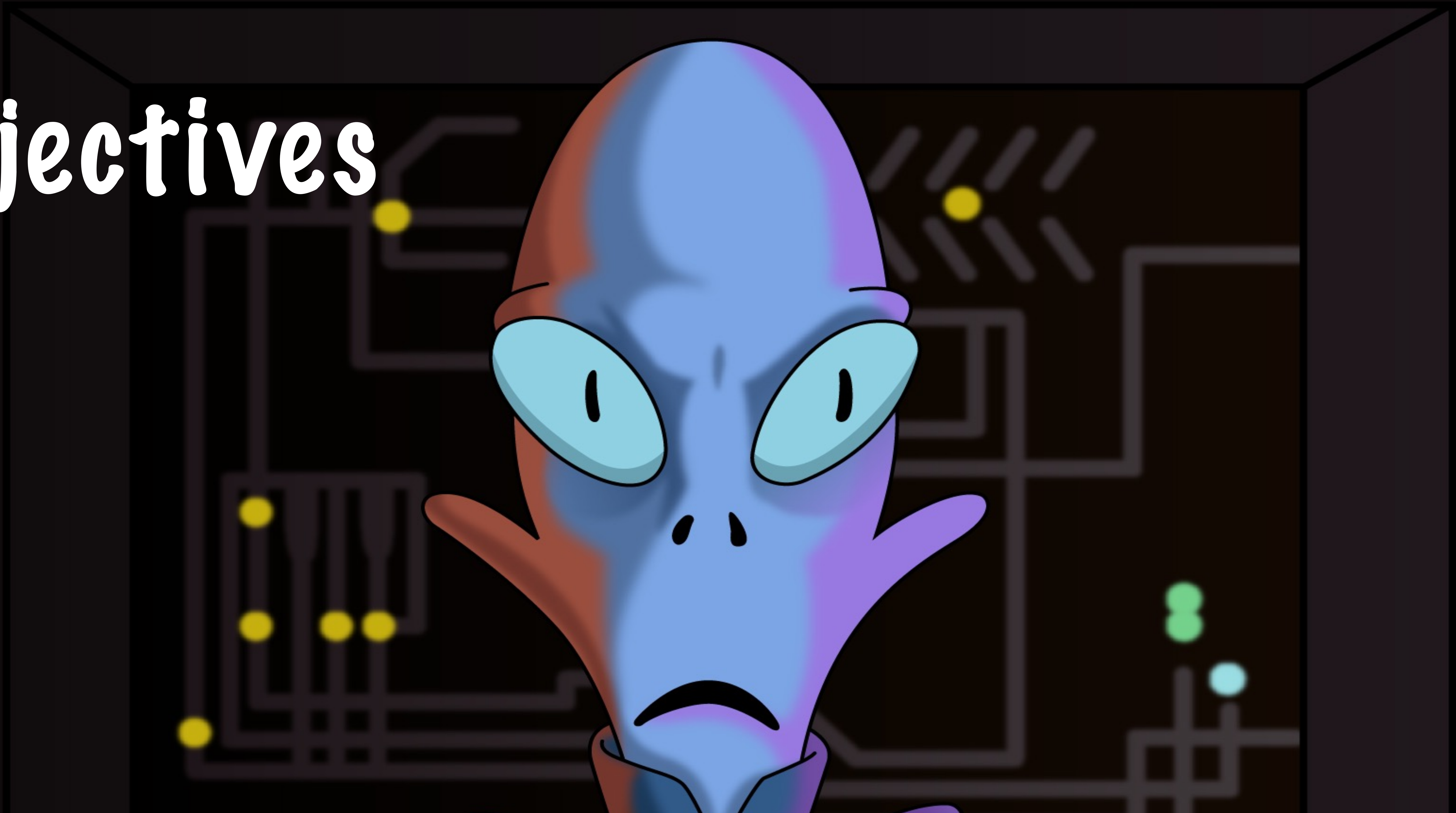
Chapter 8.3

Inverse of a square matrix

Homework

Video Notes
p608 1-71 every other odd
8.3 Worksheet

Objectives



Students will know how to find the inverse of a matrix and use inverse matrices to solve systems of linear equations.

Identity Matrix

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- The n x n square matrix with 1's down the main diagonal from upper left to lower right and 0's elsewhere is called the **multiplicative identity matrix of order n**. This matrix is designated by I_n.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Matrix

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- Let A be an n x n matrix. If there exists an n x n matrix A-1 (A inverse) such that:

  $AA^{-1} = I_n$ and $A^{-1}A = I_n$ Then A^{-1} is the **multiplicative inverse** of A.

- If a square matrix has a multiplicative inverse, it is said to be **invertible** or **nonsingular**.

- If a square matrix has no multiplicative inverse, it is called **singular**.

