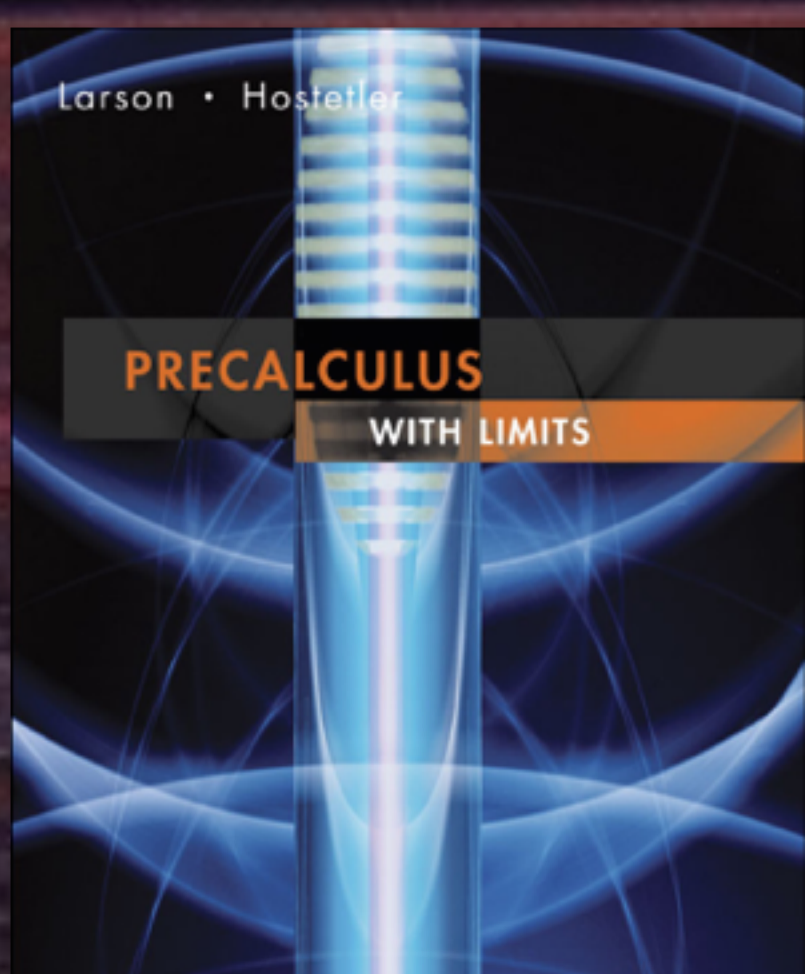
Chapter 9

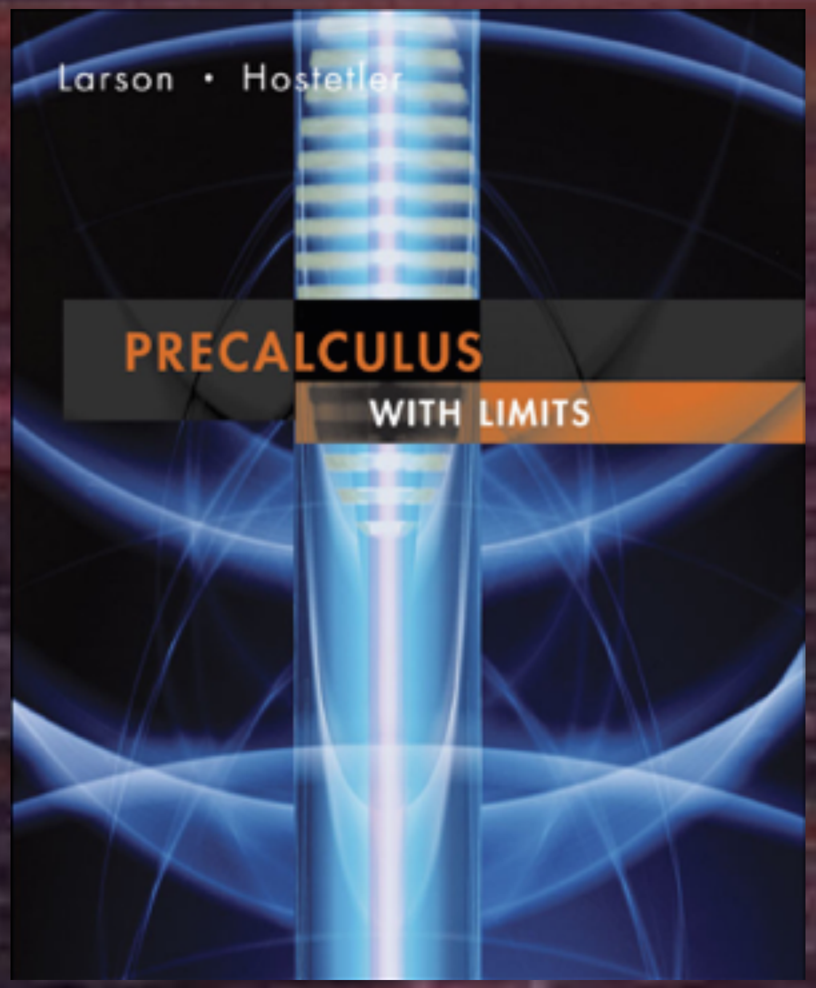
★ Sequences, and Series



★ 9.5 Binomial Theorem

Chapter 10

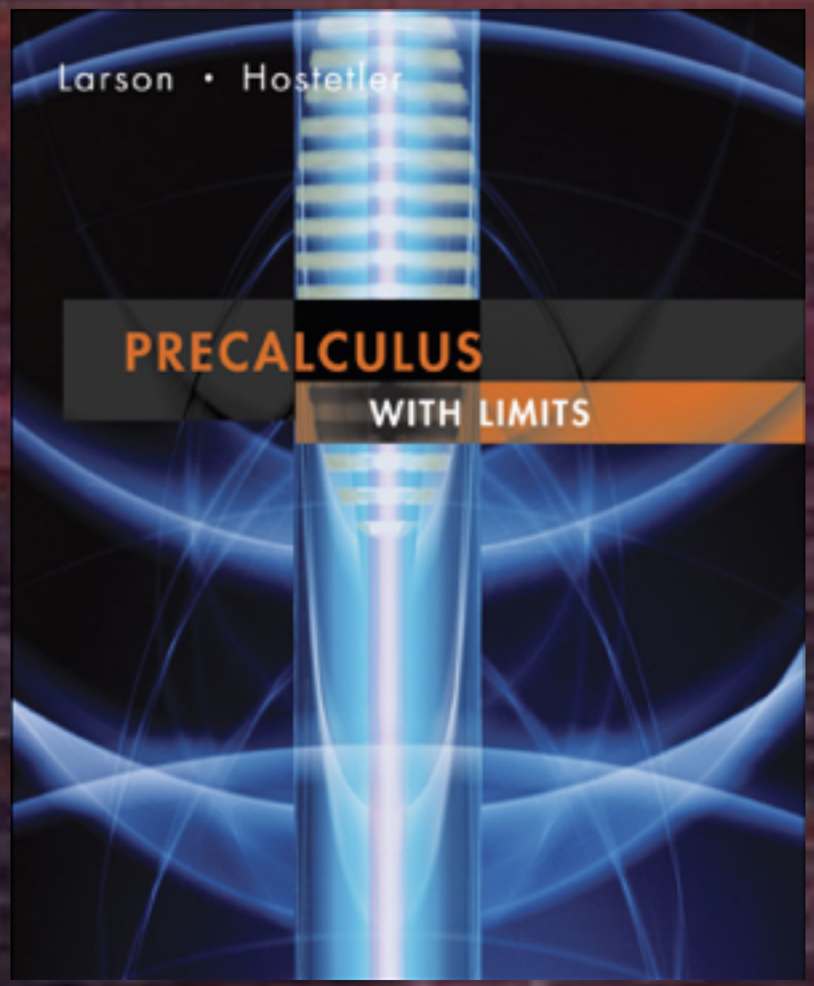
- Homework



- Read Sec 9.5
- Do p688 5, 9, 25, 31, 43, 51, 57, 61, 65, 77

Chapter 9

★ Objectives



- ★ Find Evaluate a binomial coefficient.
- ★ Expand a binomial raised to a power.
- ★ Find a particular term in a binomial expansion.

Combination

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

★ For nonnegative integers n and r, with n ≥ r, the expression $\binom{n}{r}$ (or ${}_nC_r$) called a binomial coefficient (combination) and is defined by:

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

★ Remember n! = n•(n-1)•(n-2)•(n-3)•...•1

Example: Evaluating Binomial Coefficients

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

Evaluate each of the following:

$$_6C_3 = \binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = \frac{6 \bullet 5 \bullet 4 \bullet \cancel{3 \bullet 2 \bullet 1}}{3 \bullet 2 \bullet 1 \bullet (\cancel{3 \bullet 2 \bullet 1})} = 20$$

$$_6C_0 = \binom{6}{0} = \frac{6!}{0!(6-0)!} = \frac{6!}{0!6!} = \frac{6!}{1 \bullet 6!} = 1$$

Example: Evaluating Binomial Coefficients

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

Evaluate each of the following:

$$_{8}C_{2} = \binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{8!}{2!6!} = \frac{8 \bullet 7 \bullet \cancel{6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}}{2 \bullet 1 \bullet (\cancel{6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1})} = 28$$

$$_{3}C_{3} = \binom{3}{3} = \frac{3!}{3!(3-3)!} = \frac{3!}{0!3!} = 1$$

Example: Evaluating Binomial Coefficients

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

★ There are a couple of special cases you should remember.

