Chapter 9

Sequences, and Series

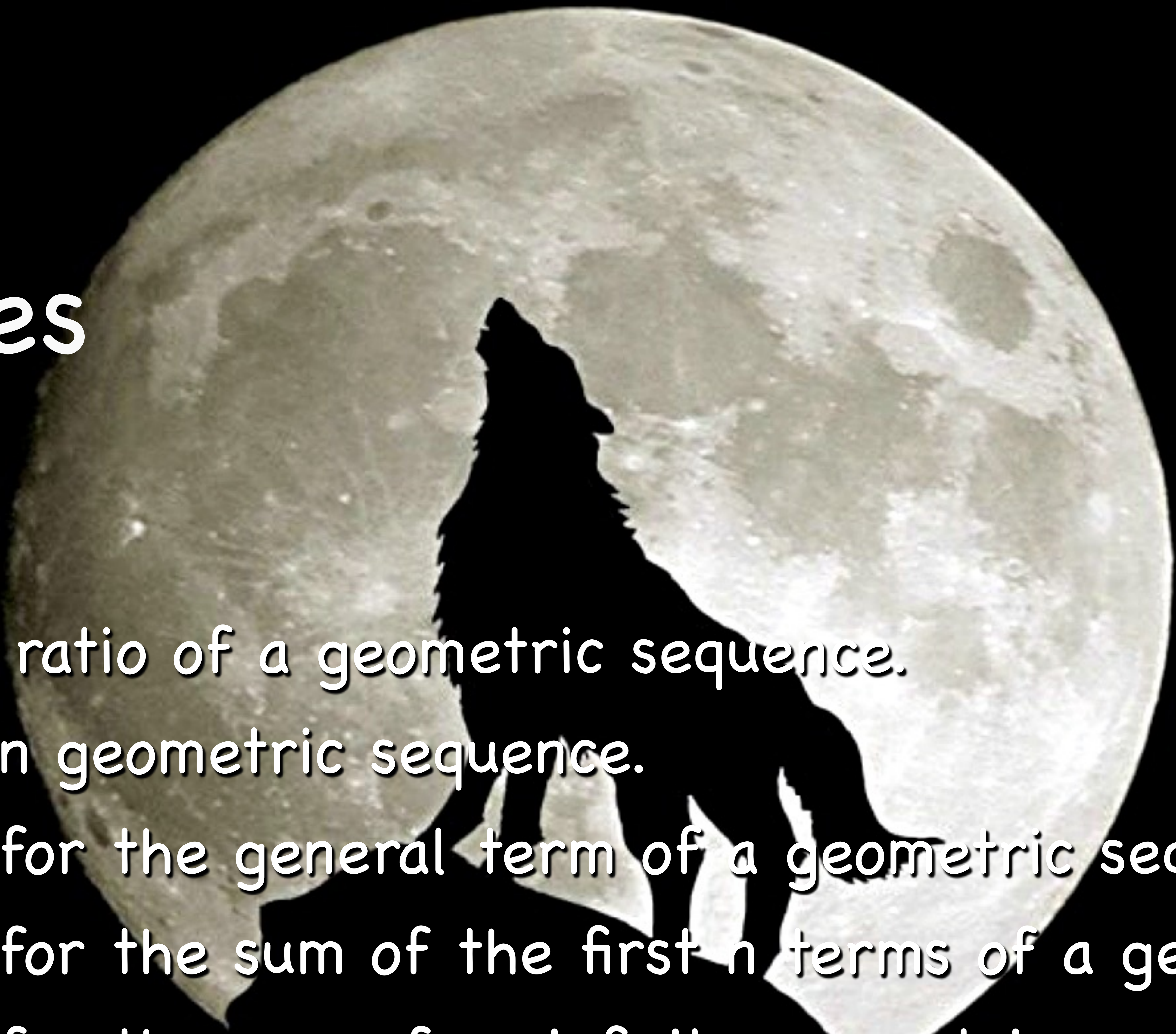
9.3 Geometric Sequences

Chapter 9

- Homework

- Read Sec 9.3
- Complete Reading Notes
- Do p669 Ex 1, 5, 9, 15, 19, 23, 29, 35, 41, 55, 67, 75, 81
- Do 9-3 Worksheet

Chapter 9-3



Objectives

- Find the common ratio of a geometric sequence.
- Write terms of an geometric sequence.
- Use the formula for the general term of a geometric sequence
- Use the formula for the sum of the first n terms of a geometric sequence.
- Use the formula for the sum of an infinite geometric sequence.