- A non-square matrix will not have an inverse.

Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

If $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, show that B is the multiplicative inverse of A

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 1\cdot -1 & 2\cdot -1 + 1\cdot 2 \\ 1\cdot 1 + 1\cdot -1 & 1\cdot -1 + 1\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + -1\cdot 1 & 1\cdot 1 + -1\cdot 1 \\ -1\cdot 2 + 2\cdot -1 & -1\cdot 1 + 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

AB = BA = I_2, thus B is the multiplicative inverse of A. There is no need to do both, if one works the reverse order will work.

Multiplicative Inverse

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- To find the inverse of a matrix we must find the matrix that when multiplied results in an identity matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} d & e \\ f & g \end{bmatrix}$ and B is the multiplicative inverse of A, then

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

thus $\begin{cases} ae + bg = 1 \\ ce + dg = 0 \end{cases}$ $\begin{cases} af + bh = 0 \\ cf + dh = 1 \end{cases}$

Multiplicative Inverse

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

$$\begin{cases} ae + bg = 1 \\ ce + dg = 0 \end{cases} \qquad \begin{cases} af + bh = 0 \\ cf + dh = 1 \end{cases}$$

these can be solved using matrices

$$\begin{bmatrix} e \\ g \end{bmatrix}$$ = reduced row-echelon of the augmented matrix $$\left[\begin{array}{cc|c} a & b & 1 \\ c & d & 0 \end{array}\right]$$

$$\begin{bmatrix} f \\ h \end{bmatrix}$$ = reduced row-echelon of the augmented matrix $$\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 1 \end{array}\right]$$

Multiplicative Inverse

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- Find the multiplicative inverse of $A = \begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix}$

$$\left[\begin{array}{cc|c} a & b & 1 \\ c & d & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} a & b & 0 \\ c & d & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 4 & 5 & 1 \\ 7 & 9 & 0 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{cc|c} 1 & \frac{5}{4} & \frac{1}{4} \\ 7 & 9 & 0 \end{array}\right] \xrightarrow{-7R_1 + R_2} \left[\begin{array}{cc|c} 1 & \frac{5}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{7}{4} \end{array}\right] \xrightarrow{-5R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & \frac{1}{4} & -\frac{7}{4} \end{array}\right] \xrightarrow{4R_2} \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & -7 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 4 & 5 & 0 \\ 7 & 9 & 1 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{cc|c} 1 & \frac{5}{4} & 0 \\ 7 & 9 & 1 \end{array}\right] \xrightarrow{-7R_1 + R_2} \left[\begin{array}{cc|c} 1 & \frac{5}{4} & 0 \\ 0 & \frac{1}{4} & 1 \end{array}\right] \xrightarrow{-5R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & \frac{1}{4} & 1 \end{array}\right] \xrightarrow{4R_2} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 4 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 9 & -5 \\ -7 & 4 \end{bmatrix}$$

- Notice we applied the same row operations in both cases. Perhaps we can do them at the same time.

Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- Find the multiplicative inverse of $A = \begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 9 & -5 \\ -7 & 4 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 4 & 5 & 1 & 0 \\ 7 & 9 & 0 & 1 \end{array}\right] \quad \frac{1}{4}R_1 \quad \left[\begin{array}{cc|cc} 1 & \frac{5}{4} & \frac{1}{4} & 0 \\ 7 & 9 & 0 & 1 \end{array}\right] \quad -7R_1 + R_2 \quad \left[\begin{array}{cc|cc} 1 & \frac{5}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{7}{4} & 1 \end{array}\right] \quad -5R_2 + R_1 \quad \left[\begin{array}{cc|cc} 1 & 0 & 9 & -5 \\ 0 & \frac{1}{4} & -\frac{7}{4} & 1 \end{array}\right]$$

$$4R_2 \quad \left[\begin{array}{cc|cc} 1 & 0 & 9 & -5 \\ 0 & 1 & -7 & 4 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 9 & -5 \\ -7 & 4 \end{bmatrix}$$

Shortcut for 2x2

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- A shortcut for finding the multiplicative inverse of a 2x2 matrix

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

switch signs

Determinant

switch positions

- Of course the preceding is true iff ad - bc (determinant) ≠ 0.

- If ad - bc = 0, A has no multiplicative inverse.

Multiplicative Inverse

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

Find the multiplicative inverse of $A = \begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{4 \cdot 9 - 5 \cdot 7}\begin{bmatrix} 9 & -5 \\ -7 & 4 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 9 & -5 \\ -7 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 9 & -5 \\ -7 & 4 \end{bmatrix}$$

switch signs

Determinant

switch positions

Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- Find the multiplicative inverse of $A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3 \cdot 1 - -2 \cdot -1}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Finding the Multiplicative Inverse

- To find the multiplicative inverse A^{-1} of an invertible matrix A

1. Form the augmented matrix $[A \mid I_n]$, where I_n is the multiplicative Identity matrix of the same order as the given matrix A.

