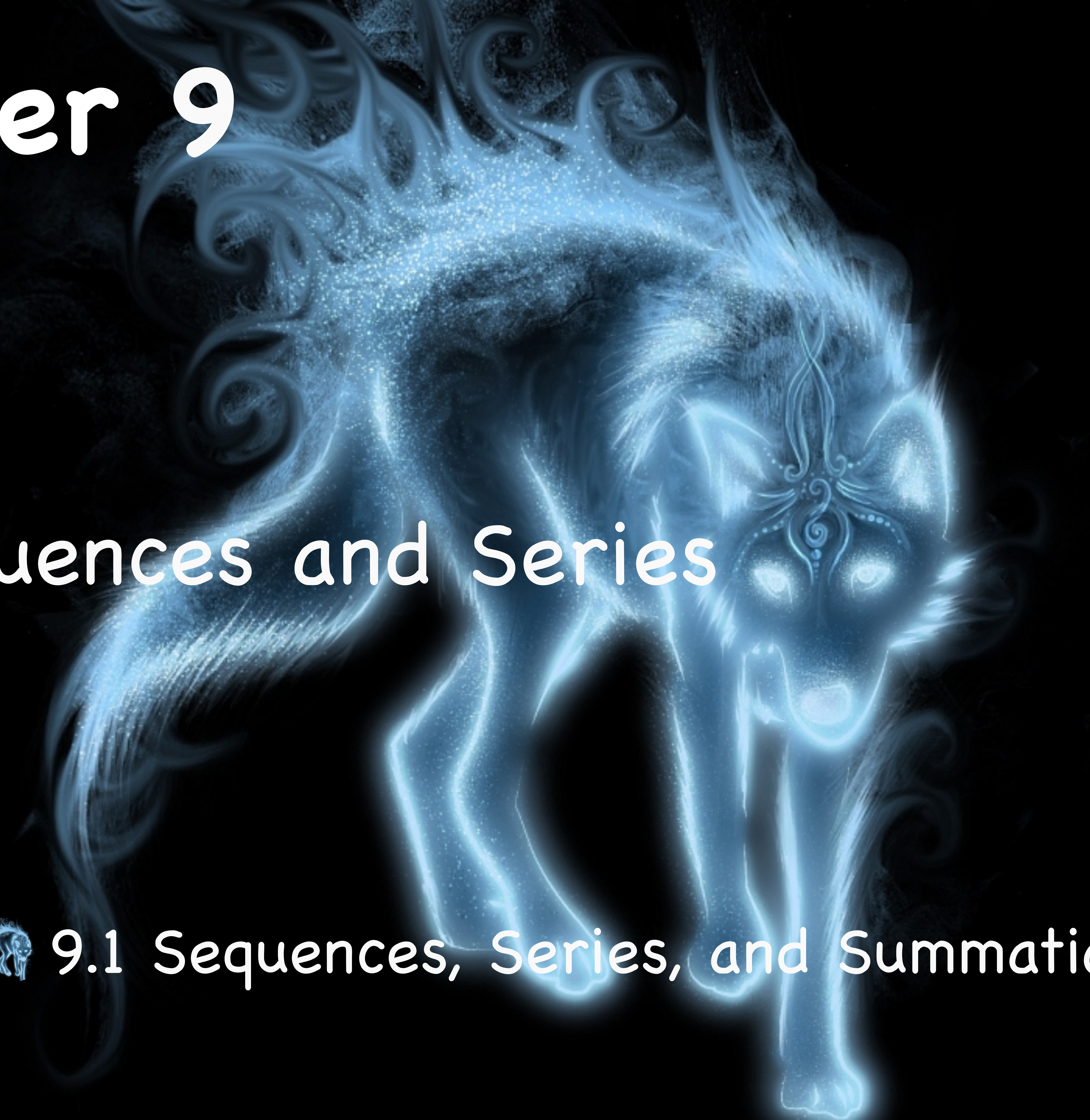
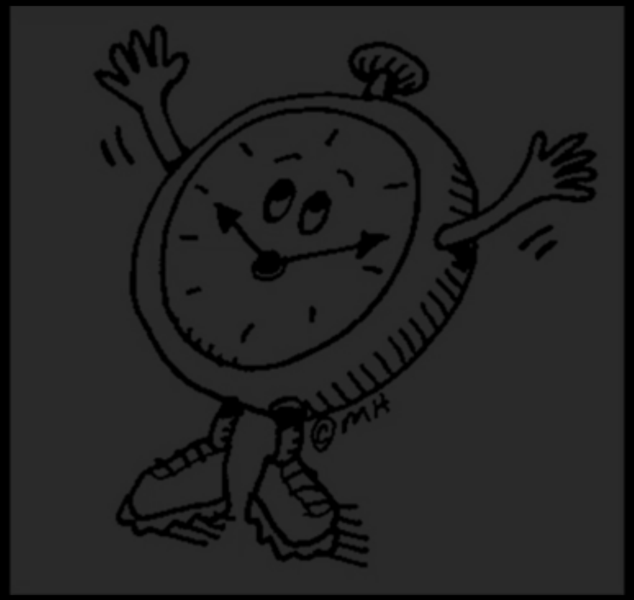
Chapter 9