Definition of a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- A **geometric sequence** is a sequence in which each term after the first is obtained by multiplying the preceding term by a fixed nonzero constant. The amount by which we multiply each time is called the **common ratio** of the sequence.

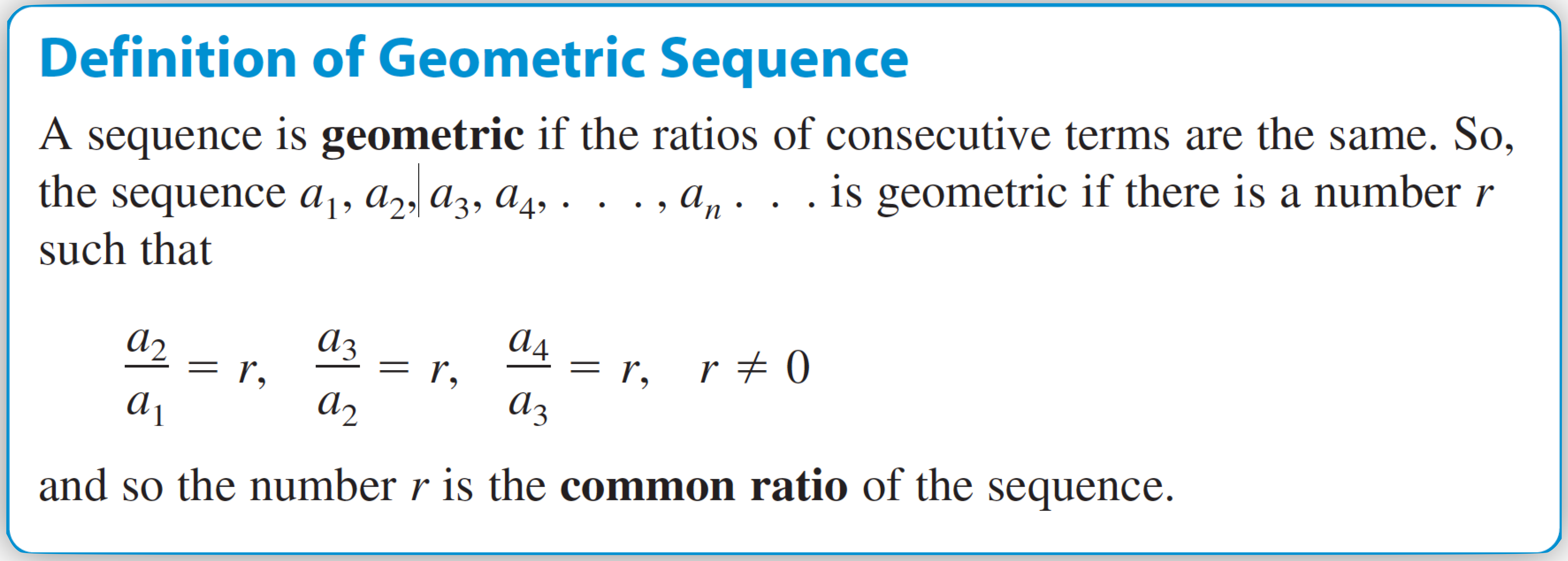
  - In short, a geometric sequence satisfies the recursive condition: $a_n = a_{n-1}r$

    - Where r is the **common ratio** and $\dfrac{a_n}{a_{n-1}} = r$

Definition of a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- From your book



Definition of Geometric Sequence

A sequence is **geometric** if the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \ldots, a_n \ldots$ is geometric if there is a number r such that

$$\frac{a_2}{a_1} = r, \quad \frac{a_3}{a_2} = r, \quad \frac{a_4}{a_3} = r, \quad r \neq 0$$

and so the number r is the **common ratio** of the sequence.

