CHAPTER 8.4

DETERMINANT

HOMEWORK

READ SEC 8.4

COMPLETE READING NOTES

COMPLETE WORKSHEETS

P616 1-83 EVERY OTHER ODD

OBJECTIVES

STUDENTS WILL KNOW HOW TO FIND MINORS, COFACTORS, AND DETERMINANTS OF SQUARE MATRICES.

DETERMINANT

- We have solved systems of linear equations through the use of matrices. It turns out that the solution to the system $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$

  - has solution $\left(\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \right)$ $x = \dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$ and $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

- Interestingly, the denominator of the solution coordinates (**$a_1b_2 - a_2b_1$**) is the **determinant** of the coefficient matrix of the system. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

DETERMINANT

- Each square matrix has a singular value called the determinant of the matrix.
  - The determinant of a 2 x 2 matrix (order 2) is found by subtracting the products of the diagonals.
  - If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A, denoted by det[A] or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is defined as …

$$\det\left[A\right] = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

  - The value of the second order determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is ad – bc.

- Note the difference in the notations, [ ] denotes matrix, | | denotes determinant.

EXAMPLE

- Find $\begin{vmatrix} 10 & 9 \\ 6 & 5 \end{vmatrix}$

$$\begin{vmatrix} 10 & 9 \\ 6 & 5 \end{vmatrix} = 10 \cdot 5 - 9 \cdot 6 = -4$$

- Evaluate the determinant of the matrix $A = \begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix}$

$$\begin{vmatrix} 2 & 4 \\ -3 & -5 \end{vmatrix} = 2 \cdot -5 - 4 \cdot -3 = 2$$

THIRD ORDER DETERMINANT

- One method for finding the determinant of a 3 x 3 matrix (order 3) is found by subtracting the sums of the products of diagonals.

- If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, the determinant of A, denoted by $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is defined as …

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + chd) - (ceg + bdi + ahf)$$

$$= (aei + bfg + chd) - ceg - bdi - ahf$$

- You may have seen this method for finding the determinant when you were shown Cramer's Rule for solving a system of 3 linear equations in a previous course.

THIRD ORDER DETERMINANT

- You may have seen the method shown slightly differently:

- If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, the determinant of A, denoted by $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is defined as ...

- We augment the original matrix with the first two columns tacked on.

$$\left| \begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array} \right| = (aei + bfg + chd) - (ceg + ahf + bdi)$$

$$= (aei + bfg + chd) - ceg - ahf - bdi$$

THIRD ORDER DETERMINANT

- Another method for calculating the determinant of a square matrix (actually the method that provides the results in the prior slides) is to use **co-factors** and **minors**.

sign alternates

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

numerical factor

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Minor = $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$ Minor = $\begin{vmatrix} b & c \\ h & i \end{vmatrix}$ Minor = $\begin{vmatrix} b & c \\ e & f \end{vmatrix}$

co-factor = $(-1)^2(ei - fh)$ co-factor = $(-1)^3(bi - ch)$ co-factor = $(-1)^4(bf - ce)$

CO-FACTORS AND MINORS

- Let us define matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, where element a_{ij} is in **row i** and **column j.**

- If A is a square matrix of order n, the **minor** of a_{ij}, denoted by M_{ij}, is the determinant of the square matrix of order $n - 1$ formed by deleting the ***ith*** row and the ***jth*** column of A. The **co-factor** of a_{ij}, denoted by C_{ij}, is $C_{ij} = (-1)^{i+j}(M_{ij})$

  - Note: the factor $(-1)^{i+j}$ alternates the sign of the **co-factor** as seen in the previous slide.

$$\begin{bmatrix} a_{11} & & a_{13} \\ a_{21} & & a_{23} \\ & a_{32} & \end{bmatrix} \quad M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = (-1)^{i+j}(M_{ij}) = -1^{3+2}\left(a_{11} \cdot a_{23} - a_{13} \cdot a_{21}\right)$$

$$= -\left(a_{11} \cdot a_{23} - a_{13} \cdot a_{21}\right)$$

CO-FACTORS AND MINORS

- You will note that the signs of the co-factors alternates. The alternating signs are created by the factor $(-1)^{i+j}$. When the sum of row and column is odd the co-factor will be negative.

