Chpt 7

Systems of Equations and Inequalities

7.4 Partial Fractions

Chpt 7.4

Homework

- Read Sec 7.4
- Complete Reading Notes
- Complete Worksheet
- Do p539 1-4, 5, 7, 9, 15-27 odd,
  39, 41, 43, 45, 47, 51

Chpt 7.4

Objectives

- Recognize partial fraction decompositions of rational expressions.
- Find partial fraction decompositions of rational expressions.
- Decompose $\frac{P}{Q}$ where Q has only distinct linear factors.
- Decompose $\frac{P}{Q}$ where Q has only repeated linear factors.

Partial Fraction Decomposition

Find partial fraction decompositions of rational expressions.

- There are times when a rational expression is more easily dealt with when it has been written as a sum of simpler rational expressions (think Calculus).
  - Partial Fraction Decomposition is the reverse of adding rational expressions. We take a rational expression and break it up into the sum of partial fractions. Hence **Partial Fraction Decomposition.**

Remember:

- A rational expression is of the form: $\dfrac{P(x)}{Q(x)}$ or in your text: $\dfrac{N(x)}{D(x)}$
- To do partial fraction decomposition, the numerator $P(x)$ must be of lower degree than the denominator $Q(x)$. If it is not, divide and work on the remainder.

Steps in Partial Fraction Decomposition

Find partial fraction decompositions of rational expressions.

Procedure for Partial Fraction Decomposition. $\dfrac{P(x)}{Q(x)}$

1. Factor the denominator $Q(x)$ into linear factors $ax + b$.
2. Set up the partial fraction decomposition with unknown constants A, B, C,…, in the numerators of the decomposition and the linear factors in the denominators.
3. Multiply both sides of the resulting equation by the least common denominator.
4. Simplify the both sides of the equation. Pay special attention to the right side.
5. Write both sides in descending powers, equate coefficients of like powers of x, and equate constant terms. Write as a linear system.
6. Solve the linear system for A, B, C, etc.
7. Substitute the values for A, B, C, etc., into the equation from step 1.

From the Text

Find partial fraction decompositions of rational expressions.

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. *Divide if improper:* If $N(x)/D(x)$ is an improper fraction [degree of $N(x) \geq$ degree of $D(x)$], divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{polynomial}) + \frac{N_1(x)}{D(x)}$$

and apply Steps 2, 3, and 4 below to the proper rational expression $N_1(x)/D(x)$. Note that $N_1(x)$ is the remainder from the division of $N(x)$ by $D(x)$.

2. *Factor the denominator:* Completely factor the denominator into factors of the form

$$(px + q)^m \quad \text{and} \quad (ax^2 + bx + c)^n$$

where $(ax^2 + bx + c)$ is irreducible.

3. *Linear factors:* For *each* factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

STUDY TIP

Section A.4, shows you how to combine expressions such as

$$\frac{1}{x - 2} + \frac{-1}{x + 3} = \frac{5}{(x - 2)(x + 3)}.$$

The method of partial fractions shows you how to reverse this process.

$$\frac{5}{(x - 2)(x + 3)} = \frac{?}{x - 2} + \frac{?}{x + 3}$$

The Partial Fraction Decomposition of $\dfrac{P(x)}{Q(x)}$: Q(x) Has Distinct Linear Factors

The form of the partial fraction decomposition for a rational expression with distinct linear factors in the denominator is;

$$\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)(a_3x+b_3)\ldots(a_nx+b_n)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \ldots + \frac{A_n}{(a_nx+b_n)}$$

Partial Fraction Decomposition with Distinct Linear Factors

Find partial fraction decompositions of rational expressions.

Find the partial fraction decomposition of $\dfrac{5x-1}{x^2+x-12}$

1. Factor the denominator.

$$\frac{5x-1}{x^2+x-12} = \frac{5x-1}{(x-3)(x+4)}$$

2. Set up the partial fraction decomposition with the unknown constants A, B, C, etc., in the numerators of the decomposition.

$$\frac{5x-1}{(x-3)(x+4)} = \frac{A}{(x-3)} + \frac{B}{(x+4)}$$

3. Multiply both sides of the resulting equation by the least common denominator.

$$\frac{5x-1}{(x-3)(x+4)}\big((x-3)(x+4)\big) = \left[\frac{A}{(x-3)} + \frac{B}{(x+4)}\right]\big((x-3)(x+4)\big)$$

4. Simplify.

$$5x-1 = A(x+4) + B(x-3)$$

Partial Fraction Decomposition with Distinct Linear Factors

Find partial fraction decompositions of rational expressions.