Example: Writing the General Term for a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- Write the general (nth) term for the geometric sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

First term = 1

Common ratio = $r = \frac{\frac{1}{2}}{1} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$

$$a_n = a_{n-1}\left(\frac{1}{2}\right)$$

- This is the recursive rule for the nth term

Example: Writing the General Term for a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- Write the recursive rule for the nth term for the geometric sequence

$$3, 6, 12, 24, 48, \ldots$$

- First term = a_1 = 3 Common ratio = $r = \frac{6}{3} = \frac{12}{6} = \frac{24}{12} = \frac{48}{24} = 2$

$$a_n = 2a_{n-1}$$

Example: Writing the Terms of a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- Write the first six terms of the geometric sequence with first term 12 and common ratio $\frac{1}{2}$
  - First term $a_1 = 12 \quad r = \frac{1}{2}$
  - Second term $a_2 = \frac{1}{2} \bullet 12 = 6$
  - Third term $a_3 = \frac{1}{2} \bullet 6 = 3$

Fourth term $a_4 = \frac{1}{2} \bullet 3 = \frac{3}{2}$

Fifth term $a_5 = \frac{1}{2} \bullet \frac{3}{2} = \frac{3}{4}$

Sixth term $a_6 = \frac{1}{2} \bullet \frac{3}{4} = \frac{3}{8}$

First six terms $= 12, 6, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$

Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- The nth term (the general term) of a geometric sequence with first term a_1 and common ratio r is $a_n = a_1 r^{n-1}$

$$a_2 = a_1 r \qquad a_3 = a_2 r = (a_1 r)r = a_1 r^2 \qquad a_4 = a_3 r = (a_1 r^2)r = a_1 r^3 \quad \ldots$$

The previous example $a_1 = 12,\ r = \dfrac{1}{2}$

$$a_2 = 12\left(\frac{1}{2}\right) = 6 \qquad a_3 = 12\left(\frac{1}{2}\right)^2 = 3$$

$$a_4 = 12\left(\frac{1}{2}\right)^3 = 12\left(\frac{1}{8}\right) = \frac{3}{2} \qquad a_5 = 12\left(\frac{1}{2}\right)^4 = 12\left(\frac{1}{16}\right) = \frac{3}{4}$$

Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

The *n*th Term of a Geometric Sequence

The *n*th term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$\begin{array}{cccccccc}
a_1, & a_2, & a_3, & a_4, & a_5, & \ldots\ldots, & a_n, & \ldots\ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
a_1, & a_1 r, & a_1 r^2, & a_1 r^3, & a_1 r^4, & \ldots, & a_1 r^{n-1}, & \ldots
\end{array}$$

Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- Find the nth term for the geometric sequence with first term 5 and common ratio 2.
  - The nth term (the general term) of a geometric sequence with first term a_1 and common ratio r is $a_n = a_1 r^{n-1}$

$$a_n = 5 \bullet 2^{n-1}$$

nth Term of a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- Find the seventh term of the geometric sequence whose first term is 5 and whose common ratio is −3.
  - The nth term (the general term) of a geometric sequence with first term a_1 and common ratio r is $a_n = a_1 r^{n-1}$

$$a_7 = 5(-3)^{7-1} = 5(-3)^6 = 5(729) = 3645$$

- The seventh term of the geometric sequence is 3645.

nth Term of a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- Find the twentieth term of the geometric sequence 1, 3, 9, 27, …

First term $a_1 = 1$ $r = \frac{3}{1} = \frac{9}{3} = \frac{27}{9} = 3$

$$a_n = a_1 r^{n-1} \qquad a_{20} = 1(3)^{20-1} = 3^{19} = 1{,}162{,}261{,}467$$

- The twentieth term of the geometric sequence is 1,162,261,467.

nth Term of a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- Find the fifteenth term of the geometric sequence with $a_3 = \frac{5}{4}$, and $a_6 = \frac{5}{32}$