2. Perform row operations on $[A \mid I_n]$ to obtain a matrix of the form $[I_n \mid B]$. This is equivalent to using Gauss-Jordan elimination to change A into the identity matrix.

3. Matrix $B = A^{-1}$

4. Verify the result by showing that $AA^{-1} = I_n = A^{-1}A$

Finding an Inverse Matrix

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \ \vdots \ I]$.
2. If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \ \vdots \ I]$. The result will be the matrix $[I \ \vdots \ A^{-1}]$. If this is not possible, A is not invertible.
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- Find the multiplicative inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

1. Form the augmented matrix $[A \mid I_n]$, where I_n is the multiplicative Identity matrix of the same order as the given matrix A.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

2. Perform row operations on $[A \mid I_n]$ to obtain a matrix of the form $[I_n \mid B]$. This is equivalent to using Gauss-Jordan elimination to change A into the identity matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_1 + R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \quad -1R_1 + R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 2 & 5 & 1 & 1 & 0 \end{array}\right] \quad -1R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 2 & 5 & 1 & 1 & 0 \end{array}\right] \quad -2R_2 + R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

$$-2R_3 + R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

Find the multiplicative inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

4. Verify the result by showing that $AA^{-1} = I_n = A^{-1}A$

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 3+0\cdot 3+2\cdot -1 & 1\cdot -2+0\cdot -2+2\cdot 1 & 1\cdot -4+0\cdot -5+2\cdot 2 \\ -1\cdot 3+2\cdot 3+3\cdot -1 & -1\cdot -2+2\cdot -2+3\cdot 1 & -1\cdot -4+2\cdot -5+3\cdot 2 \\ 1\cdot 3+-1\cdot 3+0\cdot -1 & 1\cdot -2+-1\cdot -2+0\cdot -1 & 1\cdot -4+-1\cdot -5+0\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3\cdot 1+-2\cdot -1+-4\cdot 1 & 3\cdot 0+-2\cdot 2+-4\cdot -1 & 3\cdot 2+-2\cdot 3+-4\cdot 0 \\ 3\cdot 1+-2\cdot -1+-5\cdot 1 & 3\cdot 0+-2\cdot 2+-5\cdot -1 & 3\cdot 2+-2\cdot 3+-5\cdot 0 \\ -1\cdot 1+1\cdot -1+2\cdot 1 & -1\cdot 0+1\cdot 2+2\cdot -1 & -1\cdot 2+1\cdot 3+2\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- Find the multiplicative inverse of $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & 14 \\ -1 & 4 & -3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 3 & 7 & 14 & 0 & 1 & 0 \\ -1 & 4 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$-3R_1 + R_2 \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ -1 & 4 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 + R_3 \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 6 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$-2R_2 + R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 7 & -2 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 6 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$-6R_2 + R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 7 & -2 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 0 & 8 & 19 & -6 & 1 \end{array}\right]$$

$$\frac{1}{8}R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 7 & -2 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 0 & 1 & \frac{19}{8} & \frac{-6}{8} & \frac{1}{8} \end{array}\right]$$

$$R_3 + R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 7 & -2 & 0 \\ 0 & 1 & 0 & \frac{-5}{8} & \frac{2}{8} & \frac{1}{8} \\ 0 & 0 & 1 & \frac{19}{8} & \frac{-6}{8} & \frac{1}{8} \end{array}\right]$$

$$-7R_3 + R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-77}{8} & \frac{26}{8} & \frac{-7}{8} \\ 0 & 1 & 0 & \frac{-5}{8} & \frac{2}{8} & \frac{1}{8} \\ 0 & 0 & 1 & \frac{19}{8} & \frac{-6}{8} & \frac{1}{8} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{-77}{8} & \frac{26}{8} & \frac{-7}{8} \\ \frac{-5}{8} & \frac{2}{8} & \frac{1}{8} \\ \frac{19}{8} & \frac{-6}{8} & \frac{1}{8} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -77 & 26 & -7 \\ -5 & 2 & 1 \\ 19 & -6 & 1 \end{bmatrix}$$

Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- Verify the result is the inverse matrix

$$A=\begin{bmatrix}1&2&5\\3&7&14\\-1&4&-3\end{bmatrix}\quad \frac{1}{8}\begin{bmatrix}-77&26&-7\\-5&2&1\\19&-6&1\end{bmatrix}$$

$$\begin{bmatrix}1&2&5\\3&7&14\\-1&4&-3\end{bmatrix}\left(\frac{1}{8}\begin{bmatrix}-77&26&-7\\-5&2&1\\19&-6&1\end{bmatrix}\right)=\frac{1}{8}\begin{bmatrix}1&2&5\\3&7&14\\-1&4&-3\end{bmatrix}\begin{bmatrix}-77&26&-7\\-5&2&1\\19&-6&1\end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix}1\cdot-77+2\cdot-5+5\cdot19 & 1\cdot26+2\cdot2+5\cdot-6 & 1\cdot-7+2\cdot1+5\cdot1\\3\cdot-77+7\cdot-5+14\cdot19 & 3\cdot26+7\cdot2+14\cdot-6 & 3\cdot-7+7\cdot1+14\cdot1\\-1\cdot-77+4\cdot-5+-3\cdot19 & -1\cdot26+4\cdot2+-3\cdot-6 & -1\cdot-7+4\cdot1+-3\cdot1\end{bmatrix}=\frac{1}{8}\begin{bmatrix}8&0&0\\0&8&0\\0&0&8\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

TI-84

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

Remember, you will be required to show all the steps for finding the inverse. You are certainly free to verify your result by using the TI-84. When asked to solve a system of 3 equations in 3 unknowns, use the calculator to find the inverse matrix. I told you the calculator would be your friend.

2nd x^{-1} ➢ EDIT ⋎ 1:[A] ENTER Enter Dimensions 3 ➢ 3

Enter the elements of matrix A $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ QUIT 2nd MODE

Select the matrix you just entered. 2nd x^{-1} ➢ NAME ⋎ 1:[A] ENTER

Find the inverse. [A] x^{-1} $A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

Find the multiplicative inverse of $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & 14 \\ -1 & 4 & -3 \end{bmatrix}$

2nd x⁻¹ ➢ EDIT ⋎ 1:[A] ENTER Enter Dimensions 3 ➢ 3

Enter the elements of matrix A $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & 14 \\ -1 & 4 & -3 \end{bmatrix}$ QUIT 2nd MODE

Select the matrix you just entered. 2nd x⁻¹ ➢ NAME ⋎ 1:[A] ENTER

Find the inverse. [A] x⁻¹ $A^{-1} = \begin{bmatrix} -9.625 & 3.25 & -.875 \\ -.625 & .25 & .125 \\ 2.375 & -.75 & .125 \end{bmatrix}$

That is ugly MATH 1:▸FRAC ENTER AHHHH, much better.

$$A^{-1} = \begin{bmatrix} -\frac{77}{8} & \frac{26}{8} & -\frac{7}{8} \\ -\frac{5}{8} & \frac{2}{8} & \frac{1}{8} \\ \frac{19}{8} & -\frac{6}{8} & \frac{1}{8} \end{bmatrix}$$

Solving Matrix Equations

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- The matrix equation $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is abbreviated $AX = B$.

$$\underbrace{\quad}_{A}\ \underbrace{\quad}_{X} = \underbrace{\quad}_{B}$$

A is the **coefficient matrix** of the system and X and B are matrices containing one column, called **column matrices**. The matrix B is called the **constant matrix**.

- If $AX = B$ has a unique solution, then $X = A^{-1}B$. To solve a linear system of equations, multiply A^{-1} and B to find X.

Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

Solve the system $\begin{cases} x+2z=6 \\ -x+2y+3z=-5 \\ x-y=6 \end{cases}$ by using A^{-1}, the inverse of the coefficient matrix.

Re-write the system as $\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}}_{B}$

$X = A^{-1}B$

$A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}}_{A^{-1}}\underbrace{\begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}}_{B} = \begin{bmatrix} 3\cdot 6+-2\cdot -5+-4\cdot 6 \\ 3\cdot 6+-2\cdot -5+-5\cdot 6 \\ -1\cdot 6+1\cdot -5+2\cdot 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

The solution is (4, −2, 1).

A Simple Example

Find the inverse of matrices and use inverse matrices to solve systems of linear equations.

- Solve the system by using A^{-1}, the inverse of the coefficient matrix.

$$\begin{cases} 2x - 5y = 8 \\ 3x - 8y = 1 \end{cases} \qquad \begin{bmatrix} 2 & -5 \\ 3 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

$$A \quad X = B$$

- You have 3 ways to find the inverse matrix.

- $$A^{-1} = \frac{1}{2 \cdot -8 - -5 \cdot 3}\begin{bmatrix} -8 & 5 \\ -3 & 2 \end{bmatrix} \text{ or } \left[\begin{array}{cc|cc} 2 & -5 & 1 & 0 \\ 3 & -8 & 0 & 1 \end{array}\right] \text{ or TI } A^{-1} = \begin{bmatrix} 8 & -5 \\ 3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \cdot 8 + -5 \cdot 1 \\ 3 \cdot 8 + -2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 59 \\ 22 \end{bmatrix}$$

$$X = A^{-1} \quad B$$

- The solution is (59, 22).