  - This pattern can be seen as

$$\begin{bmatrix} +_{11} & -_{12} \\ -_{21} & +_{22} \end{bmatrix} \qquad \begin{bmatrix} +_{11} & -_{12} & +_{13} \\ -_{21} & +_{22} & -_{23} \\ +_{31} & -_{32} & +_{33} \end{bmatrix} \qquad \begin{bmatrix} +_{11} & -_{12} & +_{13} & -_{14} \\ -_{21} & +_{22} & -_{23} & +_{24} \\ +_{31} & -_{32} & +_{33} & -_{34} \\ -_{41} & +_{42} & -_{42} & +_{44} \end{bmatrix}$$

  - We will not be examining larger matrices, but the pattern continues.

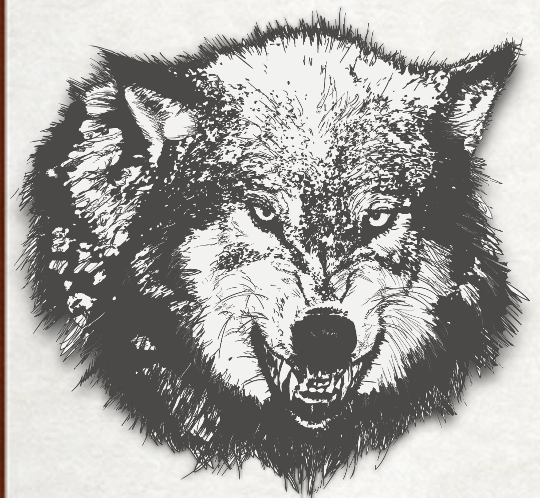
CO-FACTORS AND MINORS

Find the minor M_{23} and co-factor C_{23} from $A = \begin{bmatrix} 2 & -1 & 5 \\ 8 & -3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

- $M_{23} = \begin{vmatrix} 2 & -1 & \\ & & 2 \\ 1 & -2 & \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$
- $C_{23} = (-1)^{2+3}(2 \bullet -2 - -1 \bullet 1) = (-1)(-3) = 3$

Find the minor M_{22} and co-factor C_{22} from $A = \begin{bmatrix} 2 & -1 & 5 \\ 8 & -3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

- $M_{22} = \begin{vmatrix} 2 & & 5 \\ & -3 & \\ 1 & & 3 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$
- $C_{22} = (-1)^{2+2}(2 \bullet 3 - 5 \bullet 1) = (1)(1) = 1$



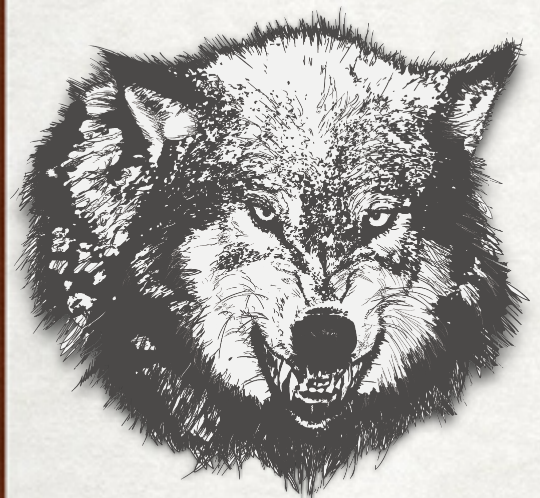
CO-FACTORS AND MINORS

Find the minor M_{13} and co-factor C_{13} from $A = \begin{bmatrix} 2 & -1 & 5 \\ 8 & -3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

- $M_{13} = \begin{vmatrix} & & 5 \\ 8 & -3 & \\ 1 & -2 & \end{vmatrix} = \begin{vmatrix} 8 & -3 \\ 1 & -2 \end{vmatrix}$
- $C_{13} = (-1)^{1+3}(8 \bullet -2 - -3 \bullet 1) = (1)(-13) = -13$

Find the minor M_{33} and co-factor C_{33} from $A = \begin{bmatrix} 2 & -1 & 5 \\ 8 & -3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

- $M_{33} = \begin{vmatrix} 2 & -1 & \\ 8 & -3 & \\ & & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 8 & -3 \end{vmatrix}$
- $C_{33} = (-1)^{3+3}(2 \bullet -3 - -1 \bullet 8) = (1)(2) = 2$



FINDING THE DETERMINANT OF A 3 X 3 MATRIX

- 1. Each of the three terms in the calculation of the determinant contains two factors - a numerical factor and a co-factor from a second order (2 x 2) determinant (**minor**).
- 2. The numerical factor in each term is an element from a row or column (i.e. the first column or the first row) of the third-order determinant.
- 3. The minus sign precedes the second term (e.g. first row, second column or first column, second row $(-1)^{1+2}$).
- 4. The second-order determinant that appears in each term is obtained by crossing out the row and the column containing the numerical factor.
- The **minor** of an element is the determinant of the minor matrix that remains after deleting the row and column of that element from a higher order matrix. This method is known as "**expansion by minors**" or "**expansion by co-factors**". There are a lot of possible minors.
- The method can be used by selecting the elements from any row or any column. For the purpose of ease and accuracy, we often select the first column or first row.