Method 1

$$a_n = a_1 r^{n-1}$$

$$a_3 = a_1 r^{3-1} \qquad \frac{5}{4} = a_1 r^2 \qquad a_1 = \frac{5}{4r^2}$$

$$a_6 = a_1 r^{6-1} \qquad \frac{5}{32} = a_1 r^5 \qquad a_1 = \frac{5}{32r^5}$$

$$\frac{5}{4r^2} = \frac{5}{32r^5} \qquad 4r^2 = 32r^5 \qquad r^3 = \frac{1}{8} \qquad r = \frac{1}{2}$$

$$a_1 = \frac{5}{4\left(\frac{1}{2}\right)^2} = 5$$

$$a_{15} = 5\left(\frac{1}{2}\right)^{14} = \frac{5}{16384}$$

- The fifteenth term of the geometric sequence is $\frac{5}{16384}$

nth Term of a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- Find the fifteenth term of the geometric sequence with $a_3 = \frac{5}{4}$, and $a_6 = \frac{5}{32}$

Method 2 $a_m = a_n r^{m-n}$

$$a_6 = a_3 r^{6-3} \qquad \frac{5}{32} = \frac{5}{4} r^3 \qquad r^3 = \frac{1}{8} \qquad r = \frac{1}{2}$$

$$a_3 = a_1 r^{3-1} \qquad \frac{5}{4} = a_1 \left(\frac{1}{2}\right)^2 \qquad \frac{5}{4} = \frac{a_1}{4} \qquad a_1 = 5$$

$$a_{15} = 5\left(\frac{1}{2}\right)^{14} = \frac{5}{16384}$$

- The fifteenth term of the geometric sequence is $\frac{5}{16384}$

Objective: Identify, write, and manipulate geometric sequences.

Find the fifteenth term of the geometric sequence with $a_3 = \frac{5}{4}$, and $a_6 = \frac{5}{32}$