Find the partial fraction decomposition of $\dfrac{5x-1}{x^2+x-12}$

5. Write both sides in descending powers, equate coefficients of like powers of x, and equate constant terms. Write as a system of equations.

$$5x-1 = A(x+4)+B(x-3)$$
$$5x-1 = Ax+4A+Bx-3B$$
$$5x-1 = (A+B)x+(4A-3B)$$
$$5x = (A+B)x$$
$$-1 = (4A-3B)$$
$$\begin{cases} A+B=5 \\ 4A-3B=-1 \end{cases}$$

6. Solve the linear system for A, B, C, etc.

$$\begin{cases} 3A+3B=15 \\ 4A-3B=-1 \end{cases} \qquad 14 = 7A \quad A = 2 \qquad 5 = 2+B \quad B = 3$$

7. Substitute the values for A, B, C, etc., into the equation from step 1.

$$\frac{5x-1}{(x-3)(x+4)} = \frac{A}{(x-3)}+\frac{B}{(x+4)} = \frac{2}{(x-3)}+\frac{3}{(x+4)}$$

Partial Fraction Decomposition

Find partial fraction decompositions of rational expressions.

Another approach to finding values for A, B, C, etc., is to substitute convenient values for x that eliminate terms. This simplifies the equation to a single variable. A value for x that is nearly always convenient is a value that creates a **factor of 0.**

Find the partial fraction decomposition of $\dfrac{5x-1}{x^2+x-12}$

$$\frac{5x-1}{(x-3)(x+4)} = \frac{A}{(x-3)} + \frac{B}{(x+4)}$$

Steps **1,2,3,&4** remain the same.

$$5x-1 = A(x+4) + B(x-3)$$

5. Substitute convenient values for x, such as the roots of the denominator.

replace x with -4.

$$5(-4)-1 = A(-4+4) + B(-4-3)$$

$$-21 = A(0) + B(-7)$$

$$B = 3$$

Replace x with 3.

$$5(3)-1 = A(3+4) + B(3-3)$$

$$14 = A(7) + B(0)$$

$$A = 2$$

Same result.

$$\frac{5x-1}{(x-3)(x+4)} = \frac{2}{(x-3)} + \frac{3}{(x+4)}$$

Partial Fraction Decomposition with Distinct Linear Factors

Find partial fraction decompositions of rational expressions.

Find the partial fraction decomposition of $\dfrac{3x+1}{x^2+2x-3}$

1. Factor the denominator.

$$\frac{3x+1}{x^2+2x-3} = \frac{3x+1}{(x-1)(x+3)}$$

2. Set up the partial fraction decomposition with the unknown constants A, B, C, etc., in the numerators of the decomposition.

$$\frac{3x+1}{(x-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+3)}$$

3. Multiply both sides of the resulting equation by the least common denominator.

$$\frac{3x+1}{(x-1)(x+3)}\Big((x-1)(x+3)\Big) = \left[\frac{A}{(x-1)} + \frac{B}{(x+3)}\right]\Big((x-1)(x+3)\Big)$$

4. Simplify.

$$3x+1 = A(x+3) + B(x-1)$$

Example: Partial Fraction Decomposition with Distinct Linear Factors

Find partial fraction decompositions of rational expressions.

Find the partial fraction decomposition of $\dfrac{3x+1}{x^2+2x-3}$

5. Choose convenient values of x (zeros of the denominator) and substitute.

$$3x+1 = A(x+3) + B(x-1)$$

Let x = 1

$$3(1)+1 = A(1+3) + B(1-1)$$
$$4 = A(4) + B(0)$$
$$A = 1$$

Let x = -3

$$3(-3)+1 = A(-3+3) + B(-3-1)$$
$$-8 = A(0) + B(-4)$$
$$B = 2$$

$$\frac{3x+1}{(x-1)(x+3)} = \frac{1}{(x-1)} + \frac{2}{(x+3)}$$

OR

5. Write both sides in descending powers, equate coefficients of like powers of x, and equate constant terms. Write as a system of equations.

$$3x+1 = A(x+3) + B(x-1)$$
$$3x+1 = Ax + 3A + Bx - B$$
$$3x+1 = \left(A+B\right)x + \left(3A-B\right)$$

$$3x = \left(A+B\right)x \qquad 3 = A+B$$
$$1 = \left(3A-B\right) \qquad 1 = 3A-B$$

$$\begin{cases} A+B=3 \\ 3A-B=1 \end{cases}$$

Example: Partial Fraction Decomposition with Distinct Linear Factors

Find partial fraction decompositions of rational expressions.