EXAMPLE

- Find the determinant of the matrix $A = \begin{bmatrix} 2 & -1 & 5 \\ 8 & -3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

- First Column

$$\begin{vmatrix} 2 & -1 & 5 \\ 8 & -3 & 2 \\ 1 & -2 & 3 \end{vmatrix} = 2\bullet(-1)^{1+1}\begin{vmatrix} -3 & 2 \\ -2 & 3 \end{vmatrix} + 8\bullet(-1)^{2+1}\begin{vmatrix} -1 & 5 \\ -2 & 3 \end{vmatrix} + 1\bullet(-1)^{3+1}\begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix}$$

$$= 2(-3\bullet 3 - 2\bullet -2) \ -8(-1\bullet 3 - 5\bullet -2) \ +1(-1\bullet 2 - 5\bullet -3)$$

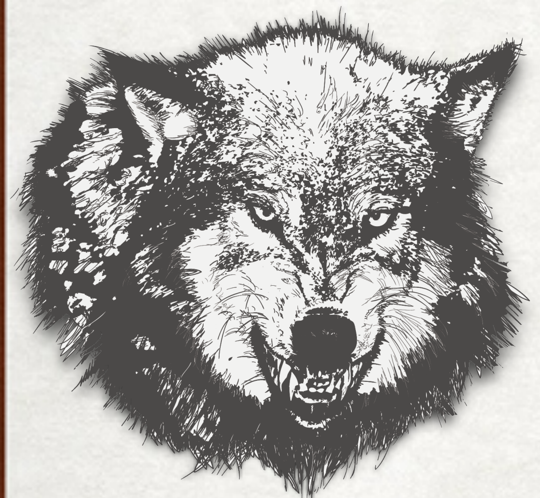
$$= 2(-5) - 8(7) + 1(13) \quad = -53$$

- First Row

$$\begin{vmatrix} 2 & -1 & 5 \\ 8 & -3 & 2 \\ 1 & -2 & 3 \end{vmatrix} = 2\bullet(-1)^{1+1}\begin{vmatrix} -3 & 2 \\ -2 & 3 \end{vmatrix} + (-1)\bullet(-1)^{1+2}\begin{vmatrix} 8 & 2 \\ 1 & 3 \end{vmatrix} + 5\bullet(-1)^{1+3}\begin{vmatrix} 8 & -3 \\ 1 & -2 \end{vmatrix}$$

$$= 2(-3\bullet 3 - 2\bullet -2) \ - (-1)(8\bullet 3 - 2\bullet 1) \ + 5(8\bullet -2 - -3\bullet 1)$$

$$= 2(-5) + 1(22) + 5(-13) = -53$$



EXAMPLE

Evaluate the determinant of $A = \begin{bmatrix} 2 & 1 & 7 \\ -5 & 6 & 0 \\ -4 & 3 & 1 \end{bmatrix}$

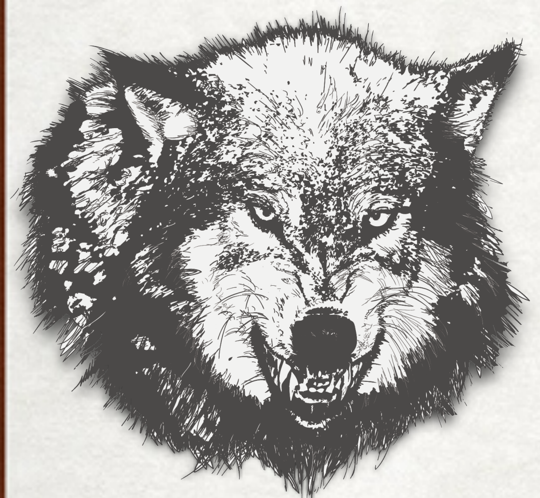
$$\begin{vmatrix} 2 & 1 & 7 \\ -5 & 6 & 0 \\ -4 & 3 & 1 \end{vmatrix} = 2\begin{vmatrix} 6 & 0 \\ 3 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 1 & 7 \\ 3 & 1 \end{vmatrix} + (-4)\begin{vmatrix} 1 & 7 \\ 6 & 0 \end{vmatrix} = 2(6 \bullet 1 - 0 \bullet 3) + 5(1 \bullet 1 - 7 \bullet 3) + (-4)(1 \bullet 0 - 7 \bullet 6)$$