Sequences and Series

9.1 Sequences, Series, and Summation Notation

Chapter 9



Homework

- Read 9.1 Complete Notes
- Do p649 1, 23, 27, 33, 35, 37, 41, 51, 55, 59, 69, 71, 73, 79, 85, 93, 99, 103

Chapter 9

Objectives

- Find particular terms of a sequence from the general term.
- Use recursion formulas.
- Use factorial notation.
- Use summation notation.
- Find the sum of an infinite series.

Definition of a Sequence

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- An **infinite sequence** $\{a_n\}$ is a function whose domain is the set of positive integers. The function values, or **terms**, of the sequence are represented by

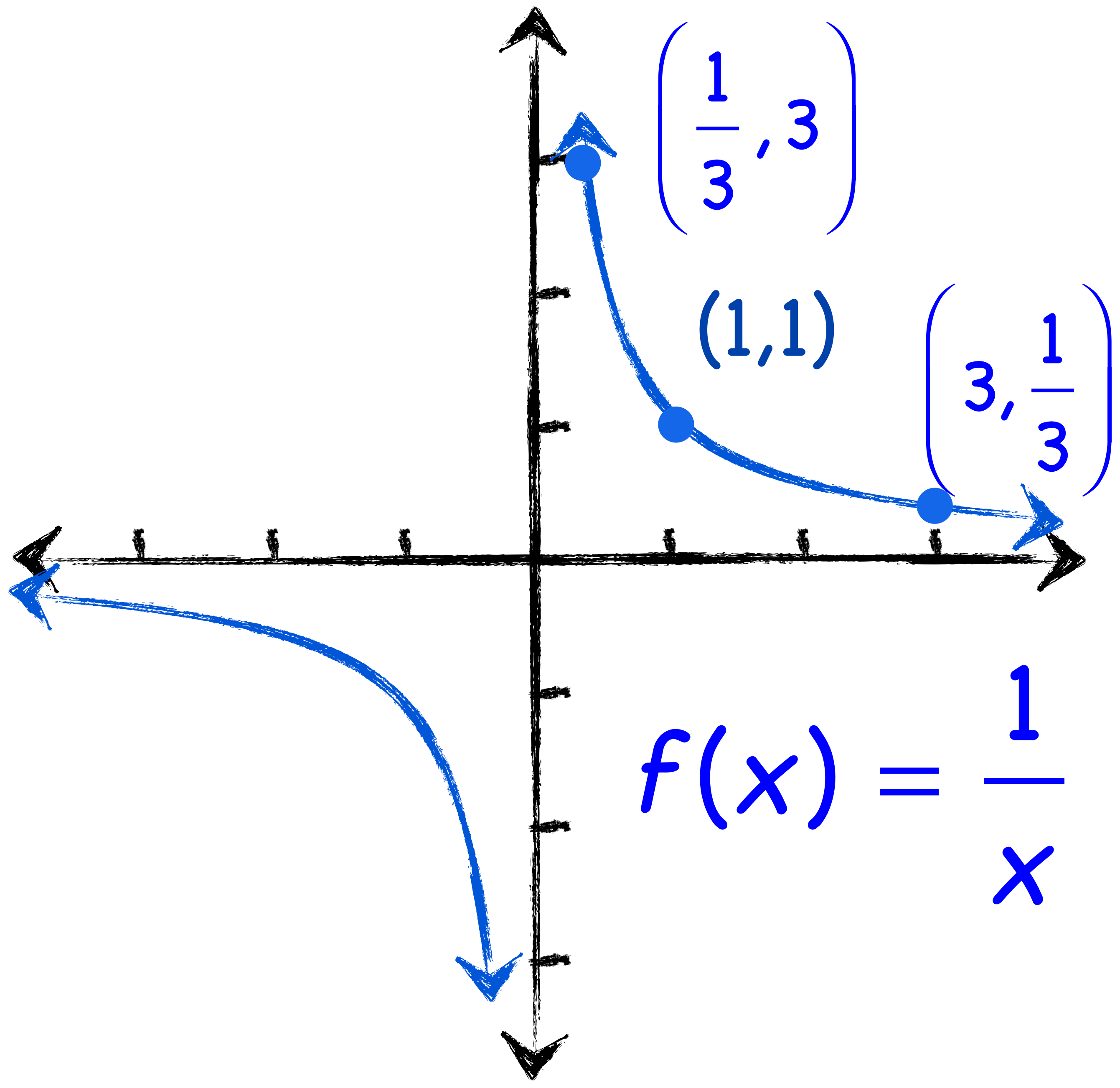
$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