Find the partial fraction decomposition of $\dfrac{3x+1}{x^2+2x-3}$

6. Solve the linear system for A, B, C, etc.

$$\begin{cases} A+B=3 \\ 3A-B=1 \end{cases} \qquad \begin{aligned} 4 &= 4A \\ A &= 1 \end{aligned} \qquad \begin{aligned} 3 &= 1+B \\ B &= 2 \end{aligned}$$

7. Substitute the values for A, B, C, etc., into the equation from step 1.

$$\frac{3x+1}{x^2+2x-3} = \frac{A}{(x-1)} + \frac{B}{(x+3)}$$

$$= \frac{1}{(x-1)} + \frac{2}{(x+3)}$$

The form of the partial fraction decomposition for a rational expression containing the linear factor ax + b occurring n times as its denominator is:

$$\frac{P(x)}{(ax+b)^n} = \frac{A_1}{(ax+b)^1} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \ldots + \frac{A_n}{(ax+b)^n}$$

Include one fraction with a constant numerator for each power of ax + b.

The Partial Fraction Decomposition of Has Repeated Linear Factors

Find the partial fraction decomposition of $\dfrac{x+2}{x(x-1)^2}$

1. Already factored for us.

2. Set up the partial fraction decomposition with the unknown constants A, B, C, etc., in the numerators of the decomposition.

$$\frac{x+2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

3. Multiply both sides of the resulting equation by the least common denominator.

$$\frac{x+2}{x(x-1)^2}\left(x(x-1)^2\right) = \left[= \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}\right]\left(x(x-1)^2\right)$$

4. Simplify.

$$x+2 = A(x-1)^2 + Bx(x-1) + Cx$$

The Partial Fraction Decomposition of Has Repeated Linear Factors

- Find the partial fraction decomposition of $\dfrac{x+2}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{(x-1)} + \dfrac{C}{(x-1)^2}$
  - Select convenient values for x.

$$x+2 = A(x-1)^2 + Bx(x-1) + Cx$$

Let x = 0

$$0+2 = A(0-1)^2 + B(0)(0-1) + C(0)$$

$$2 = A(-1)^2 + B(0)(-1) + C(0)$$

$$2 = A$$

Let x = 1, A = 2

$$1+2 = 2(1-1)^2 + B(1)(1-1) + C(1)$$

$$3 = A(0)^2 + B(1)(0) + C(1)$$

$$3 = C$$

Let x = 2, A = 2, C = 3

$$2+2 = 2(2-1)^2 + B(2)(2-1) + 3(2)$$

$$4 = 2(1)^2 + B(2)(1) + 6$$

$$4 = 2 + 2B + 6$$

$$-4 = 2B$$

$$-2 = B$$

$$\frac{x+2}{x(x-1)^2} = \frac{2}{x} + \frac{-2}{(x-1)} + \frac{3}{(x-1)^2}$$

The Partial Fraction Decomposition of Has Repeated Linear Factors

Find the partial fraction decomposition of $\dfrac{x+2}{x(x-1)^2}$

Alternate method of system of equations.

5. Expand and write both sides in descending powers, equate coefficients of like powers of x, and equate constant terms. Write as a system of equations.

$$x+2 = A(x-1)^2 + Bx(x-1) + Cx$$
$$x+2 = A(x^2-2x+1) + Bx^2 - Bx + Cx$$
$$x+2 = Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$$
$$x+2 = Ax^2 + Bx^2 - 2Ax - Bx + Cx + A$$
$$x+2 = (A+B)x^2 - (2A-B+C)x + A$$

$$0x^2 + 1x + 2 = (A+B)x^2 - (2A-B+C)x + A$$

$$0x^2 = (A+B)x^2$$
$$0 = A+B$$

$$1x = (-2A-B+C)x$$
$$1 = -2A-B+C$$

$$A = 2$$

$$\begin{cases} -2A - B + C = 1 \\ A + B = 0 \\ A = 2 \end{cases}$$

Find the partial fraction decomposition of $\dfrac{x+2}{x(x-1)^2}$

6. Solve the linear system for A, B, C, etc.

$$\begin{cases} -2A - B + C = 1 \\ A + B = 0 \\ A = 2 \end{cases} \qquad \begin{aligned} 2 + B &= 0 \\ B &= -2 \end{aligned} \qquad \begin{aligned} -2(2) - (-2) + C &= 1 \\ C &= 3 \end{aligned}$$

7. Substitute the values for A, B, C, etc., into the equation from step 1.

$$\frac{x+2}{x(x-1)^2} = \frac{2}{x} + \frac{-2}{(x-1)} + \frac{3}{(x-1)^2} = \frac{2}{x} - \frac{2}{(x-1)} + \frac{3}{(x-1)^2}$$

The Partial Fraction Decomposition of Has Repeated Linear Factors

Find the partial fraction decomposition of $\dfrac{2x+4}{(x+1)^3}$

1. Obviously factored denominator.

2. Set up the partial fraction decomposition with the unknown constants A, B, C, etc., in the numerators of the decomposition.

$$\frac{2x+4}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

3. Multiply both sides of the resulting equation by the least common denominator

$$\frac{2x+4}{(x+1)^3}(x+1)^3 = \left[\frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}\right](x+1)^3$$

4. Simplify

$$2x+4 = A(x+1)^2 + B(x+1) + C$$

The Partial Fraction Decomposition of Has Repeated Linear Factors

Find partial fraction decompositions of rational expressions.

