Chapter 8

- 8.2 Matrix Operations

Chapter 8.2

Homework

- Read Sec 8.2
- Complete Reading Notes
- Do p597 1-69 every other odd
- 8.2 WS
- 8.2 WS 2

Chapter 8

Objectives

- Use matrix notation.
- Understand what is meant by equal matrices.
- Add and subtract matrices.
- Perform scalar multiplication.
- Multiply matrices.
- Model applied situations with matrix operations.

Matrix Notations

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- An array of numbers, arranged in rows and columns and placed in brackets, is called a matrix. We can represent a matrix in two different ways:

- A capital letter, such as A, B, or C, can denote a matrix.

- A lowercase letter enclosed in brackets, such as $[a_{ij}]$ can be used to denote a matrix.

- A general element in matrix A is denoted by a_{ij}. This refers to the element in the ith row and jth column.

Matrix Notations

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- A matrix of order $m \times n$ has m rows and n columns.
    - If $m = n$, a matrix has the same number of rows as columns and is called a square matrix.

- Let $A = \begin{bmatrix} 5 & -2 \\ -3 & \pi \\ 1 & 6 \end{bmatrix}$

- The order of A is ... 3×2
- a_{12} is ... -2
- Identify a_{31} 1

Matrix Equality

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Two matrices A and B are equal ($A = B$) if and only if they have the same order and corresponding elements are equal.
  - i.e. if $A = B$, and A is $m \times n$, then B is $m \times n$; and $a_{ij} = b_{ij}$ for all $i = 1,2,\ldots$ and all $j = 1,2,\ldots$
    - Let $A = \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
    - If $A = B$, then $a = -1$, $b = -2$, $c = 8$, $d = 5$.

Matrix Addition

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ and both are m x n matrices.
  - Then $A + B = [a_{ij} + b_{ij}]$.
    - Matrices of the same order are added by adding the elements in the corresponding positions.
  - And $A - B = [a_{ij} + -b_{ij}]$.
    - Matrices of the same order are subtracted by adding the opposite of elements in the corresponding positions.
  - You cannot add matrices with differing dimensions.

Adding Matrices

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Find $\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 5 - -4 & 4 - 8 \\ -3 - 6 & 7 - 0 \\ 0 - -5 & 1 - 3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -9 & 7 \\ 5 & -2 \end{bmatrix}$$

Matrix Addition Properties

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let 0 be an m x n zero matrix. ($a_{ij} = 0$, for every element)
- Let A, B, and C be m x n matrices.

1. $A + B = B + A$ Commutative Property of matrix Addition.
2. $(A + B) + C = A + (B + C)$ Associative Property of matrix Addition.
3. $A + 0 = 0 + A = A$ Matrix Additive Identity
4. $A + -A = -A + A = 0$ Matrix Additive Inverse

Matrix Scalar Multiplication

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- If $A = [a_{ij}]$ is a matrix of order $m \times n$ and c is a scalar, then the matrix cA is an $m \times n$ matrix found by $cA = [ca_{ij}]$

  - The product of a matrix with a scalar is found by multiplying each element of A by the real number c. cA is a scalar multiple of A.

- Let $A = \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$

- Find $-6A$ $\quad -6A = \begin{bmatrix} -6(-1) & -6(-2) \\ -6(8) & -6(5) \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -48 & -30 \end{bmatrix}$

Example

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let $A = \begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$

Find 3A + 2B

$$3A + 2B = 3\begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 3(-4) & 3(1) \\ 3(3) & 3(0) \end{bmatrix} + \begin{bmatrix} 2(-1) & 2(-2) \\ 2(8) & 2(5) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 3 \\ 9 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} -12 + -2 & 3 + -4 \\ 9 + 16 & 0 + 10 \end{bmatrix} = \begin{bmatrix} -14 & -1 \\ 25 & 10 \end{bmatrix}$$

Properties of Scalar Multiplication

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let A and B be m x n matrices, c and d are scalars. Then:

1. $(cd)A = c(dA)$ Associative Property of scalar multiplication.
2. $1A = A$ Scalar Multiplicative Identity
3. $c(A + B) = cA + cB$ Distributive Property
4. $(c + d)A = cA + dA$ Distributive Property

Matrix Multiplication

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- For the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix.
  - The product of an m x n matrix, A, and and n x p matrix, B, is an m x p matrix AB, whose elements are found by:
  - The element in the ith row and jth column of AB is found by multiplying each element in the ith row of A by the corresponding element in the jth column of B and adding the products.

Matrix Multiplication

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

- Then

ARow 1 x BCol 1

ARow 1 x BCol 2

$$AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

ARow 2 x BCol 1

ARow 2 x BCol 2

Example

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$ Find AB

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 3 \cdot 1 & 1 \cdot 6 + 3 \cdot 0 \\ 2 \cdot 4 + 5 \cdot 1 & 2 \cdot 6 + 5 \cdot 0 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 13 & 12 \end{bmatrix}$$

Example

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let $A = \begin{bmatrix} -1 & 0 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 & 7 \\ 1 & 0 & 3 \end{bmatrix}$ Find AB

$$AB = \begin{bmatrix} -1 & 0 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 5 & -2 & 7 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 \cdot 5 + 0 \cdot 1 & -1 \cdot -2 + 0 \cdot 0 & -1 \cdot 7 + 0 \cdot 3 \\ 6 \cdot 5 + 2 \cdot 1 & 6 \cdot -2 + 2 \cdot 0 & 6 \cdot 7 + 2 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 & -7 \\ 32 & -12 & 48 \end{bmatrix}$$

Example

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$

If possible, Find AB

$$AB = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 2 + 3\cdot 0 & 1\cdot 3 + 3\cdot 5 & 1\cdot -1 + 3\cdot 4 & 1\cdot 6 + 3\cdot 1 \\ 0\cdot 2 + 2\cdot 0 & 0\cdot 3 + 2\cdot 5 & 0\cdot -1 + 2\cdot 4 & 0\cdot 6 + 2\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 18 & 11 & 9 \\ 0 & 10 & 8 & 2 \end{bmatrix}$$

Example

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let $A = \begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ If possible, Find AB

- For the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix.