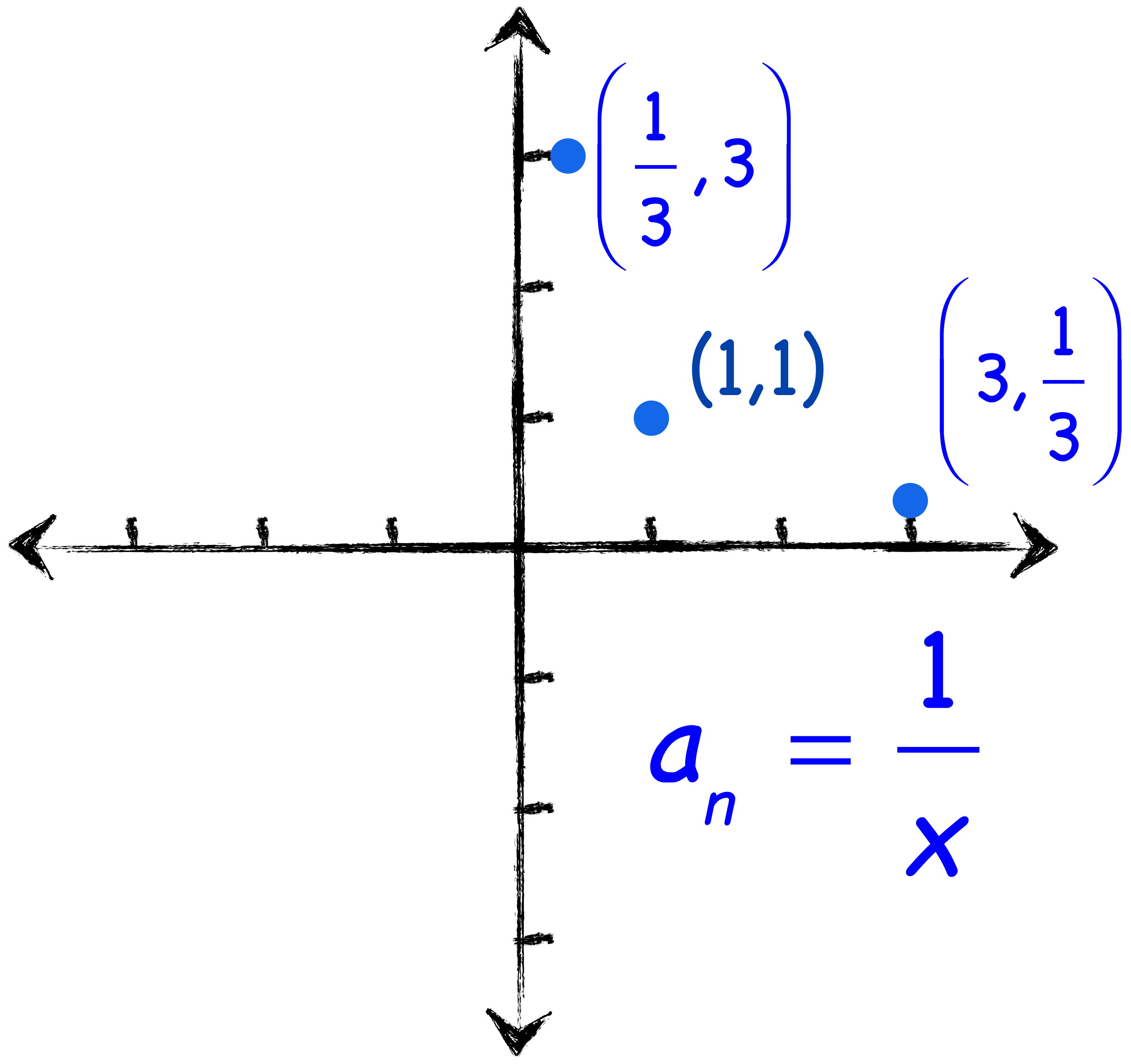
- Sequences whose domains consist only of the first **n** positive integers are called **finite sequences**.