Evaluate each of the following:

$$_8C_2 = \binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{8!}{2!6!} = 28$$

$$\binom{n}{r} = \binom{n}{n-r}$$

$$_nC_r = {_nC_{n-r}}$$

$$_8C_6 = \binom{8}{6} = \frac{8!}{6!(8-6)!} = \frac{8!}{6!2!} = 28$$

Example: Evaluating Binomial Coefficients

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

★ There are a couple of special cases you should remember.

Evaluate each of the following:

$${}_6C_0 = \binom{6}{0} = \frac{6!}{0!(6-0)!} = \frac{6!}{0!6!} = \frac{6!}{1 \bullet 6!} = 1 \qquad \binom{n}{0} = \binom{n}{n}$$

$${}_6C_6 = \binom{6}{6} = \frac{6!}{6!(6-6)!} = \frac{6!}{6!0!} = \frac{6!}{6! \bullet 1} = 1 \qquad {}_nC_0 = {}_nC_n$$

Binomial Expansion

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

★ Expanding powers of binomials

Note the Coefficients

$(a+b)^2 = (a+b)(a+b) = 1a^2 + 2ab + 1b^2$ 1 2 1

$(a+b)^3 = (a+b)(a+b)(a+b) = (a^2 + 2ab + b^2)(a+b)$

$= 1a^3 + 3a^2b + 3ab^2 + 1b^3$ 1 3 3 1

$(a+b)^4 = (a+b)(a+b)(a+b)(a+b) = (a^3 + 3a^2b + 3ab^2 + b^3)(a+b)$

$= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$ 1 4 6 4 1

Pascal's Triangle

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

Row 0 → 1

Row 1 → 1 1

Row 2 → 1 2 1

Row 3 → 1 3 3 1

Row 4 → 1 4 6 4 1

Row 5 → 1 5 10 10 5 1

Note the Coefficients

1 2 1

1 3 3 1

1 4 6 4 1

★ It turns out, the rows of Pascal's triangle form the coefficients of the binomial expansion. The nth row forms the coefficients for the expansion of $(a + b)^n$.

Using Pascal

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

$$(x+y)^0 = 1$$ 0th row

$$(x+y)^1 = 1x + 1y$$ 1st row

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$ 2nd row

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$ 3rd row

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$ ⋮

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$$(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

$$(x+y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

Binomial Expansion

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

★ When expanding a binomial $(a + b)^n$ using Pascal's triangle.

1. There are n + 1 terms.
2. The coefficients are the numbers from the nth row of Pascal's triangle
3. The exponent of a is n in the first term, and the exponent decreases by 1 in each successive term.
4. The exponent of b is 0 in the first term, and the exponent increases by 1 in each successive term.
5. The sum of the exponents in any term is n.

The Binomial Theorem

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

★ For any positive integer n,

$$(a+b)^n = \sum_{i=1}^{n} {}_nC_r\, a^{n-r} b^r$$

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n} a^0 b^n$$

★ a and b can represent any form of number; (i.e. integer, variable, or variable expression).

★ Note the pattern of the exponents for each term.

The Binomial Theorem

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r\, x^{n-1}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote binomial coefficients.

Example: Using the Binomial Theorem

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

Expand $(x - 2y)^5$ $a = x$, $b = -2y$, $n = 5$

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}a^0 b^n$$

$$(x-2y)^5 = \binom{5}{0}x^5(-2y)^0 + \binom{5}{1}x^4(-2y)^1 + \binom{5}{2}x^3(-2y)^2 +$$
$$+ \binom{5}{3}x^2(-2y)^3 + \binom{5}{4}x^1(-2y)^4 + \binom{5}{5}x^0(-2y)^5$$

Example: Using the Binomial Theorem

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

Expand $(x - 2y)^5$ $a = x$, $b = -2y$, $n = 5$

$$(x-2y)^5 = \binom{5}{0}x^5(-2y)^0 + \binom{5}{1}x^4(-2y)^1 + \binom{5}{2}x^3(-2y)^2 + \binom{5}{3}x^2(-2y)^3 + \binom{5}{4}x^1(-2y)^4 + \binom{5}{5}x^0(-2y)^5$$

$$(x-2y)^5 = \frac{5!}{0!(5-0)!}x^5(-2y)^0 + \frac{5!}{1!(5-1)!}x^4(-2y)^1 + \frac{5!}{2!(5-2)!}x^3(-2y)^2 + \frac{5!}{3!(5-3)!}x^2(-2y)^3 + \frac{5!}{4!(5-4)!}x^1(-2y)^4 + \frac{5!}{5!(5-5)!}x^0(-2y)^5$$