- Since that is not the case in this example, multiplication is not possible. The product is undefined.

- You will note that matrix multiplication is not commutative.

Example

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

If possible, Find AB

$$AB = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} 32 \end{bmatrix}$$

If possible, Find BA

$$BA = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 5 \cdot 1 & 5 \cdot 2 & 5 \cdot 3 \\ 6 \cdot 1 & 6 \cdot 2 & 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

- It will not often happen that you can multiply both AB and BA

Example

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

If possible, Find AB

$$AB = \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 1 + -1\cdot 1 + 4\cdot 1 & 1\cdot 1 + -1\cdot 2 + 4\cdot -1 & 1\cdot 0 + -1\cdot 4 + 4\cdot 3 \\ 4\cdot 1 + -1\cdot 1 + 3\cdot 1 & 4\cdot 1 + -1\cdot 2 + 3\cdot -1 & 4\cdot 0 + -1\cdot 4 + 3\cdot 3 \\ 2\cdot 1 + 0\cdot 1 + -2\cdot 1 & 2\cdot 1 + 0\cdot 2 + -2\cdot -1 & 2\cdot 0 + 0\cdot 4 + -2\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 & 8 \\ 6 & -1 & 5 \\ 0 & 4 & -6 \end{bmatrix}$$

Identity Matrix

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- The identity matrix is an n x n matrix with elements of the main diagonal equal to 1 and all other elements equal 0.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity Matrix

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Find AI_2

$$AI_2 = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 3 \cdot 0 & 1 \cdot 0 + 3 \cdot 1 \\ 2 \cdot 1 + 5 \cdot 0 & 2 \cdot 0 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$AI_3 = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 + 3 \cdot 0 + 5 \cdot 0 & 1 \cdot 0 + 3 \cdot 1 + 5 \cdot 0 & 1 \cdot 0 + 3 \cdot 0 + 5 \cdot 1 \\ 2 \cdot 1 + 4 \cdot 0 + 6 \cdot 0 & 2 \cdot 0 + 4 \cdot 1 + 6 \cdot 0 & 2 \cdot 0 + 4 \cdot 0 + 6 \cdot 1 \end{bmatrix}$$

Properties of Matrix Multiplication

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Let A, B, and C be matrices, and c is a scalar. Then:

1. $I_mA = AI_n = A$ — Matrix Multiplicative Identity $A = m \times n$
2. $(AB)C = A(BC)$ — Associative Property of matrix multiplication.
3. $A(B + C) = AB + AC$ — Distributive Property of matrix multiplication
4. $(A + B)C = AC + BC$ — Distributive Property of matrix multiplication
5. $c(AB) = (cA)B$ — Associative Property of scalar multiplication

Solving a Matrix Equation

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Solve for X in the matrix equation 3X + A = B, where

$$A = \begin{bmatrix} 2 & -8 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -10 & 1 \\ -9 & 17 \end{bmatrix}$$

$$3X + \begin{bmatrix} 2 & -8 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ -9 & 17 \end{bmatrix} \qquad 3X = \begin{bmatrix} -10 & 1 \\ -9 & 17 \end{bmatrix} - \begin{bmatrix} 2 & -8 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -12 & 9 \\ -9 & 13 \end{bmatrix}$$

$$X = \frac{1}{3}\begin{bmatrix} -12 & 9 \\ -9 & 13 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -3 & \frac{13}{3} \end{bmatrix}$$

System of Equations

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Write the following system of equations as a matrix equation.

$$\begin{cases} 2x + 3y + 4z = 3 \\ 3x + 9y - 5z = 0 \\ 5x - 3y + 5z = 9 \end{cases}$$

- Let's define this as a matrix equation AX = B, with A = coefficient matrix, X = solution matrix, and B = constant matrix.

$$AX = B \rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 3 & 9 & -5 \\ 5 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix}$$

System of Equations

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- Write the following system of equations as a matrix equation.

$$\begin{cases} 2x + 3y + 4z = 3 \\ 3x + 9y - 5z = 0 \\ 5x - 3y + 5z = 9 \end{cases} \qquad \begin{bmatrix} 2 & 3 & 4 \\ 3 & 9 & -5 \\ 5 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix}$$

- You can now solve the system by using Gauss-Jordan.

- Create augmented matrix [A|B]
- Gauss-Jordan gives matrix $[I_3|X]$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 3 \\ 3 & 9 & -5 & 0 \\ 5 & -3 & 5 & 9 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.33696 \\ 0 & 1 & 0 & -.28261 \\ 0 & 0 & 1 & .29348 \end{array}\right]$$

Objectives: Add and subtract matrices, multiply matrices by real numbers, and multiply two matrices.

- From the previous section you should be able to perform elementary row operations on matrices.
  - You can also perform matrix operations; add, subtract, and multiply simply by selecting the matrices and performing the operations as you would with any values.

$A + B$ — [2nd] [x^{-1}] (MATRIX) [▶] 1:[A] [ENTER] [+] [2nd] [x^{-1}] (MATRIX) [▶] 2:[B]

$A \times B$ — [2nd] [x^{-1}] (MATRIX) [▶] 1:[A] [ENTER] [X] [2nd] [x^{-1}] (MATRIX) [▶] 2:[B]