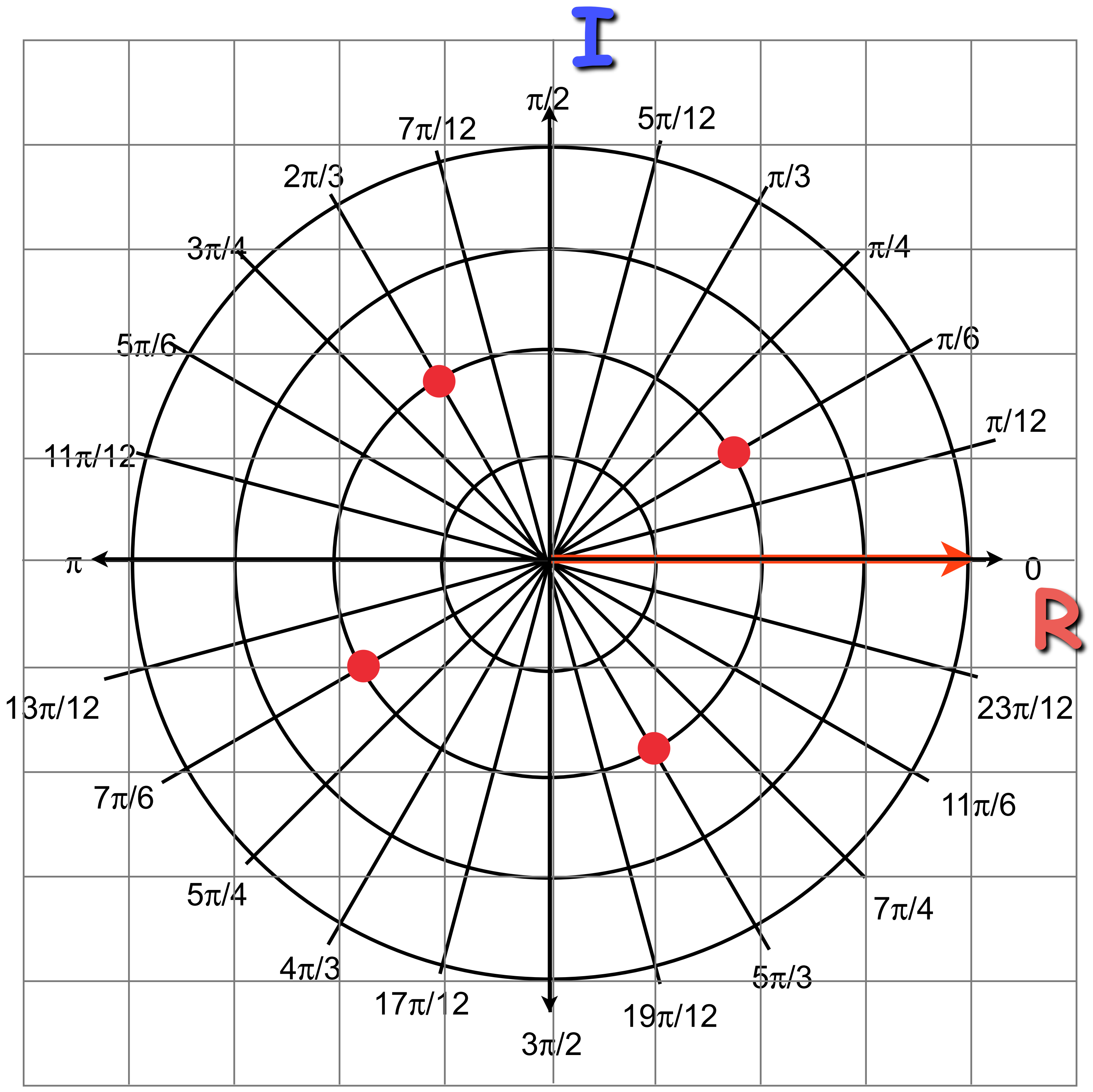
Graphing

$$= 2\left[\cos 30^\circ + i\sin 30^\circ\right] = \sqrt{3} + i$$

$$= 2\left[\cos 120^\circ + i\sin 120^\circ\right] = -1 + i\sqrt{3}$$

$$= 2\left[\cos 210^\circ + i\sin 210^\circ\right] = -\sqrt{3} - i$$

$$= 2\left[\cos 300^\circ + i\sin 300^\circ\right] = 1 - i\sqrt{3}$$



Example: Finding the Roots of a Complex Number

Objective: Multiply and divide complex numbers in trigonometric form and find powers and nth roots of complex numbers.

Find all the complex fourth roots of **16**. Write roots in polar form, with θ in radians.

$$16 = 16\left[\cos(0) + i\sin(0)\right]$$

$$\sqrt[4]{16} = \sqrt[4]{16}\left[\cos\left(\frac{0+k2\pi}{4}\right) + i\sin\left(\frac{0+k2\pi}{4}\right)\right]$$

$$z_0 = \sqrt[4]{16}\left[\cos\left(\frac{0+2\pi \bullet 0}{4}\right) + i\sin\left(\frac{0+2\pi \bullet 0}{4}\right)\right] = 2\left[\cos 0 + i\sin 0\right] = 2$$

$$z_1 = \sqrt[4]{16}\left[\cos\left(\frac{0+2\pi \bullet 1}{4}\right) + i\sin\left(\frac{0+2\pi \bullet 1}{4}\right)\right] = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 2i$$

$$z_2 = \sqrt[4]{16}\left[\cos\left(\frac{0+2\pi \bullet 2}{4}\right) + i\sin\left(\frac{0+2\pi \bullet 2}{4}\right)\right] = 2\left[\cos\pi + i\sin\pi\right] = -2$$

$$z_3 = \sqrt[4]{16}\left[\cos\left(\frac{0+2\pi \bullet 3}{4}\right) + i\sin\left(\frac{0+2\pi \bullet 3}{4}\right)\right] = 2\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right] = -2i$$