$$= 2(6) + 5(-20) - 4(-42) \quad = 80$$

$$\begin{vmatrix} 2 & 1 & 7 \\ -5 & 6 & 0 \\ -4 & 3 & 1 \end{vmatrix} = 7\begin{vmatrix} -5 & 6 \\ -4 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} + (1)\begin{vmatrix} 2 & 1 \\ -5 & 6 \end{vmatrix}$$

$$= 7(-5 \bullet 3 - 6 \bullet (-4)) + 0(2 \bullet 3 - 1 \bullet (-4)) + 1(2 \bullet 6 - 1 \bullet (-5))$$

$$= 7(9) + 0(10) + 1(17) \quad = 80$$



FINDING THE DETERMINANT OF AN N X N MATRIX

- The determinant of an n x n matrix is the sum of the elements of any row or column multiplied by their respective co-factors.

$$\det\left[A\right] = \left|A\right| = \sum_{i=1}^{n} a_{ij} \bullet c_{ij}$$

- for **any** column j

$$\det\left[A\right] = \left|A\right| = \sum_{j=1}^{n} a_{ij} \bullet c_{ij}$$

for **any** row i

FINDING THE DETERMINANT OF AN N X N MATRIX

- Evaluate the determinant of $A = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 2 & 0 & -5 & 4 \\ -1 & 1 & 9 & -3 \\ -4 & 0 & -5 & 2 \end{bmatrix}$

  - Notice column 2. Is that not a nice column? Let us use that column, shall we?

  - The 3 0 elements will eliminate 3 minors. Only 1 remains, $a_{32} = 1$.

$$\det A = \begin{vmatrix} 1 & 0 & 3 & 5 \\ 2 & 0 & -5 & 4 \\ -1 & 1 & 9 & -3 \\ -4 & 0 & -5 & 2 \end{vmatrix} = (0)(-1)^{2+1}(M_{21}) + (0)(-1)^{2+2}(M_{22}) + (1)(-1)^{2+3}(M_{23}) + (0)(-1)^{2+4}(M_{24})$$

$$= (0) + (0) + (1)(-1)^{2+3}(M_{23}) + (0) \quad = (-1)^{2+3}(M_{23})$$

FINDING THE DETERMINANT OF AN N X N MATRIX

- Evaluate the determinant of $A = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 2 & 0 & -5 & 4 \\ -1 & 1 & 9 & -3 \\ -4 & 0 & -5 & 2 \end{bmatrix} = (-1)^{3+2}(M_{32})$

$$\begin{vmatrix} 1 & 0 & 3 & 5 \\ 2 & 0 & -5 & 4 \\ -1 & 1 & 9 & -3 \\ -4 & 0 & -5 & 2 \end{vmatrix} = 1 \cdot (-1)^{3+2} \begin{vmatrix} 1 & 3 & 5 \\ 2 & -5 & 4 \\ -4 & -5 & 2 \end{vmatrix} = (-1)\left(1\begin{vmatrix} -5 & 4 \\ -5 & 2 \end{vmatrix} - (3)\begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix} + (5)\begin{vmatrix} 2 & -5 \\ -4 & -5 \end{vmatrix}\right)$$

$$= -1\left(1(-5 \cdot 2 - 4 \cdot -5) - 3(2 \cdot 2 - 4 \cdot -4) + 5(2 \cdot -5 - -5 \cdot -4)\right)$$

$$= -1\left(1(10) - 3(20) + 5(-30)\right) \quad = 200$$

- You should be able to see how complicated this can get if we did not a have a row of mostly 0s. We would have 3 more 3rd order determinate, each with 3 second order determinants for a total of 12 minors and 12 co-factors.

TI-84

- Evaluate the determinant of $A = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 2 & 0 & -5 & 4 \\ -1 & 1 & 9 & -3 \\ -4 & 0 & -5 & 2 \end{bmatrix}$

- Of course, you can find the determinant of an n x n matrix on the TI-84.

MATRIX

2nd x^{-1} ➢ EDIT ⋎ 1:[A] ENTER Enter Dimensions 4 ➢ 4 ENTER

MATRIX[A] 4 x 4

$$\begin{matrix} 1 & 0 & 3 & 5 \\ 2 & 0 & -5 & 4 \\ -1 & 1 & 9 & -3 \\ -4 & 0 & -5 & 2 \end{matrix}$$

QUIT

2nd MODE

MATRIX MATRIX

2nd x^{-1} ➢ MATH ⋎ 1:det(ENTER 2nd x^{-1} ⋎ 1:[A]) ENTER

200