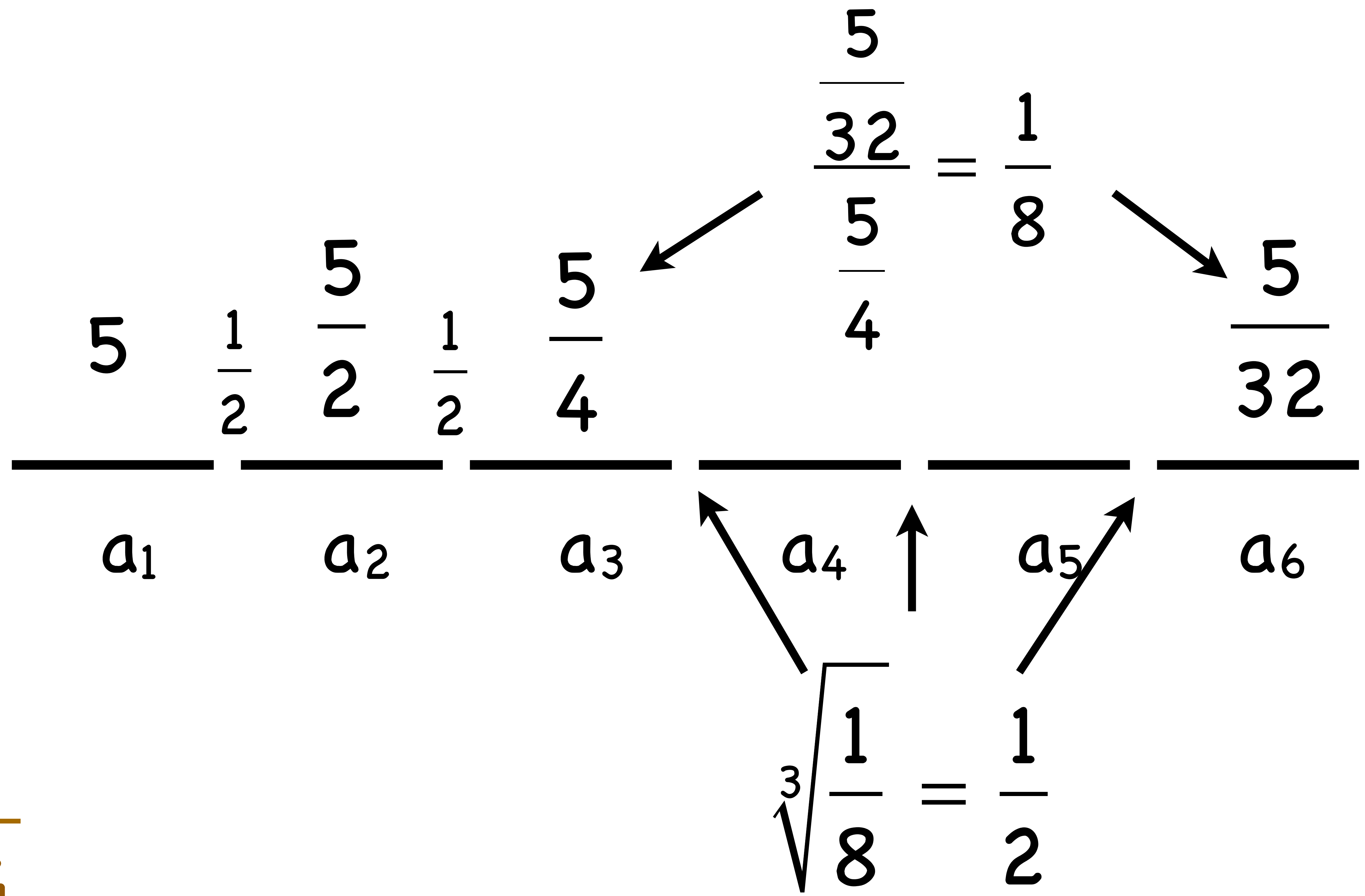
Method 3

$a_1 = 5 \quad r = \frac{1}{2}$

$$a_n = a_1 r^{n-1}$$

$$a_{15} = a_1 r^{15-1}$$

$$a_{15} = 5\left(\frac{1}{2}\right)^{14} = \frac{5}{16384}$$



$$\frac{\frac{5}{32}}{\frac{5}{4}} = \frac{1}{8}$$

a_1		a_2		a_3	a_4	a_5	a_6
5	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{5}{4}$			$\frac{5}{32}$

$$\sqrt[3]{\frac{1}{8}} = \frac{1}{2}$$

The Sum of the First *n* Terms of a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- The sum, S_n, of the first n terms of a geometric sequence is given by

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

a_1 is the first term

r is the common ratio ($r \neq 1$).

Let us take the series $S_5 =$ 2 + 6 + 18 + 54 + 162

Multiply by r = 3 $3S_5 =$ 6 + 18 + 54 + 162 + 486

Subtract the two $(1\text{–}3)S_5 =$ 2 − 486

$$(1\text{–}3)S_5 = 2 - 486 = 2(1 - 243)$$

$$S_5 = \frac{2(1-243)}{1-3} = \frac{2(1-3^5)}{1-3} = \frac{a_1(1-r^n)}{1-r}$$

Sum of Partial Geometric Series

Objective: Identify, write, and manipulate geometric sequences.

- The sum, S_n, of the first n terms of a geometric sequence is given by

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

a_1 is the first term

r is the common ratio ($r \neq 1$).

Let us take the series S_n = $a_1 + ra_1 + r^2a_1 + \ldots + r^{n-1}a_1$

Multiply by r rS_n = $ra_1 + r^2a_1 + \ldots + r^{n-1}a_1 + r^na_1$

Subtract the two $(1-r)S_n$ = $a_1 \qquad - r^na_1$

$$(1-r)S_n = a_1 - r^na_1 \qquad S_n = \frac{a_1 - r^na_1}{1-r} = \frac{a_1\left(1-r^n\right)}{1-r}$$

Objective: Identify, write, and manipulate geometric sequences.

- The sum, S_n, of the first n terms of a geometric sequence is given by

$$a_1 + a_2 + a_3 + \ldots + a_n \quad = a_1 + ra_1 + r^2a_1 + \ldots + r^{n-1}a_1$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1} = \frac{a_1(1-r^n)}{1-r}$$

  - With the summation notation, be careful with the exponent. Keep in mind the first term a_1 is multiplied by 1 or r^0.

Sum of a Finite Geometric Series

Objective: Identify, write, and manipulate geometric sequences.

The Sum of a Finite Geometric Sequence

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ .\ .\ .\ ,\ a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$.