Definition of a Sequence

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- The graph of a sequence is a **set of discrete points**. The graph of the sequence $a_x = \frac{1}{x}$ is similar to $f(x) = \frac{1}{x}$ except a sequence only contains the points whose x-coordinates are positive integers. The domain of the sequence is the set of natural numbers.





Writing Terms of a Sequence from a General Term

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- Write the first four terms of the sequence whose **n**th term, or general term, is given by the **explicit** formula: $a_n = 2n + 5$

  - To find the first four terms, replace **n** in the formula with **1, 2, 3,** and **4**.

$$a_1 = 2(1) + 5 = 7$$

$$a_2 = 2(2) + 5 = 9$$

$$a_3 = 2(3) + 5 = 11$$

$$a_4 = 2(4) + 5 = 13$$

The first four terms of the sequence are 7, 9, 11, 13.

- Write the first four terms of the sequence whose **n**th term, or general term, is given by the explicit formula: $b_n = 3^{n-1}$

  - To find the first four terms, replace **n** in the formula with **1, 2, 3,** and **4**.

$$b_1 = 3^{1-1} = 3^0 = 1$$

$$b_2 = 3^{2-1} = 3^1 = 3$$

$$b_3 = 3^{3-1} = 3^2 = 9$$

$$b_4 = 3^{4-1} = 3^3 = 27$$

The first four terms of the sequence are 1, 3, 9, 27.

Writing Terms of a Sequence from a General Term

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- Write the first four terms of the sequence whose **n**th term, or general term, is given by the explicit formula:

$$c_n = \frac{(-1)^n}{n^2+1}$$

  - To find the first four terms, replace **n** in the formula with **1, 2, 3,** and **4**.

$$c_1 = \frac{(-1)^1}{1^2+1} = -\frac{1}{2}$$

$$c_2 = \frac{(-1)^2}{2^2+1} = \frac{1}{5}$$

$$c_3 = \frac{(-1)^3}{3^2+1} = -\frac{1}{10}$$

$$c_4 = \frac{(-1)^4}{4^2+1} = \frac{1}{17}$$

The first four terms of the sequence are...

$$-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \frac{1}{17}$$

Note the alternating signs.

Recursion Formulas

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- A **recursion (recursive) formula** defines the **n**th term of a sequence as a **function of a previous term**.

  - Write the first four terms of the sequence in which $a_1 = 3$ and $a_n = 2a_{n-1} + 5$

$$a_1 = 3$$

$$a_2 = 2(a_1) + 5 = 2(3) + 5 = 11$$

$$a_3 = 2(a_2) + 5 = 2(11) + 5 = 27$$

$$a_4 = 2(a_3) + 5 = 2(27) + 5 = 59$$

The first four terms of the sequence are 3, 11, 27, 59.

