Finding all Zeros of a Polynomial Function

When solving polynomial equations, use the rational zero test to find all possible rational zeros, then use Descarte's Rule of Signs to help narrow down the choices if possible. The fundamental theorem of Algebra plays a major role in this.

The Fundamental Theorem of Algebra

Every polynomial equation of degree n with complex coefficients has n roots in the complex numbers.

In other words, if you have a 5th degree polynomial equation, it has 5 roots.

Example: Find all zeros of the polynomial function $f_{(x)} = 2x^4 + 7x^3 - 4x^2 - 27x - 18$.

$$2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$$

Begin by setting the function equal to zero.

Find all possible rational zeros.

$$\pm 1,\quad \pm 2,\quad \pm 3,\quad \pm 6,\quad \pm 9,\quad \pm 18,\quad \pm\frac{1}{2},\quad \pm\frac{3}{2},\quad \pm\frac{9}{2}$$

Once again, there are 18 possible zeros to the function. If Descarte's Rule of Signs is used, it may or may not help narrow down the choices for synthetic division.

For this equation, there is 1 possible positive zero, and either 3 or 1 possible negative zeros.

This information was found in a previous example. Based on this, a chart may be constructed showing the possible combinations. Remember, this is a 4th degree polynomial, so each row must add up to 4.

+	-	*i*
1	***3***	***0***
1	***1***	***2***

$$\pm 1,\quad \pm 2,\quad \pm 3,\quad \pm 6,\quad \pm 9,\quad \pm 18,\quad \pm\frac{1}{2},\quad \pm\frac{3}{2},\quad \pm\frac{9}{2}$$

Now set up a synthetic division problem, and begin checking each zero until a root of the equation is found..

$$\begin{array}{|lllll}
2 & 7 & -4 & -27 & -18 \\
0 & & & & \\
\hline
\end{array}$$

$$\begin{array}{r|rrrrr}
-1 & 2 & 7 & -4 & -27 & -18 \\
 & 0 & -2 & -5 & 9 & 18 \\
\hline
 & 2 & 5 & -9 & -18 & 0
\end{array}$$

-1 works as a zero of the function. There are now 3 zeros left. We can continue to test each zero, but we need to first rewrite the new polynomial.

$2x^3 + 5x^5 - 9x - 18$

The reason this must be done is to check using the rational zero test again. Using the rational zero test again could reduce the number of choices to work with, or the new polynomial may be factorable.

$$\begin{array}{r|rrrr}
-3 & 2 & 5 & -9 & -18 \\
 & 0 & -6 & 3 & 18 \\
\hline
 & 2 & -1 & -6 & 0
\end{array}$$

Here, we found that -3 works. The reason negative numbers are being used first here is because of the chart above. The chart says there is a greater chance of one of the negatives working rather than a positive, since there are potentially 3 negative zeros here and only one positive. Notice the new equation was used for the division.

$$2x^2 - x - 6$$
$$(2x+3)(x-2) = 0$$
$$x = -3/2 \quad and \quad x = 2$$

This is now a factorable polynomial. Solve by factoring.

We now have all zeros of the polynomial function. They are $-3/2,\ -1,\ -3\ \ and\ \ 2$

Be aware, the remaining polynomial may not be factorable. In that case, it will be necessary to either use the quadratic formula, or complete the square.
Find all <u>real</u> zeros of the following functions (no complex numbers). Remember, if there is no constant with which to use the rational zero test, factor out a zero first, then proceed.

A) $f_{(x)} = x^3 - 6x^2 + 11x - 6$

B) $f_{(x)} = x^3 - 9x^2 + 27x - 27$

C) $f_{(x)} = x^3 - 9x^2 + 20x - 12$

D) $f_{(x)} = x^4 - 7x^2 + 12$

E) $f_{(x)} = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

F) $f_{(x)} = x^4 - 13x^2 - 12x$

There is a difference between the questions: "find all real zeros" and "find all zeros." Be careful to pay attention to which is being asked.

The following questions, denoted by *, show up in the* "Writing a Polynomial Function in Factored Form" *topic. Since these functions show up later, the zeros of these functions only need to be found once. As we move on to that topic,, the answers found below will be used to rewrite the polynomial in the manner indicated.

***Find <u>all</u> zeros of the following functions. (Include any complex solutions)**

A) $f_{(x)} = x^4 - 81$

B) $f_{(x)} = x^4 - 7x^2 + 12$

C) $f_{(x)} = x^3 - x + 6$

D) $f_{(x)} = x^6 + 4x^4 - 41x^2 + 36$

E) $f_{(x)} = x^4 + 10x^2 + 9$

F) $f_{(x)} = x^4 - x^3 + 25x^2 - 25x$

G) $f_{(x)} = x^4 - x^3 - 2x^2 - 4x - 24$

H) $f_{(x)} = x^4 - x^3 - 29x^2 - x - 30$

I) $f_{(x)} = x^3 - x^2 - 3x + 3$

J) $f_{(x)} = x^4 - 7x^2 + 10$

K) $f_{(x)} = x^3 - 6x^2 + 13x - 10$

L) $f_{(x)} = x^5 + 15x^3 - 16x$

Writing a Polynomial Function in Factored Form

Once all zeros of a polynomial function are found, the function can be rewritten in one of several different ways.

A polynomial function may be written in one of the following ways.

- ***As a product of factors that are irreducible over the rationals.***
  This means only rational numbers may be used in the factors.

- ***As a product of factors that are irreducible over the reals.***
  Irrational numbers may be used as long as they are real, i.e. $\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$.

- ***In completely factored form. This may also be written as a product of linear factors.***
  Complex numbers may be used in the factors, i.e. $(x+2i)(x-2i)$.

This section involves writing polynomials in one of the factored forms illustrated above. These are the same problems that were solved on the previous page, so there is no need to solve them again. Use the solutions previously found, to write the polynomial in the desired form.

For example, a polynomial function that has zeros of 3 ***and*** $2\pm\sqrt{3}$ ***would look like the following; in completely factored form.***

$$f_{(x)} = (x-3)\left(x-2+\sqrt{3}\right)\left(x-2-\sqrt{3}\right)$$

Notice each variable x is to the first power, so these are linear factors.

When polynomial functions are written like this, it is obvious where the x intercepts lie.

***Write the polynomial function as a product of factors that are irreducible over the reals.**

A) $f_{(x)} = x^4 - 81$ **B)** $f_{(x)} = x^4 - 7x^2 + 12$ **C)** $f_{(x)} = x^3 - x + 6$

D) $f_{(x)} = x^6 + 4x^4 - 41x^2 + 36$ **E)** $f_{(x)} = x^4 + 10x^2 + 9$ **F)** $f_{(x)} = x^4 - x^3 + 25x^2 - 25x$

G) $f_{(x)} = x^4 - x^3 - 2x^2 - 4x - 24$ **H)** $f_{(x)} = x^4 - x^3 - 29x^2 - x - 30$ **I)** $f_{(x)} = x^3 - x^2 - 3x + 3$

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***Write the polynomial functions in completely factored form. (Remember, this can also be asked in the form "Write polynomial as a product of linear factors.")**

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C) $f_{(x)} = x^3 - x + 6$

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Write $x^4 - 81$ **as a product of linear factors.**

Write $x^4 - 16$ **as a product of linear factors.**

How can you identify linear factors?