Finding the Sum of the First n Terms of a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- Find the sum of the first nine terms of the geometric sequence: **2, -6, 18, -54, …**

$$a_1 = 2 \qquad r = \frac{-6}{2} = \frac{18}{-6} = \frac{-54}{18} = -3 \qquad S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$S_9 = \frac{a_1(1 - r^n)}{1 - r} = \frac{2(1 - (-3)^9)}{1 - -3} = \frac{2(1 - (-19683)}{4} = \frac{2(1 + 19683)}{4} = 9842$$

- The sum is 9842

Finding the Sum of the First *n* Terms of a Geometric Sequence

Objective: Identify, write, and manipulate geometric sequences.

- Evaluate $\sum_{i=1}^{20} 2\left(0.1\right)^{i}$ $\qquad S_n = \sum_{i=1}^{n} a_1 r^{i-1} = \dfrac{a_1(1-r^n)}{1-r}$

$$\sum_{i=1}^{20} 2\left(0.1\right)^{i} = \sum_{i=1}^{20} 2\left(0.1\right)\left(0.1\right)^{i-1} = \sum_{i=1}^{20} 0.2\left(0.1\right)^{i-1} \qquad a_1 = .2 \qquad r = .1$$

$$= \frac{.2\left(1-.1^{20}\right)}{1-.1} = \frac{.2\left(.99999999999999999999\right)}{.9} \approx \frac{2}{9}$$

- If that got past you, we simply need the first term and r.

$$a_1 = 2\left(0.1\right)^1 = .2 \qquad a_2 = 2\left(0.1\right)^2 = 2\left(0.01\right) = .02 \qquad r = \frac{.02}{.2} = .1$$

The Sum of an Infinite Geometric Series

Objective: Identify, write, and manipulate geometric sequences.

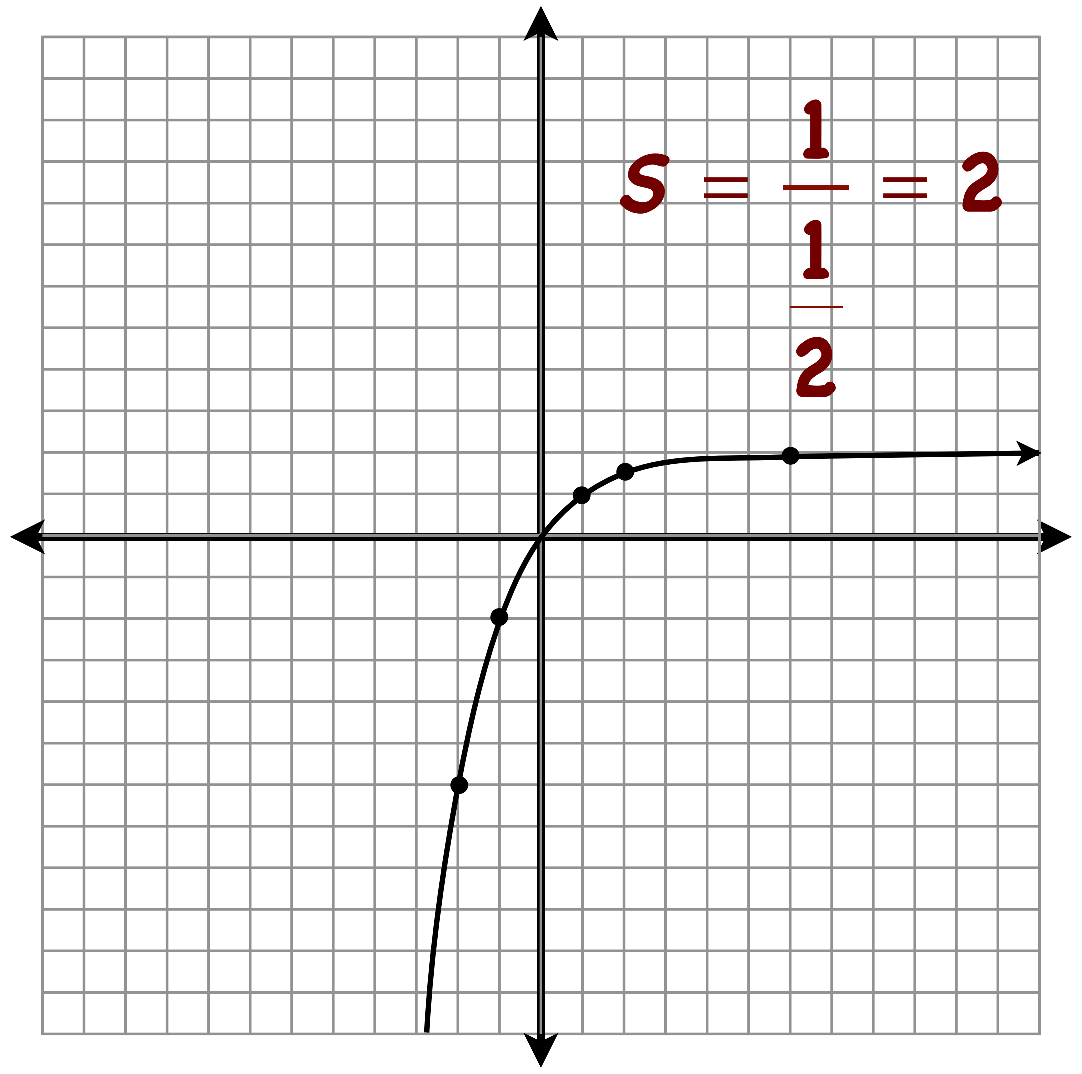
- Let us revisit the sum of a finite geometric series. $S_n = \dfrac{a_1(1-r^n)}{1-r}$