Find the partial fraction decomposition of $\dfrac{2x+4}{(x+1)^3} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3}$

5. Expand and write both sides in descending powers, equate coefficients of like powers of x, and equate constant terms. Write as a system of equations.

$$2x+4 = A(x+1)^2 + B(x+1) + C$$
$$2x+4 = A(x^2+2x+1) + B(x+1) + C$$
$$2x+4 = Ax^2 + 2Ax + A + Bx + B + C$$
$$2x+4 = Ax^2 + 2Ax + Bx + A + B + C$$
$$2x+4 = Ax^2 + (2A+B)x + A + B + C$$
$$0x^2 + 2x + 4 = Ax^2 + (2A+B)x + A + B + C$$

$$\begin{cases} A+B+C=4 \\ 2A+B=2 \\ A=0 \end{cases}$$

6. Solve the linear system for A, B, C, etc.

$$A = 0$$

$$2A + B = 2$$
$$2(0) + B = 2$$
$$B = 2$$

$$A + B + C = 4$$
$$0 + 2 + C = 4$$
$$C = 2$$

7. Substitute the values for A, B, C, etc., into the equation from step 1.

$$\frac{2x+4}{(x+1)^3} = \frac{2}{(x+1)^2} + \frac{2}{(x+1)^3}$$

“Improper” Fraction

Find partial fraction decompositions of rational expressions.

- Find the partial fraction decomposition of $\dfrac{x^3+2x^2-x+1}{x^2+3x-4}$
  - We have a new issue with which we must deal. The numerator is of greater degree than the denominator. That cannot stand.
  - No problem, we simply divide. Keeping the quotient in mind we focus on the remainder.

$$\frac{x^3+2x^2-x+1}{x^2+3x-4} = \begin{array}{r|l} & \;x-1 \\ \cline{2-2} x^2+3x-4 & x^3+2x^2-x+1 \\ & \underline{x^3+3x^2-4x} \\ & -x^2+3x+1 \\ & \underline{-x^2-3x+4} \\ & 6x-3 \end{array} = x-1+\frac{6x-3}{x^2+3x-4}$$

  - We do not discard the quotient. The quotient function will become part of the decomposition with the decomposition of the remainder.

"Improper Fraction"

Find partial fraction decompositions of rational expressions.

Find the partial fraction decomposition of $\dfrac{x^3+2x^2-x+1}{x^2+3x-4} = x-1+\dfrac{6x-3}{x^2+3x-4}$

Now we proceed as before with just the remainder function.

1. Factor the denominator of the remainder.

$$\frac{6x-3}{x^2+3x-4} = \frac{6x-3}{(x-1)(x+4)}$$

2. Set up the partial fraction decomposition with the unknown constants A, B, C, etc., in the numerators of the decomposition.

$$\frac{6x-3}{(x-1)(x+4)} = \frac{A}{(x-1)} + \frac{B}{(x+4)}$$

3. Multiply both sides of the resulting equation by the least common denominator

$$\frac{6x-3}{(x-1)(x+4)}\big((x-1)(x+4)\big) = \left[\frac{A}{(x-1)} + \frac{B}{(x+4)}\right]\big((x-1)(x+4)\big)$$

4. Simplify

$$6x-3 = A(x+4) + B(x-1)$$

"Improper Fraction"

Find partial fraction decompositions of rational expressions.

Find the partial fraction decomposition of $\dfrac{x^3+2x^2-x+1}{x^2+3x-4} = x-1+\dfrac{6x-3}{x^2+3x-4}$

5. Select convenient values for x, the zeros of the remainder denominator.

$$6x-3 = A(x+4)+B(x-1)$$

Let x = 1

$$6(1)-3 = A(1+4)+B(1-1)$$
$$3 = A(5)+B(0)$$
$$A=\frac{3}{5}$$

Let x = -4

$$6(-4)-3 = A(-4+4)+B(-4-1)$$
$$-27 = A(0)+B(-5)$$
$$B=\frac{27}{5}$$

6. Substitute the values for A, B, C, etc., into the equation from step 1.

$$\frac{x^3+2x^2-x+1}{x^2+3x-4} = x-1+\frac{A}{(x-1)}+\frac{B}{(x+4)} = x-1+\frac{\frac{3}{5}}{(x-1)}+\frac{\frac{27}{5}}{(x+4)} = x-1+\frac{3}{5(x-1)}+\frac{27}{5(x+4)}$$