Example: Using the Binomial Theorem

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

Expand $(x - 2y)^5$ $\quad a = x,\ b = -2y,\ n = 5$

$$(x-2y)^5 = \frac{5!}{0!(5-0)!}x^5(-2y)^0 + \frac{5!}{1!(5-1)!}x^4(-2y)^1 + \frac{5!}{2!(5-2)!}x^3(-2y)^2 +$$

$$+ \frac{5!}{3!(5-3)!}x^2(-2y)^3 + \frac{5!}{4!(5-4)!}x^1(-2y)^4 + \frac{5!}{5!(5-5)!}x^0(-2y)^5$$

$$(x-2y)^5 = 1x^5(-2y)^0 + 5x^4(-2y)^1 + 10x^3(-2y)^2 +$$

$$10x^2(-2y)^3 + 5x^1(-2y)^4 + 1x^0(-2y)^5$$

Example: Using the Binomial Theorem

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

Expand $(x - 2y)^5$ $a = x$, $b = -2y$, $n = 5$

$$(x-2y)^5 = 1x^5(-2y)^0 + 5x^4(-2y)^1 + 10x^3(-2y)^2 + 10x^2(-2y)^3 + 5x^1(-2y)^4 + 1x^0(-2y)^5$$

$$(x-2y)^5 = 1x^5(1) + 5x^4(-2y) + 10x^3(4y^2) + 10x^2(-8y^3) + 5x^1(16y^4) + 1x^0(-32y^5)$$

Example: Using the Binomial Theorem

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

Expand $(x - 2y)^5$ $a = x$, $b = -2y$, $n = 5$

$$(x-2y)^5 = 1x^5(1) + 5x^4(-2y) + 10x^3(4y^2) + 10x^2(-8y^3) + 5x^1(16y^4) + 1x^0(-32y^5)$$

$$(x-2y)^5 = x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$$

Summary

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

- When expanding a binomial $(a + b)^n$, keep in mind:
  - The first term of $(a+b)^n = a^n$.
  - The last term of $(a+b)^n$ is b^n.
  - The exponent of a decreases by one each term.
  - The exponent of b increases by one each term.
  - The expansion of $(a+b)^n$ has $n+1$ terms.
  - Each term has degree n. The exponents add to n.
  - DO NOT forget to raise the coefficient of each term to the appropriate exponent.

Example

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

★ Expand $(2x - 3y)^4$

$$(2x-3y)^4 = \binom{4}{0} \qquad + \binom{4}{1} \qquad + \binom{4}{2} \qquad + \binom{4}{3} \qquad + \binom{4}{4}$$

$$(2x-3y)^4 = 16x^4 + 4(8x^3)(-3y) + 6(4x^2)(9y^2) + 4(2x)(-27y^3) + 81y^4$$

$$(2x-3y)^4 = 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$$

Finding a specific term

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

- Expand $(2x - 3y)^{19}$
- How many terms are there?
- I am certain you do not want to find all 20 terms, so how 'bout we just find the 12th term?
- In the previous example note the patterns in the terms.

$$(2x-3y)^4 = \binom{4}{0}(2x)^4(-3y)^0 + \binom{4}{1}(2x)^3(-3y)^1 + \binom{4}{2}(2x)^2(-3y)^2 + \binom{4}{3}(2x)^1(-3y)^3 + \binom{4}{4}(2x)^0(-3y)^4$$

Term 1 Term 2 Term 3 Term 4 Term 5

$$\binom{4}{0}(2x)^{4-0}(-3y)^0 \quad \binom{4}{1}(2x)^{4-1}(-3y)^1 \quad \binom{4}{2}(2x)^{4-2}(-3y)^2 \quad \binom{4}{3}(2x)^{4-3}(-3y)^3 \quad \binom{4}{4}(2x)^{4-4}(-3y)^4$$

Finding a Particular Term in a Binomial Expansion

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

The (r+1)st term of the expansion of $(a + b)^n$ is

$$\binom{n}{r} a^{n-r} b^{r}$$

The 5th term of $(a+b)^{12}$ is $\binom{12}{4} a^{12-4} b^{4} = 495a^{8}b^{4}$

Example: Finding a Single Term of a Binomial Expansion

Objective: Use the Binomial Theorem and Pascal's Triangle to calculate binomial coefficients and binomial expansions.

Find the fifth term in the expansion of $(2x + y)^9$

$a = 2x$, $b = y$, $n = 9$, 5th term, $r = 4$

$$\binom{9}{4}a^5b^4 = 126(2x)^5y^4 = 126(32x^5)y^4$$

$$= 4032x^5y^4$$