  Consider the case where n → ∞.

  If $|r| > 1$, r^n will continue to get larger, without end.

  But if $|r| < 1$, r^n will get smaller, in fact, as $n \to \infty$, $r^n \to 0$.

  Try graphing $y = 1 \bullet \dfrac{1-\left(\frac{1}{2}\right)^x}{1-\frac{1}{2}} = \dfrac{1-\left(\frac{1}{2}\right)^x}{\frac{1}{2}}$ as x ➛ ∞, the graph ➛ 2



  so as $n \to \infty$, $S_\infty = \dfrac{a_1(1-0)}{1-r} \rightarrow S = \dfrac{a_1}{1-r}$

Infinite Geometric Series

Objective: Identify, write, and manipulate geometric sequences.

- The sum of an infinite geometric series when |r| < 1 is

$$S = \sum_{i=1}^{\infty} a_1 r^{i-1} = \sum_{i=0}^{\infty} a_1 r^{i} = \frac{a_1}{1-r}$$

- If |r| > 1, the infinite geometric series does not have a sum.

Sum of an Infinite Geometric Series

Objective: Identify, write, and manipulate geometric sequences.

The Sum of an Infinite Geometric Series

If $|r| < 1$, the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdot \cdot \cdot + a_1r^{n-1} + \cdot \cdot \cdot$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1 - r}.$$

Example: Finding the Sum of an Infinite Geometric Series

Objective: Identify, write, and manipulate geometric sequences.

- Find the sum of the infinite geometric series: $3+2+\frac{4}{3}+\frac{8}{9}+\ldots$

$a_1 = 3$

$$r=\frac{2}{3}=\frac{\frac{4}{3}}{2}=\frac{\frac{8}{9}}{\frac{4}{3}}=\frac{2}{3}$$

$$S=\frac{a_1}{1-r}=\frac{3}{1-\frac{2}{3}}=\frac{3}{\frac{1}{3}}=9$$

Example

Objective: Identify, write, and manipulate geometric sequences.