Recursive Form

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- Everyone knows this one … $1, 1, 2, 3, 5, 8, 13, \ldots$
  - Write the sequence using recursive notation.

$a_1 = 1$ $\quad a_2 = 1 + nada = a_1 + nothing$ $\quad a_3 = 1 + 1 = a_2 + a_1$

$a_4 = 2 + 1 = a_3 + a_2$ $\quad a_5 = 3 + 2 = a_4 + a_3$

- The famous (infamous?) Fibonacci Sequence $a_k = a_{k-1} + a_{k-2}$

Factorial Notation

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- If n is a positive integer, the notation **n!** (read "**n** factorial") is the **product** of all positive integers from **n** down through 1.

$$n! = n(n-1)(n-2)\bullet \ldots \bullet 2 \bullet 1$$

$0! = 1$ 0! (zero factorial), by definition is 1.

$$1! = 1$$

$$2! = 2 \bullet 1 = 2$$

$$3! = 3 \bullet 2 \bullet 1 = 3$$

$$\vdots$$

$$8! = 8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 = 40320$$

$$(2n)! = 2n(2n-1)(2n-2)\bullet \ldots \bullet 2 \bullet 1$$

Finding Terms of a Sequence Involving Factorials

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- Write the first four terms of the sequence whose nth term is $a_n = \dfrac{20}{(n+1)!}$

$$a_1 = \frac{20}{(1+1)!} = \frac{20}{2!} = \frac{20}{2} = 10$$

$$a_2 = \frac{20}{(2+1)!} = \frac{20}{3!} = \frac{20}{6} = \frac{10}{3}$$

$$a_3 = \frac{20}{(3+1)!} = \frac{20}{4!} = \frac{20}{24} = \frac{5}{6}$$

$$a_4 = \frac{20}{(4+1)!} = \frac{20}{5!} = \frac{20}{120} = \frac{1}{6}$$

The first four terms of the sequence are $10, \dfrac{10}{3}, \dfrac{5}{6}, \dfrac{1}{6}$

Example: Finding Terms of a Sequence Involving Factorials

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- Write the first four terms of the sequence whose nth term is $a_n = \dfrac{n^2}{n!}$

$$a_1 = \frac{1^2}{1!} = \frac{1}{1} = 1$$

$$a_2 = \frac{2^2}{2!} = \frac{4}{2 \cdot 1} = 2$$

$$a_3 = \frac{3^2}{3!} = \frac{9}{3 \cdot 2 \cdot 1} = \frac{9}{6} = \frac{3}{2}$$

$$a_4 = \frac{4^2}{4!} = \frac{16}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{16}{24} = \frac{2}{3}$$

The first four terms of the sequence are $1, 2, \frac{3}{2}, \frac{2}{3}$

Finding Terms of a Sequence Involving Factorials

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- Evaluate the factorial $\dfrac{n!}{(n+1)!}$

$$\frac{n!}{(n+1)!} = \frac{n!}{(n+1)(n)(n-1)(n-2)\ldots} = \frac{n!}{(n+1)n!} = \frac{1}{n+1}$$

Finding Terms of a Sequence Involving Factorials

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

Write the most likely nth term of the following sequences using explicit notation.

$0, 3, 8, 15, \ldots$ $\qquad 0 = 1^2 - 1,\ 3 = 2^2 - 1,\ 8 = 3^2 - 1,\ 15 = 4^2 - 1$

$$a_n = n^2 - 1$$

$\frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$ $\qquad \frac{1}{2} = \frac{1}{2!},\ \frac{1}{6} = \frac{1}{3!},\ \frac{1}{24} = \frac{1}{4!},\ \frac{1}{120} = \frac{1}{5!}$

$$a_n = \frac{1}{(n+1)!}$$

Summation Notation

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- The sum of the first n terms of a sequence is a series.