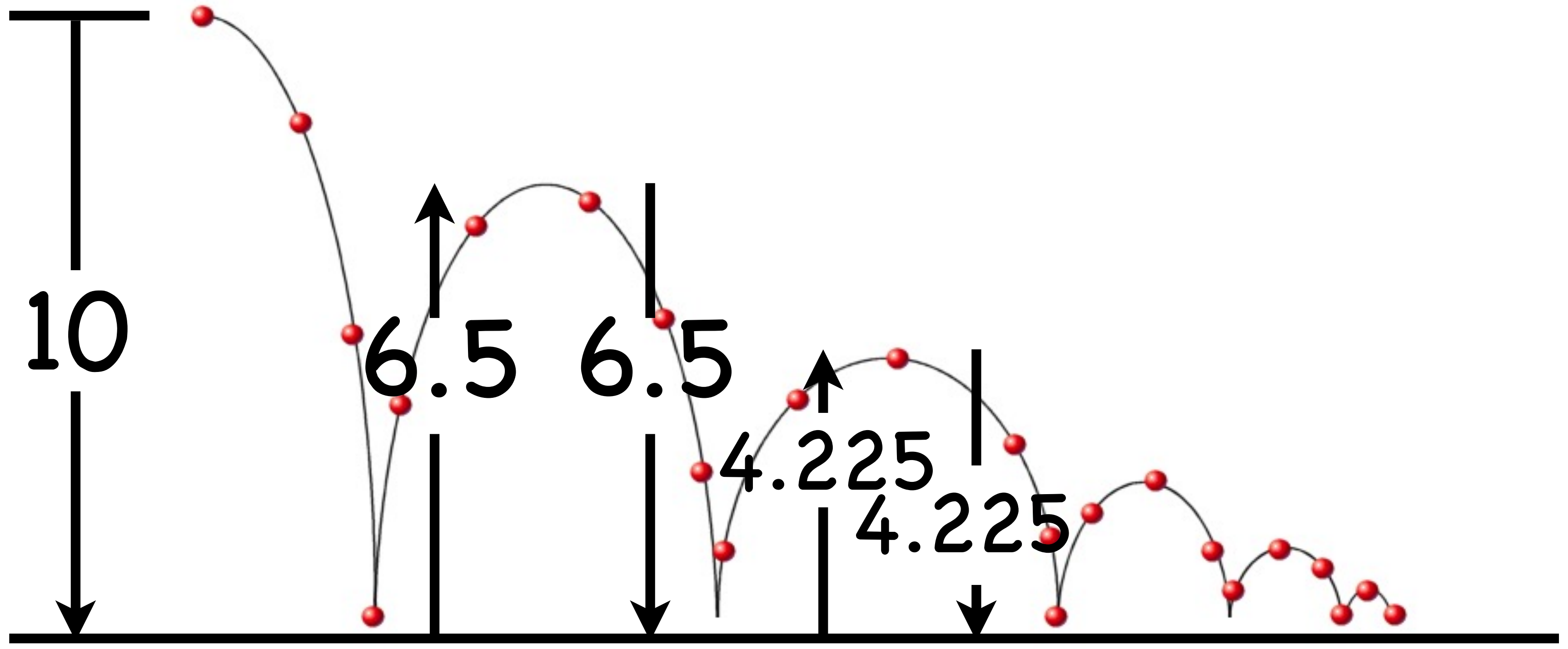
- Evaluate $\sum_{n=1}^{\infty} 2(0.4)^{n-1}$

$$\sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{1-r} \qquad a_1 = 2(0.4)^{1-1} = 2 \qquad a_2 = 2(0.4)^{2-1} = .8 \qquad r = \frac{.8}{2} = .4$$

$$\sum_{n=1}^{\infty} 2(0.4)^{n-1} = \frac{2}{1-0.4} = \frac{2}{.6} = \frac{10}{3}$$

Objective: Identify, write, and manipulate geometric sequences.

- A ball is dropped from a height of 10 feet. Each time it bounces back up, it bounces 0.65 times as high as it did on the previous bounce. What is the total distance traveled by the ball?



$$10 + 2(10)(.65) + 2[(10)(.65)](.65) + \ldots$$

$$= 10 + 2(10)(.65) + 2(10)(.65)^2 + \ldots$$

$$= 10 + 2[(10)(.65) + (10)(.65)^2 + \ldots]$$

$$= 10 + 2\sum_{n=1}^{\infty} 10\left(0.65\right)^{n} = 10 + 2\sum_{n=1}^{\infty} 10\left(.65\right)\left(0.65\right)^{n-1} = 10 + 2\sum_{n=1}^{\infty} 6.5\left(0.65\right)^{n-1}$$

$$= 10 + 2\left(\frac{6.5}{1-.65}\right) = 10 + 2\left(\frac{6.5}{.35}\right) = 10 + 2\left(\frac{6.5}{.35}\right) \approx 47.1429\ ft$$