- The sum of the first n terms of a sequence is represented by **summation notation**

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 \ldots + a_{n-2} + a_{n-1} + a_n$$

where

i is the **index of summation,**

n is the **upper limit of summation**,

1 is the **lower limit of summation**.

Example: Using Summation Notation

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

Expand and evaluate the sum: $\sum_{i=1}^{6} 2i^2$

$$\sum_{i=1}^{6} 2i^2 = 2(1^2) + 2(2^2) + 2(3^2) + 2(4^2) + 2(5^2) + 2(6^2)$$

$$= 2(1) + 2(4) + 2(9) + 2(16) + 2(25) + 2(36)$$

$$= 2 + 8 + 18 + 32 + 50 + 72$$

$$= 182$$

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Example: Using Summation Notation

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

Expand and evaluate the sum: $\sum_{k=1}^{5}(2^k - 3)$

$$\sum_{k=1}^{5}(2^k - 3) = (2^1 - 3) + (2^2 - 3) + (2^3 - 3) + (2^4 - 3) + (2^5 - 3)$$

$$= (2 - 3) + (4 - 3) + (8 - 3) + (16 - 3) + (32 - 3)$$

$$= -1 + 1 + 5 + 13 + 29$$

$$= 47$$

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Using Summation Notation

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- Expand and evaluate the sum: $\sum_{i=1}^{5}(-1)^{i-1}i!$

$$\sum_{i=1}^{5}(-1)^{i-1}i! = (-1)^{1-1}1!+(-1)^{2-1}2!+(-1)^{3-1}3!+(-1)^{4-1}4!+(-1)^{5-1}5!$$

$$= 1(1)+(-1)(2)+(1)(6)+(-1)(24)+(1)(120)$$

$$= 1+-2+6+-24+120$$

$$= 101$$

Using Summation Notation

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- Expand and evaluate the sum: $\sum_{i=1}^{5} 4$

$$\sum_{i=1}^{5} 4 = 4 + 4 + 4 + 4 + 4 = 4(5) = 20$$

Properties of Sums

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

1. $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + \ldots + ca_{n-1} + ca_n$$

$$= c(a_1 + a_2 + \ldots + a_{n-1} + a_n)$$

$$= c\sum_{i=1}^{n} a_i$$

★ Revisit slide 17

Properties of Sums

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

2 $$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n}(a_i + b_i) = (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_{n-1} + b_{n-1}) + (a_n + b_n)$$

$$= (a_1 + a_2 + \ldots + a_{n-1} + a_n) + (b_1 + b_2 + \ldots + b_{n-1} + b_n)$$

$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

3 $$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

★ Revisit slide 18

Examples using Properties of Sums

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

Expand and evaluate

$$\sum_{k=1}^{5}(k^2-3) = \sum_{k=1}^{5}k^2 - \sum_{k=1}^{5}3 = (1^2+2^2+3^2+4^2+5^2)-(3\bullet 5) = 55-15 = 40$$

$$\sum_{i=1}^{5}(i+1) = \sum_{i=1}^{5}i + \sum_{i=1}^{5}1 = (1+2+3+4+5)+(1\bullet 5) = 15+5 = 20$$

$$\sum_{i=1}^{5}4(i+1) = 4\left(\sum_{i=1}^{5}(i+1)\right) = 4\left(\sum_{i=1}^{5}i + \sum_{i=1}^{5}1\right) = 4(15+5) = 80$$

Series

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

- If a_1, a_2, a_3, … is an infinite sequence, then $\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_{n-1} + a_n$ is the **nth partial sum** of the sequence.

- The nth partial sums of the sequence themselves form an infinite sequence. This type of infinite sequence is called an **infinite series**, and is denoted by

$$\sum_{i=1}^{\infty} a_i$$

Series and Partial Sum

Objectives: Use sequence, factorial, and summation notation to write the terms and sum of a sequence, and how to find the sum of an infinite series

Find the sum of the series $\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^k$

$$\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^k=\left(\frac{1}{10}\right)^1+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+\left(\frac{1}{10}\right)^4+\ldots$$

$$=\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\ldots$$

$$=.1+.01+.001+.0001+\ldots$$

$$=.1111\ldots=\frac{1}{9}$$